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SLLN and annealed CLT for random walks in I.I.D. random environment on Cayley trees

Siva Athreya[a], Antar Bandyopadhyay[b,c,*], Amites Dasgupta[c], Neeraja Sahasrabudhe[d]

[a] *Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore, 560059, India*

[b] *Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi Centre, 7 S. J. S. Sansanwal Marg, New Delhi 110016, India*

[c] *Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Kolkata, 203 B. T. Road, Kolkata 700108, India*

[d] *Department of Mathematical Sciences, Indian Institute of Science Education and Research, Mohali, Knowledge City, Sector 81, Manauli, Sahibzada Ajit Singh Nagar, Punjab 140306, India*

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Abstract

We consider the random walk in an independent and identically distributed (i.i.d.) random environment on a Cayley graph of a finite *free product* of copies of $\mathbb{Z}$ and $\mathbb{Z}_2$. Such a Cayley graph is readily seen to be a regular tree. Under a uniform ellipticity assumption on the i.i.d. environment we show that the walk has positive speed and establish the annealed central limit theorem for the graph distance of the walker from the starting point.

Keywords: Random walk on free group; Random walk in random environment; Trees; Transience; Central limit theorem; Positive speed

1. Introduction

In this article we consider a *Random Walk in Random Environment (RWRE)* model on an infinite regular tree where the environment at each vertex is independent and are also

* Corresponding author at: Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi Centre, 7 S. J. S. Sansanwal Marg, New Delhi 110016, India.

E-mail addresses: athreya@isibang.ac.in (S. Athreya), antar@isid.ac.in (A. Bandyopadhyay), amites@isical.ac.in (A. Dasgupta), neeraja@iisermohali.ac.in (N. Sahasrabudhe).

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"*identically*" distributed. To make this notion of i.i.d.-ness of the environment rigorous we adopt the framework developed in [1] and define the model on a finitely generated non-abelian free group and then transfer it back to an appropriate even degree regular tree (Section 2). As in any (static) RWRE model, we will first choose an *environment* by above specified random mechanism and keep it fixed throughout the time evolution. A walker then moves randomly on the vertex set of a regular tree in such a way that given the environment, its position forms a time-homogeneous Markov chain whose transition probabilities depend only on the environment. Under a uniform ellipticity assumption we prove that the walk has positive speed (Theorem 2.1) and obeys an annealed central limit theorem (Theorem 2.2).

RWRE on the integer lattice $\mathbb{Z}^d$ for $d \geq 1$ as well as on the infinite d-regular tree $\mathbb{T}_d$ for $d \geq 3$ (also known as *Bethe lattice* [4]) has been studied in great detail and there is a vast literature. RWRE model on the one-dimensional integer lattice $\mathbb{Z}$ was first introduced by Solomon in [17] where he gave explicit criteria for the recurrence and transience of the walk for *independent and identically distributed (i.i.d.)* environment distribution. Since then a large variety of results have been discovered for RWRE in $\mathbb{Z}^d$, we refer the reader to [18,19] for a review of the subject.

Perhaps the earliest known results for RWRE on trees are by Pementle and Lyons [12]. In that paper they consider a model on rooted tress, which later came to be known as *random conductance model*. In that model, the random conductances along each path from vertices to the root are assumed to be independent and identically distributed. The random walk is then shown to be recurrent or transient depending on how large the value of the average conductance is. Later [10] considered the same model under additional assumption of the jump probabilities are also i.i.d. and have studied the speed of the walk in the recurrent regime. There have been several work also on random walk on random trees, particularly on random walk on Galton–Watson trees [7,11,13–15]. In [6] the authors developed a technique that provided a lower bound on the speed of transient random walk in a random environment on regular trees. They also used bounds on the first regeneration level and regeneration time to obtain an annealed invariance principle. Their methods also applied to once-reinforced random walk.

Motivated by these, [1] considered a RWRE model on a regular tree, where the environment (or rather the transition laws) at each vertex are independent and also "identically" distributed. However, unlike in the usual RWRE models on integer lattices, such as on $\mathbb{Z}$ as introduced by [17], or the random conductance models on trees [12], it is not entirely obvious how to make the random transition laws on the vertices of a tree "identically" distributed. To make this notion of "*i.i.d. environment*" rigorous in [1] defined the model on a finite free product of copies of $\mathbb{Z}$ and $\mathbb{Z}_2$ and then transfers it back to a regular tree of appropriate degree which is essentially same as a Cayley graph associated with the group. A more detailed description is given in the following section. A similar model was also considered in [16]. In both [1,16] under differing mild non-degeneracy assumptions the authors proved that the RWRE is transient with probability one.

In this work, we prove that under a *uniform ellipticity* assumption on the i.i.d. environment (which is a stronger assumption than the one made in [1]), the walk has a positive drift away from the starting point and admits an *annealed Central Limit Theorem* under linear centring and square-root scaling. We adapt the regeneration time technique developed in [6] and obtain bounds on the moments of the regeneration levels and times. However, we will see in the sequel that the definition of the i.i.d. environment needs a group action of the underlying Cayley tree (assumption (E1)). Thus many details of the proof including the definition of the regeneration time have been worked out using different arguments than those in [6].

It is worth noting here that, the RWRE on integer lattice $\mathbb{Z}^d$ with i.i.d. random environment is often considered as a good discrete model for *random walk on random media* [19]. $\mathbb{Z}^d$ can also be viewed as the Cayley graph of the *free abelian* group with d generators. The i.i.d. environment then becomes a group invariant measure on the environment space, where the actions of the group are nothing but the usual euclidean translations. In the same vain, our model essentially complements this idea by defining it appropriately on any finitely presented *non-abelian free group*. As we shall see in the next section our model provides a group invariant measure on the environment space resulting to an i.i.d. environment on the associate (left)-Cayley graph, which in this case will turnout to be an infinite regular tree/Bethe lattice. Thus our work can be used for studying transportation problem on a random media where the underlying space is non-amenable. As noted in [5], many statistical physics problems which are studied on integer lattices, are also studied on Bethe lattices. In general, such studies are interesting on their own right, but they also throw light on the classical problems by comparing the differences which may arise on Bethe lattice. This aspect is also one of our motivation to consider our model.

The rest of the paper is organised as follows. In Section 2 we define the model precisely and state our main results. In Section 3, we prove tail bounds for regeneration levels (Proposition 3.3) and moment bounds for regeneration times (Proposition 3.4). Using the bounds obtained, in Section 4 we prove our main results.

2. Model and main results

Even though our framework is same as in [1], but for sake of completeness, we begin by providing a detailed description of the model below.

Group structure: Following [1] we will also consider a group G which is a free product of finitely many groups, say, $G_1, G_2, \ldots, G_k$ and $H_1, H_2, \ldots, H_r$, where each $G_i \cong \mathbb{Z}$ and each $H_j \cong \mathbb{Z}_2$. Let $d = 2k + r$.

Cayley graph: Let G be a group defined above. Suppose $G_i = \langle a_i \rangle$ for $1 \leq i \leq k$ and $H_j = \langle b_j \rangle$ where $b_j^2 = e$ for $1 \leq j \leq r$. Here by $\langle a \rangle$ we mean the group generated by a single element a. Let $S := \{a_1, a_2, \ldots, a_k\} \cup \left\{a_1^{-1}, a_2^{-1}, \ldots, a_k^{-1}\right\} \cup \{b_1, b_2, \ldots, b_r\}$ be a generating set for G. We note that S is a symmetric set, that is, $s \in S \iff s^{-1} \in S$. In what will follow the order of the generating elements will not be of any consequence. We will thus list the elements of S simply by $\{s_1, s_2, \ldots, s_d\}$.

We now define a graph $\bar{G}$ with vertex set G and edge set $E := \left\{\{x, y\} \,\middle|\, yx^{-1} \in S\right\}$. We will say $x \sim y$ whenever $\{x, y\} \in E$. Such a graph $\bar{G}$ is called a *(left) Cayley Graph* of G with respect to the generating set S. Since G is a free product of groups which are isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2$, it is easy to see that $\bar{G}$ is a graph with no cycles and is regular with degree d, thus it is isomorphic to the d-regular infinite tree which we will denote by $\mathbb{T}_d$. We refer the reader to [20] for a treatment of other infinite graphs and groups. We will abuse the terminology a bit and will write $\mathbb{T}_d$ for the Cayley graph of the group G. This way we essentially endow the d-regular tree, $\mathbb{T}_d$, a *group structure*, which we will make use to define an *i.i.d.* environment.

Note that for the d-dimensional Euclidean lattice, such a group structure is automatic, which is the product of d copies of the abelian free group $\mathbb{Z}$. In our case, all the difference appears due to the fact that on $\mathbb{T}_d$, a group can only be obtained through free product of several copies of $\mathbb{Z}$ and also with possible free product of groups generated by *torsion* elements.

We will consider the identity element e of G as the root of $\mathbb{T}_d$. We will write $N(x) := \left\{ y \in G \,\middle|\, yx^{-1} \in S \right\}$ for the set of all neighbours of a vertex $x \in \mathbb{T}_d$. Observe that from definition $N(e) = S$.

For $x \in G$, define the mapping $\theta_x : G \to G$ by $\theta_x(y) = yx$, then θ_x is an automorphism of $\mathbb{T}_d$. We will call θ_x the *translation by* x. For a vertex $x \in \mathbb{T}_d$ and $x \neq e$, we denote by $|x|$, the length of the unique path from the root e to x and $|e| = 0$. Further, if $x \in \mathbb{T}_d$ and $x \neq e$ then we define $\overleftarrow{x}$ as the *parent* of x, that is, the penultimate vertex on the unique path from e to x.

Random Environment: Let $\mathcal{S} := \mathcal{S}_e$ be a collection of probability measures on the d elements of $N(e) = S$. To simplify the presentation and avoid various measurability issues, we assume that $\mathcal{S}$ is a Polish space (including the possibilities that $\mathcal{S}$ is finite or countably infinite). For each $x \in \mathbb{T}_d$, $\mathcal{S}_x$ is the push-forward of the space $\mathcal{S}$ under the translation θ_x, that is, $\mathcal{S}_x := \mathcal{S} \circ \theta_x^{-1}$. Note that the probabilities on $\mathcal{S}_x$ have support on $N(x)$. That is to say, an element $\omega(x, \cdot)$ of $\mathcal{S}_x$, is a probability measure satisfying $\omega(x, y) \geq 0 \; \forall \; y \in \mathbb{T}_d$ and $\sum_{y \in N(x)} \omega(x, y) = 1$.

Let $\mathcal{B}_{\mathcal{S}_x}$ denote the Borel σ-algebra on $\mathcal{S}_x$. The *environment space* is defined as the measurable space $(\Omega, \mathcal{F})$ where $\Omega := \prod_{x \in \mathbb{T}_d} \mathcal{S}_x$ and $\mathcal{F} := \bigotimes_{x \in \mathbb{T}_d} \mathcal{B}_{\mathcal{S}_x}$.

An element $\omega \in \Omega$ will be written as $\left\{ \omega(x, \cdot) \,\middle|\, x \in \mathbb{T}_d \right\}$. An environment distribution is a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We will denote by $\mathbb{E}$ the expectation taken with respect to the probability measure $\mathbb{P}$.

Random Walk: Given an environment $\omega \in \Omega$, a random walk $(X_n)_{n \geq 0}$ is a time homogeneous Markov chain taking values in $\mathbb{T}_d$ with transition probabilities given by $(\omega(x, y))_{x, y \in \mathbb{T}_d}$. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each $\omega \in \Omega$, we denote by $\mathbf{P}_\omega^x$ the law induced by $(X_n)_{n \geq 0}$ on $\left((\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{G} \right)$, where $\mathcal{G}$ is the σ-algebra generated by the cylinder sets, such that $\mathbf{P}_\omega^x(X_0 = x) = 1$. The probability measure $\mathbf{P}_\omega^x$ is called the *quenched law* of the random walk $(X_n)_{n \geq 0}$, starting at x. We will use the notation $\mathbf{E}_\omega^x$ for the expectation under the quenched measure $\mathbf{P}_\omega^x$.

Following Zeitouni [19], we note that for every $B \in \mathcal{G}$, the function $\omega \mapsto \mathbf{P}_\omega^x(B)$ is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^x$ on $\left(\Omega \times (\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{F} \otimes \mathcal{G} \right)$ by the relation

$$\mathbb{P}^x(A \times B) = \int_A \mathbf{P}_\omega^x(B) \, \mathbb{P}(d\omega), \quad \forall \; A \in \mathcal{F}, \; B \in \mathcal{G}.$$

With a slight abuse of notation, we also denote the marginal of $\mathbb{P}^x$ on $(\mathbb{T}_d)^{\mathbb{N}_0}$ by $\mathbb{P}^x$, whenever no confusion occurs. This probability distribution is called the *annealed law* of the random walk $(X_n)_{n \geq 0}$, starting at x. We will use the notation $\mathbb{E}^x$ for the expectation under the annealed measure $\mathbb{P}^x$. For simplicity, when $x = e$, we will drop the superscript from the notations $\mathbf{P}_\omega^e, \mathbf{E}_\omega^e, \mathbb{P}^e$ and $\mathbb{E}^e$ and simply write $\mathbf{P}_\omega, \mathbf{E}_\omega, \mathbb{P}$ and $\mathbb{E}$ respectively.

Assumptions: Throughout this paper we will assume the following hold,

(E1) $\mathbb{P}$ is a product measure on $(\Omega, \mathcal{F})$ with "*identical*" marginals, that is, under $\mathbb{P}$ the random probability laws $\left\{ \omega(x, \cdot) \,\middle|\, x \in \mathbb{T}_d \right\}$ are independent and "identically" distributed in the sense that

$$\mathbb{P} \circ \theta_x^{-1} = \mathbb{P}, \tag{1}$$

for all $x \in G$.

(E2) There exists $\epsilon > 0$ such that

$$\mathbb{P}\left(\omega\left(e, s_i\right) > \epsilon \ \forall\ 1 \le i \le d\right) = 1. \tag{2}$$

Remark 1. The assumption (E1) above essentially says that the random transition laws $\left\{\omega\left(x, \cdot\right) \,\middle|\, x \in \mathbb{T}^d\right\}$ are *independent and identically distributed (i.i.d.)*. But on $\mathbb{T}_d$ to define *identically distributed* we introduce the group structure and make the probability law $\mathbb{P}$ invariants under the *translations* by the group elements. This is different than the much studied *random conductance model* with i.i.d. conductances introduced in [12]. In fact the only example where the two models agree is the deterministic environment of the *simple symmetric walk* on $\mathbb{T}_d$. It is worth to point out here that a random walk on a Galton–Watson tree [11] satisfies the assumption (E1) and so does a random walk on a multi-type Galton–Watson tree [8].

We are now ready to state our main results. We begin with a law of large numbers result for $|X_n|$ which also establishes that the speed of walk is positive.

Theorem 2.1. *Assume (E1) and (E2). Then there exists* $v > 0$, *such that,*

$$\lim_{n\to\infty} \frac{|X_n|}{n} = v, \tag{3}$$

almost surely with respect to $\mathbb{P}^e$.

In [1], it was pointed out that under (E1), (E2), $\liminf_{n\to\infty} \frac{|X_n|}{n} > 0$, if $\epsilon > \frac{1}{2(d-1)}$. The above result not only establishes that the walk on $\mathbb{T}_d$ has a *positive speed*, but also shows that the corresponding limit exists almost surely. Our next result is an annealed central limit theorem for $\mid X_n \mid$.

Theorem 2.2. *Assume (E1)and (E2). Then there exists* $\sigma^2 > 0$ *such that, under* $\mathbb{P}^e$,

$$\sqrt{n}\left(\frac{|X_n|}{n} - v\right) \xrightarrow{d} Z, \tag{4}$$

with $Z \sim N(0, \sigma^2)$.

We note here that although we define the walk starting at $X_0 = e$, the root, results hold for starting at any vertex x of $\mathbb{T}_d$. This is because the environment is invariant under the translation by the group G. Indeed it will be evident from the proofs that the constants v and σ^2 are also independent of the starting position. Thus Theorems 2.1 and 2.2 hold for any initial distribution of X_0 on the vertex set of $\mathbb{T}_d$.

3. Regeneration times

In this section we shall introduce a sequence of *regeneration times* and provide moment bounds for them. We begin with some notation. For any $x, y \in \mathbb{T}_d$, $[x, y]$ denote the unique path from x to y, that is, $[x, y] = \{\{x_i\}_{i=0}^n \mid x_0 = x, x_n = y, x_i^{-1} x_{i-1} \in S, 1 \le i \le n$ and all are distinct$\}$. Let $\mathbb{T}_d(y)$ be the sub-tree rooted at y, i.e., $\{x \in \mathbb{T}_d : y \in [e, x]\}$ and $\mathbb{T}_d^n = \{x \in \mathbb{T}_d :\mid x \mid= n\}$. The type of $x \in \mathbb{T}_d, x \neq e$ is $s \in S$ if $x \overleftarrow{x}^{-1} = s$ and we shall denote it by s_x. We define

$$T(y) := \inf\{n \ge 0 \colon X_n = y\} \tag{5}$$

and

$$R(y) := \inf\{n \geq 1 : X_{n-1} \in \mathbb{T}_d(y),\ X_n = y\}, \tag{6}$$

be the hitting time of y and the return time to y, respectively. We also define,

$$T_n := \inf\{k \geq 0 : X_k \in \mathbb{T}_d^n\} \tag{7}$$

and

$$R := \inf\{n \geq 1 : X_n = X_0\}, \tag{8}$$

to be the hitting time of level n and return time of the walk to its starting point, respectively.

The *first regeneration level* is then defined as $l_1 := \inf\{k \geq 1 : R(X_{T_k}) = \infty\}$, and the *$n-th$ regeneration level* for $n \geq 2$ is defined recursively as $l_n := \inf\{k > l_{n-1} : R(X_{T_k}) = \infty\}$. Regeneration times for $n \geq 1$, are defined by

$$\tau_n := \begin{cases} T_{l_n} & \text{if } l_n < \infty; \\ \\ \infty & \text{otherwise.} \end{cases} \tag{9}$$

3.1. Tail bounds for the first regeneration time

In this section we will prove tail bounds for first regeneration level. We need to identify the regeneration levels with a multi-type branching process (see [2]) and obtain the required bounds. The process will follow from a colouring scheme to be placed on the vertices. The colouring scheme is an adaptation from [6].

Colouring scheme: We will construct a coupling with a Markov chain that lives on sub-trees of $\mathbb{T}_d$ and using this we will develop a colouring scheme on the vertices.

Let

$$\{h_n(x, y) \mid n \geq 1, x, y \in \mathbb{T}_d,\ \text{and } x \sim y\}$$

be i.i.d. Exponential random variables with mean 1, which are independent of the environment ω.

Suppose $X_0 = x$, an explicit a.s. construction of the walk is obtained by taking

$$X_{n+1} = \operatorname*{argmin}_{y \sim X_n} \frac{h_{n+1}(X_n, y)}{\omega(X_n, y)}.$$

Fix a finite sub-tree $\mathcal{C}$ of $\mathbb{T}_d$ with $x \in \mathcal{C}$. Let $\{Y_n^{\mathcal{C}}\}_{n \geq 0}$ be such that $Y_0^{\mathcal{C}} = x$ and

$$Y_{n+1}^{\mathcal{C}} = \operatorname*{argmin}_{y \sim Y_n^{\mathcal{C}}, y \in \mathcal{C}} \frac{h_{n+1}(Y_n^{\mathcal{C}}, y)}{\omega(Y_n^{\mathcal{C}}, y)}.$$

We claim that process $\{Y_n^{\mathcal{C}}\}_{n \geq 0}$ is a Markov chain on $\mathcal{C}$. To see this, we shall construct a coupling with the walk $\{X_n : n \geq 1\}$ so that $Y^{\mathcal{C}}$ has the same law as X whenever it visits the sub-tree $\mathcal{C}$. Define $\lambda_1 = \inf\{n \geq 0 : X_n \notin \mathcal{C}\}$, and recursively for $i \geq 1$,

$$\mu_i = \inf\{n \geq \lambda_i : X_n \in \mathcal{C}\} \text{ followed by } \lambda_{i+1} = \inf\{n \geq \mu_i : X_n \notin \mathcal{C}\}.$$

Define for each $k \geq 1$,

$$W_k = \begin{cases} X_k & \text{if } k \leq \lambda_1 - 1, \text{ or } \mu_i \leq k \leq \lambda_{i+1} \text{ for some } i \geq 1; \\ X_{\lambda_i - 1} & \text{if } \lambda_i \leq k < \mu_i, \text{ for some } i \geq 1. \end{cases}$$

Note that as $\mathbb{T}_d$ is a tree, for all $i \geq 1$, $X_{\lambda_i - 1} = X_{\mu_i} \in \mathcal{C}$. It is now easy to see that $\{W_n\}_{n\geq 0}$ is a Markov chain on $\mathcal{C}$ and has the same law as $\{Y_n^{\mathcal{C}}\}_{n\geq 0}$. We begin by colouring the root e as red. Let $k \geq 1$ and $\psi \geq 1$. A vertex $y \in \mathbb{T}_d^{k\psi}$ is coloured red if and only if

- its ancestor at level $(k-1)\psi$, say x, is coloured red, and
- $\{Y_n^{[x,y]}\}_{n\geq 0}$, started at x, hits y before returning to x.

For each $\psi \geq 1$ and $k \geq 1$, $s \in S$ let $Z_\psi(k,s)$ be the number of red vertices at level $k\psi$ of type s. Let $Z_\psi(0) = \{e\}$ and for $k \geq 1$, define

$$Z_\psi(k) := (Z_\psi(k,s))_{s\in S} \in \bar{\mathbb{N}}^d, \text{ where } \bar{\mathbb{N}} = \mathbb{N} \cup \{0\}.$$

Under the annealed measure, $\{Z_\psi\}$ is a multi-type Branching process with expected offspring matrix $M = (m_{su}(\psi))_{s,u\in S}$ is given by

$$m_{su}(\psi) = \mathbb{E}\left[\sum^{*}\left(\sum_{m=1}^{\psi-1}\prod_{j=1}^{m}\frac{\omega(x_j,x_{j-1})}{\omega(x_j,x_{j+1})}\right)^{-1}\right],$$

where $\sum^*$ is the sum over all indexes, such that $x_1, x_2, \ldots, x_\psi \in \mathbb{T}_d, s_{x_1} = s, s_{x_\psi} = u$ and $x_\psi \in \mathbb{T}_d^\psi$.

Proposition 3.1. *Assume (E1) and (E2). There exists $\psi \geq 1$ such that Z_ψ is supercritical.*

Proof. We will show that the largest eigenvalue, ρ, of the offspring matrix M is larger than 1. We observe that for $1 \leq i \leq n-1$

$$\mathbf{P}_\omega(Y_n^{[x,y]} = x_{i-1} \mid Y^{[x,y]} = x_i) = \frac{\omega(x_i,x_{i-1})}{\omega(x_i,x_{i+1}) + \omega(x_i,x_{i-1})}$$

and

$$\mathbf{P}_\omega(Y_n^{[x,y]} = x_{i+1} \mid Y^{[x,y]} = x_i) = \frac{\omega(x_i,x_{i+1})}{\omega(x_i,x_{i+1}) + \omega(x_i,x_{i-1})}.$$

Using a standard gambler's ruin chain argument we can conclude

$$\mathbf{P}_\omega(Y^{[x,y]} \text{ hits } y \text{ before returning to } x\ ) = \left(\sum_{m=1}^{n-1}\prod_{j=1}^{m}\frac{\omega(x_j,x_{j-1})}{\omega(x_j,x_{j+1})}\right)^{-1}.$$

We shall now show that for all $s \in S$, and large enough ψ

$$\sum_{u\in S} m_{su}(\psi) = \sum_{u\in S}\mathbb{E}\left[\sum^{*}\left(\sum_{m=1}^{\psi-1}\prod_{j=1}^{m}\frac{\omega(x_j,x_{j-1})}{\omega(x_j,x_{j+1})}\right)^{-1}\right] > 1, \tag{10}$$

where $\sum^*$ is defined above. Now notice that for the ψ for which Eq. (10) holds for that the row sums of the matrix M are larger than 1, this implies that the largest eigenvalue of M is bigger than 1. Thus the process Z_ψ is supercritical.

To show (10) we follow an argument very similar to the proof of Theorem 1 in [1]. Suppose $\sigma_n = \alpha_n\alpha_{n-1}\cdots\alpha_1$ where $\alpha_i \in S$ and $\alpha_{i+1} \neq \alpha_i^{-1}$ and $n = |\sigma_n|$. Further, let $x_k = \alpha_k\alpha_{k-1}\cdots\alpha_1$

where $1 \leq k \leq n$. Note that $x_k = \alpha_k x_{k-1}$ for all $1 \leq k \leq n$. Let

$$\Phi(\sigma_n) := \frac{1}{\omega(e, x_1)} \prod_{k=1}^{n-1} \frac{\omega(x_k, x_{k-1})}{\omega(x_k, x_{k+1})} = \left(\prod_{k=1}^{n-1} \frac{\omega_k\left(\alpha_k^{-1}\right)}{\omega_{k-1}(\alpha_k)} \right) \frac{1}{\omega_{n-1}(\alpha_n)}, \tag{11}$$

where we write $\omega_k(s) := \omega(x_k, s x_k)$ for any $s \in S$. We will now show that P-a.s., there is a subset of vertices of $\mathbb{T}_d^n$ with size $O\left((d-1)^{n-1}\right)$ such that the Φ-value of these vertices is strictly smaller than $(d-1)^{\frac{n}{2}}$.

Let $\mathcal{B}_{\mathbb{N}_0}$ denote the product σ-algebra on $S^{\mathbb{N}_0}$, and μ be a probability measure on $\left(S^{\mathbb{N}_0}, \mathcal{B}_{\mathbb{N}_0}\right)$ such that $(Y_n)_{n\geq 0} \in S^{\mathbb{N}_0}$ forms a Markov chain on S with

$$\mu\left(Y_n = s \,\middle|\, Y_{n-1} = t\right) = \frac{1}{d-1}, \quad s, t \in S \text{ with } s \neq t^{-1}. \tag{12}$$

It is easy to see that the chain $(Y_n)_{n\geq 0}$ is an aperiodic, irreducible and finite state Markov chain and its stationary distribution is the uniform distribution on S. We shall assume that Y_0 is uniformly distributed on S. Thus each Y_n is also uniform on S.

Let $\eta_n = Y_n Y_{n-1} \cdots Y_1$. From Eq. (12) it follows that η_n is uniformly distributed on the set of vertices $\mathbb{T}_d^n$. Now

$$\begin{aligned} \frac{1}{n} \ln \Phi(\eta_n) &= o(1) + \frac{1}{n} \sum_{k=1}^{n-1} \left(\ln \omega_k\left(Y_k^{-1}\right) - \ln \omega_{k-1}(Y_k) \right) \\ &= o(1) + \frac{1}{n} \sum_{s\in S} \sum_{j=1}^{N_{n-1}\left(s^{-1}\right)} \left(\ln \omega_{k_j(s^{-1})}\left(s^{-1}\right) - \ln \omega_{k_j(s^{-1})-1}(s) \right), \end{aligned} \tag{13}$$

where for each $s \in S$, $k_1\left(s^{-1}\right), k_2\left(s^{-1}\right) \cdots, k_{N_{n-1}(s^{-1})}\left(s^{-1}\right)$ are the time points k when $Y_k = s^{-1}$ and

$$N_n(s) = \sum_{k=1}^{n} \mathbf{1}(Y_k = s). \tag{14}$$

Now consider the product space $\left(\Omega \times S^{\mathbb{N}_0}, \mathcal{B}_\Omega \otimes \mathcal{B}_{\mathbb{N}_0}, \mathbb{P} \otimes \mu\right)$. By Theorems 6.5.5 and 6.6.1 of [9] we have $\mathbb{P} \otimes \mu$-almost surely for all $s \in S$,

$$\lim_{n\to\infty} \frac{N_n(s)}{n} = \frac{1}{d}. \tag{15}$$

Further under assumption (A2) and using the Strong Law of Large Numbers for i.i.d. random variables we have $\mathbb{P}$-almost surely, for every fixed $s \in S$,

$$\lim_{n\to\infty} \frac{1}{N_{n-1}\left(s^{-1}\right)} \sum_{j=1}^{N_{n-1}\left(s^{-1}\right)} \ln \omega_{k_j(s^{-1})}\left(s^{-1}\right) = E\left[\ln \omega_1\left(s^{-1}\right)\right],$$

and also

$$\lim_{n\to\infty} \frac{1}{N_{n-1}\left(s^{-1}\right)} \sum_{j=1}^{N_{n-1}\left(s^{-1}\right)} \ln \omega_{k_j(s^{-1})-1}(s) = E\left[\ln \omega_1(s)\right].$$

As S is a symmetric set of generators for G. Therefore, $\mathbb{P} \otimes \mu$-almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \ln \Phi(\eta_n) = \frac{1}{d} \sum_{s\in S} E\left[\ln \omega_1\left(s^{-1}\right) - \ln \omega_1\left(s\right)\right] = 0. \tag{16}$$

So there exists $1 < \tilde{c} < d-1$ such that, $\mathbb{P} \otimes \mu$-almost surely,

$$\exists\, \tilde{M} \in \mathbb{N}, \qquad \text{s.t. } \forall\, n \geq \tilde{M} \qquad \Phi(\eta_n) < \tilde{c}^n.$$

This implies that there exist $\tilde{c} < c < d-1$ and $K > 0$, such that, $\mathbb{P} \otimes \mu$-almost surely,

$$\exists\, M \in \mathbb{N}, \qquad \text{s.t. } \forall\, n \geq M, \qquad \sum_{m=1}^{n-1} \Phi(\eta_m) < K c^{n-1}. \tag{17}$$

Let $M^\omega_\mu \in \mathbb{N}$ be the minimum such M for which Eq. (17) holds. So by Fubini's theorem, it follows from (17), that almost surely for every $\omega \in \Omega$, μ-almost surely, $n \geq M^\omega_\mu \in \mathbb{N}$, such that

$$\mu\left(\sum_{m=1}^{n-1} \Phi(\eta_m) < K c^{n-1}\right) > \frac{1}{2}. \tag{18}$$

But recall that under μ, the distribution of η_n is uniform on the vertices of $\mathbb{T}^n_d$, so for all $n \geq M^\omega_\mu$

$$\frac{\#\left\{\sigma_n \in \mathbb{T}^n_d \,:\, \sum_{m=1}^{n-1} \Phi(\sigma_m) < K c^{n-1}\right\}}{d\,(d-1)^{n-1}} > \frac{1}{2}. \tag{19}$$

This readily implies that for all $n \geq M^\omega_\mu$

$$\#\left\{\sigma_n \in \mathbb{T}^n_d \,:\, \left(\sum_{m=1}^{n-1} \Phi(\sigma_m)\right)^{-1} > \frac{1}{K} c^{-(n-1)}\right\} > \frac{d\,(d-1)^{n-1}}{2}. \tag{20}$$

Now notice that

$$\begin{aligned}
\sum_{u\in S} m_{su}(n) &= \mathbb{E}\left[\sum_{u\in S} \sum^{*} \left(\sum_{m=1}^{n-1} \prod_{j=1}^{m} \frac{\omega(x_j, x_{j-1})}{\omega(x_j, x_{j+1})}\right)^{-1}\right] \\
&\geq \mathbb{E}\left[\sum_{u\in S} \sum^{*} \left(\sum_{m=1}^{n-1} \Phi(\eta_m)\right)^{-1}\right] \\
&\geq \mathbb{E}\left[\frac{d(d-1)^{n-1}}{2K c^{n-1}} \cdot \mathbf{1}\left(n \geq M^\omega_\mu\right)\right]
\end{aligned}$$

where the second inequality is because of the factor $\frac{1}{\omega(e,x_1)} > 1$ in the definition of Φ (see Eq. (11)) and the last inequality follows from (20). Now applying Fatou's Lemma we get

$$\liminf_{n\to\infty} \sum_{u\in S} m_{su}(n) \geq \mathbb{E}\left[\liminf_{n\to\infty} \frac{d(d-1)^{n-1}}{2K c^{n-1}} \mathbf{1}\left(n \geq M^\omega_\mu\right)\right] = \infty.$$

The last equality follows from the fact that $1 < c < d-1$. This completes the proof of Eq. (10). $\square$

Now, for $x \in \mathbb{T}_d$, $y \in \mathbb{T}_d(x)$ is called the *first child* of a vertex $y_0 \in \mathbb{T}_d(x)$, if

$$y = \operatorname*{argmin}_{z \sim y_0, z \neq \overleftarrow{y_0}} \frac{h_1(y_0, z)}{\omega(y_0, z)} \tag{21}$$

Often we will simply write "$y \in \mathbb{T}_d(x)$ is a *first child* " to mean y is the first child of the parent $\overleftarrow{y} \in \mathbb{T}_d(x)$. Notice that by construction of the walk $(X_n)_{n \geq 0}$ whenever the walk enters a previously unexplored domain, it does so through a first child.

For $m \geq 1$,

$$F_{m,x} = \{y \in \mathbb{T}_d(x) :| y | - | x |= m \text{ and } y \text{ is a } \textit{first child}\}.$$

Let $\psi \geq 1, \zeta \geq 1$

$$\Sigma_x = \mathbb{T}_d(x) \cap Z_\psi \cap_{k=1}^{\infty} F^c_{k\zeta\psi,x},$$

where by slight abuse of notation, we denote the set of red vertices by Z_ψ.

$$B(x) = \{\Sigma_x \text{ is finite}\}, B_0 = B(e), \text{ and } B_k = B(X_{T_{k\psi\zeta}}), \ k \geq 1.$$

Lemma 3.2. *The collection of events $\{B_i, i \geq 1\}$, are independent.*

Proof. The event $B(x) \in \sigma\{h_n(z, y) : z, y \in \mathbb{T}_d(x), n \geq 1\}$. Let $i_1 < i_2 < \cdots < i_k$ be positive integers. Note, as observed,

$$B_{i_j} \in \sigma\{h_n(z, y) : z, y \in \mathbb{T}_d(X_{T_{i_j\zeta\psi}}), n \geq 1\}.$$

Note however that $X_{T_{i_j\zeta\psi}}$ is a first child at level $i_j\zeta\psi$. and this implies $\mathbb{T}_d(X_{T_{i_j\zeta\psi}}) \cap \mathbb{T}_d(X_{T_{i_l\zeta\psi}}) = \emptyset$. Hence $\{B_{i_j} : 1 \leq j \leq k\}$ are mutually independent. $\square$

Proposition 3.3. *$\exists\ \gamma < 1$ such that for $n \geq 1$, we have*

$$\mathbb{P}(l_1 \geq n\psi\zeta) \leq \gamma^{n-1}.$$

Proof. Note that $B^c_i \subseteq$ {level $i\psi\zeta$ is a regeneration level}. Hence, using that B_i are independent we have,

$$\mathbb{P}(l_1 \geq n\psi\zeta) \leq \mathbb{P}(\bigcap_{i=1}^{n-1} B_i) = \prod_{i=1}^{n-1} \mathbb{P}(B_i).$$

For $s \in S$, let

$$B^s_i := \{X_{T_{i\zeta\psi}} \overleftarrow{X}^{-1}_{T_{i\zeta\psi}} = s\} \cap B_i.$$

Note that for $s \in S$, $\mathbb{P}(B^s_i) = \mathbb{P}(B^s_j)$ for all $1 \leq i, j \leq n$. Hence,

$$\mathbb{P}(B_i) = \mathbb{P}(\cup_{s \in S} B^s_i) = \sum_{s \in S} \mathbb{P}(B^s_i) = \gamma$$

Therefore $\mathbb{P}(l_1 \geq n\psi\zeta) \leq \prod_{i=1}^{n-1} \left(\sum_{s\in S} \mathbb{P}(B^s_i)\right) = \gamma^{n-1}$.

Now we will show that we can choose $\zeta > 0$, such that $\gamma < 1$. It is enough to show that we can choose $\zeta > 0$, such that $\mathbb{P}(B_1) < 1$. For this we follow an argument similar to the proof of Lemma 3.3 of [6]. From definition it is clear that the vertices which belong to $\Sigma_{X_{\psi\zeta}}$ are obtained as follows. The vertices at level $(\zeta - 1)\psi$ are of d-types and have a distribution with mean matrix $M^{(\zeta-1)\psi}$. Further, the vertices at level $\zeta\psi$ have a number of various types of coloured vertices and we have deleted the first child. Therefore, the expectation matrix of such vertices is $M-A$ where A is a $d\times d$-matrix with $0\le A_{su}\le 1$ and $A\mathbf{1}=\mathbf{1}$. Thus, $\gamma=\mathbb{P}(B_1)$ is at most as large as the extinction probability of a multi-type branching process with mean matrix $\tilde{M}_\zeta := M^{(\zeta-1)\psi}(M-A)$. But, from Eq. (10) it follows that

$$m_0 := \min_{s\in S}\sum_{u\in S} m_{su} > 1.$$

So for any $s\in S$, the sth row sum of $\tilde{M}_\zeta$ is at least as large as $m_0^{(\zeta-1)\psi}(m_0-1)$. Now select $\zeta\ge 1$, such that $m_0^{(\zeta-1)\psi}(m_0-1)>1$. From the argument above then we can conclude that $\gamma<1$. □

3.2. Moment bounds

We conclude this section with the required moment bounds on the regeneration times defined above. For $x\in\mathbb{T}_d$, let $T(x)$ be as in (5) and τ_1 be as in (9). Furthermore, let

$$L(x) := \sum_{j=0}^{\infty} I_{\{X_j=x\}} \text{ and } \mathcal{D} := \sum_{x\in\mathbb{T}_d} I_{\{T(x)\le\tau_1\}} \tag{22}$$

be total number of visits to x and the number of distinct vertices visited before τ_1 respectively.

In Proposition 3.4 we show moment bounds on $L(e)$, $\mathcal{D}$ and these yield the moment bounds on regeneration times. For the bound on $L(e)$ we will work with induced walk on specific subtrees and provide moment bounds for their visits to the root. For moment bounds on $\mathcal{D}$ we will need tail bounds on regeneration levels that were proved in Proposition 3.3.

Proposition 3.4. *Assume (E1) and (E2). Then for $p\ge 1$,*

(a) $\mathbb{E}[L(e)^p]<\infty$*;*
(b) $\mathbb{E}[\mathcal{D}^p]<\infty$*; and*
(c) $\mathbb{E}[\tau_1^p]<\infty$.

Proof (*a*)**.** For $n\ge 1$, let

$$\mathcal{U}_n = \{\{x^i\}_{i=1}^d : x^i\in\mathbb{T}_d^n \text{ and } [x^i,e]\cap[x^j,e]=\{e\} \text{ for all } 1\le i\ne j\le d\}.$$

We will denote any element of $\mathcal{U}_n$ by $\mathcal{A}_n$ and the smallest sub-tree in $\mathbb{T}_d$ containing $\mathcal{A}_n$ will be denoted by $\mathfrak{T}_n$. Consider the walk $\{Y_k^{\mathfrak{T}_n}: k\ge 1\}$. Define

$$\tilde{T}_{\mathcal{A}_n} = \inf\{k\ge 1 : Y_k^{\mathfrak{T}_n}\in\mathcal{A}_n\}, \text{ and } \tilde{L}(e,\tilde{T}_{\mathcal{A}_n}) = \sum_{i=0}^{\infty} I_{\{Y_i^{\mathfrak{T}_n}=e, i<\tilde{T}_{\mathcal{A}_n}\}}$$

to be the hitting time of $\mathcal{A}_n$ and the number of visits of $Y^{\mathfrak{T}_n}$ to e before the walk $Y^{\mathfrak{T}_n}$ hits $\mathcal{A}_n$ respectively. Define $\tilde{R}_n=\inf\{k\ge 1 : Y_k^{\mathfrak{T}_n}=e\}$ return time to e. Under the quenched law, $\tilde{L}(e,\tilde{T}_{\mathcal{A}_n})$ is a geometric random variable with parameter q_ω given by

$$q_w = \mathbf{P}_\omega(\tilde{T}_{\mathcal{A}_n}<\tilde{R}_n) = \sum_{i=1}^{d} w(e,s_i)\left(\sum_{j=1}^{n}\prod_{k=1}^{j-1}\frac{\omega(x_k^i,x_{k-1}^i)}{\omega(x_k^i,x_{k+1}^i)}\right)^{-1}.$$

Using standard results about geometric random variables we know that for $p > 1$, $\exists\, c_p > 0$, such that

$$\mathbb{E}[\tilde{L}(e, \tilde{T}_{\mathcal{A}_n})^p] \leq c_p \mathbb{E}[q_\omega^{-p}] = c_p \mathbb{E}\left[\left(\sum_{i=1}^{d} w(e, s_i)\left(\sum_{j=1}^{n}\prod_{k=1}^{j-1}\frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)}\right)^{-1}\right)^{-p}\right]$$
$$\leq c_p\, d^{-p}\, \mathbb{E}\left[\left(\min_{1\leq i\leq d}\left(\sum_{j=1}^{n}\prod_{k=1}^{j-1}\frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)}\right)^{-1}\right)^{-p}\right]$$
$$= c_p\, d^{-p}\, \mathbb{E}\left[\max_{1\leq i\leq d}\left(\sum_{j=1}^{n}\prod_{k=1}^{j-1}\frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)}\right)^{p}\right]. \tag{23}$$

For $x \in \mathbb{T}_d$, define $R^x := \inf\{n \geq 1 : Y_n^{\mathbb{T}_d(x)} = x\}$. We note here that the distribution of R^x under the annealed measure $\mathbb{P}$ depends on x only through its type. In particular, $\mathbb{P}\,(s_x = s, R^x < \infty)$ depends only on $s \in S$. We further define

$$H := \inf\{k \geq 1 : \exists \mathcal{A}_k \in \mathcal{U}_k \text{ such that } \mathbf{P}_\omega^x(R^x = \infty) = 1 \text{ for all } x \in \mathcal{A}_k\}.$$

Observe that for any $n \geq 1$,

$$L(e) I_{\{H=n\}} \leq \tilde{L}(e, \tilde{T}_{\mathcal{A}_n}) I_{\{H=n\}}.$$

By Hölder's inequality, with $a = 1 + \delta/p$, $b = 1 + p/\delta$, for some $\delta > 0$ and using (23) we have

$$\mathbb{E}[L(e)^p I_{\{H=n\}}] \leq \mathbb{E}[(\tilde{L}(e, \tilde{T}_{\mathcal{A}_n}))^p I_{\{H=n\}}]$$
$$\leq \mathbb{E}[\tilde{L}(e, \tilde{T}_{\mathcal{A}_n})^{pa}]^{1/a}\mathbb{P}(H = n)^{1/b}$$
$$\leq c_p\, d^{-p}\left(\mathbb{E}\left[\max_{1\leq i\leq d}\left(\sum_{j=1}^{n}\prod_{k=1}^{j-1}\frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)}\right)^{pa}\right]\right)^{1/a}\mathbb{P}(H = n)^{1/b}.$$
$$\leq c_p\, d^{-p}\left(\mathbb{E}\left[\max_{1\leq i\leq d}\left(\sum_{j=1}^{n}\prod_{k=1}^{j-1}\frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)}\right)^{pa}\right]\right)^{1/a}\mathbb{P}(H = n)^{1/b}. \tag{24}$$

Now,

$$\mathbb{P}(H = n) \leq \mathbb{P}\left(\bigcup_{s\sim e}\ \bigcap_{y\in\mathbb{T}_d(s)\cap\mathbb{T}_d^{n-1}}\{R^y < \infty\}\right) \leq \sum_{s\sim e}\mathbb{P}\left(\bigcap_{y\in\mathbb{T}_d(s)\cap\mathbb{T}_d^{n-1}}\{R^y < \infty\}\right)$$
$$= \sum_{s\sim e}\ \prod_{y\in\mathbb{T}_d(s)\cap\mathbb{T}_d^{n-1}}\mathbb{P}\left(R^y < \infty\right)$$
$$\leq d\left(\max_{s\in S, y\in\mathbb{T}_d(s)\cap\mathbb{T}_d^{n-1}}\mathbb{P}\left(s_y = s, R^y < \infty\right)\right)^{(d-1)^{n-2}}. \tag{25}$$

Let $q = \max_{s\in S, y\in\mathbb{T}_d(s)\cap\mathbb{T}_d^{n-1}}\mathbb{P}\left(s_y = s, R^y < \infty\right)$. Recall that $\mathbb{P}\left(s_y = s, R^y < \infty\right)$ depends on y only through $s \in S$, thus, $q \max_{s\in S}\mathbb{P}\left(s_y = s, R^y < \infty\right)$. Therefore, $q < 1$. From(24), (25) and using (E2) the above we have

$$
\begin{aligned}
\mathbb{E}[L(e)^p, I_{\{H=n\}}] &\le c_p\, d^{-p} \left(\mathbb{E}\left[\max_{1\le i\le d} \left(\sum_{j=1}^{n} \prod_{k=1}^{j-1} \frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)} \right)^{pa} \right] \right)^{1/a} \left(d q^{(d-1)^{n-2}} \right)^{\frac{1}{b}} \\
&\le c_p\, d^{-p+\frac{1}{b}} \left(\sum_{j=1}^{n} \frac{1}{\epsilon^{j-1}} \right) q^{\frac{(d-1)^{n-2}}{b}} \\
&\le c_p d^{-p+\frac{1}{b}} \frac{\epsilon}{1-\epsilon} \left(\frac{1}{\epsilon}\right)^{n-1} q^{\frac{(d-1)^{n-2}}{b}} \\
&\le \frac{c_p\, \epsilon^2\, d^{-p+\frac{1}{b}}}{1-\epsilon} \exp\left(-\log(\epsilon) n + \frac{1}{b} \log(q)(d-1)^{n-2} \right)
\end{aligned} \tag{26}
$$

It easily follows that

$$
\mathbb{E}[L(e)^p] = \sum_{n=1}^{\infty} \mathbb{E}[L(e)^p, I_{\{H=n\}}] < \infty.
$$

(b) From the definition of $\mathcal{D}$, we have

$$
\begin{aligned}
\mathcal{D} &= \sum_{x\in\mathbb{T}_d} I_{\{T(x)\le\tau_1\}} = 1 + \sum_{x\ne e, x\in\mathbb{T}_d} \sum_{n=1}^{\infty} I_{\{T(x)\le T_n\}}\, I_{\{l_1=n\}} \\
&= 1 + \sum_{n=1}^{\infty} \sum_{x\ne e, x\in\mathbb{T}_d} I_{\{T(x)\le T_n\}}\, I_{\{l_1=n\}} \\
&= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \sum_{x\in\mathbb{T}_d^k} I_{\{T(x)\le T_n\}} \right) I_{\{l_1=n\}} \\
&\le 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}} \right) I_{\{l_1=n\}}.
\end{aligned}
$$

For each $k \ge 1$, we may dominate the random variable $\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}}$ by a Geometric $(1-q)$ random variable (where q was defined in the previous proof). This implies

$$
\mathbb{E}\left(\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}} \right)^{2p} \le c_p (1-q)^{-2p}.
$$

Using this, Jensen's inequality, followed by Hölder's inequality we have for all $n \ge 1$,

$$
\begin{aligned}
n^{p-1} \sum_{k=1}^{n} \mathbb{E}\left(\left(\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}} \right)^{p} I_{\{l_1=n\}} \right) &\le n^{p-1} \sum_{k=1}^{n} \mathbb{E}\left(\left(\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}} \right)^{p} I_{\{l_1=n\}} \right) \\
&\le n^{p-1} \sum_{k=1}^{n} \sqrt{ \mathbb{E}\left(\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}} \right)^{2p} } \sqrt{\mathbb{P}(l_1 = n)} \\
&\le \sqrt{c_p}(1-q)^{-p} n^p \sqrt{\mathbb{P}(l_1 = n)}
\end{aligned}
$$

Then,

$$\mathbb{E}[\mathcal{D}^p] \le \sqrt{c_p}(1-\mathbb{P}(R<\infty))^{-p}\sum_{n=1}^{\infty} n^p\,\mathbb{P}(l_1=n)^{1/2}. \tag{27}$$

The result follows from Proposition 3.3.

(c) Let $\{x_i : 1 \le i \le \mathcal{D}\}$ be an enumeration of the vertices visited by the walk X before time τ_1. It is easy to see that

$$\tau_1 = \sum_{i=1}^{\mathcal{D}} L(x_i).$$

So,

$$\mathbb{E}[\tau_1^p] = \mathbb{E}\left[\left(\sum_{i=1}^{\mathcal{D}} L(x_i)\right)^p\right] \le \mathbb{E}\left[\mathcal{D}^{p-1}\sum_{i=1}^{\mathcal{D}} L(x_i)^p\right]. \tag{28}$$

Now

$$\mathcal{D}^{p-1}\sum_{i=1}^{\mathcal{D}} L(x_i)^p = \sum_{i=1}^{\infty} \mathcal{D}^{p-1} L(x_i)^p I_{\{\mathcal{D}\ge i\}}.$$

For each $i \ge 1$, using Hölder's inequality twice (first with $q = 1+\delta/p$, and $q' = 1+p/\delta$) and Chebychev's inequality, we have

$$\begin{aligned}\mathbb{E}\left[\mathcal{D}^{p-1}L(x_i)^p I_{\{\mathcal{D}\ge i\}}\right] &\le \left[\mathbb{E}[L(x_i)^{p+\delta}]\right]^{1/q}\left[\mathbb{E}[\mathcal{D}^{(p-1)q'} I_{\{\mathcal{D}\ge i\}}]\right]^{1/(q')}\\ &\le \left[\mathbb{E}[L(x_i)^{p+\delta}]\right]^{1/q}\left[\mathbb{E}[\mathcal{D}^{2(p-1)q'}]\right]^{1/(2q')}\mathbb{P}(\mathcal{D}\ge i)^{1/(2q')}\\ &\le \left[\mathbb{E}[L(x_i)^{p+\delta}]\right]^{1/q}\left[\mathbb{E}[\mathcal{D}^{2(p-1)q'}]\right]^{1/(2q')}\left[\mathbb{E}(\mathcal{D}^{4q'})\right]^{1/(2q')}\frac{1}{i^2}.\end{aligned} \tag{29}$$

By definition of L we have

$$\mathbb{E}[L(x_i)^{p+\delta}] \le \mathbb{E}[L(x_i)^{p+\delta}|X_0 = x_i] = \mathbb{E}[L(e)^{p+\delta}]. \tag{30}$$

Using (28), (29), (30), the fact that $\sum_{i=1}^{\infty}\frac{1}{i^2} < \infty$, along with part (a) and (b), we have the result. $\square$

4. Proof of main results

We are now ready to prove the main result. We will need only the first moment bound obtained in Proposition 3.4 for the regeneration time. Then a standard renewal technique will yield the result in Theorem 2.1. For Theorem 2.2, we will identify independent blocks conditioned on the type of vertex at the regeneration times. This along with proportion of visits to each time will yield the result.

Proof of Theorem 2.1. Let $\{\tau_n\}_{n\ge1}$ be the sequence of regeneration times defined in (9). By Proposition 3.4, $\mathbb{E}(\tau_1) < \infty$. So for all $n \ge 1$, there exists a (random) subsequence $\{k_n\}_{n\ge1}$ such that

$$\tau_{k_n} < n \le \tau_{k_n+1}. \tag{31}$$

It is then readily seen that

$$\frac{\mid X_n \mid}{n} = \frac{\mid X_{\tau_1} \mid + \sum_{m=1}^{k_n-1}(\mid X_{\tau_{m+1}} \mid - \mid X_{\tau_m} \mid) + \mid X_n \mid - \mid X_{\tau_{k_n}} \mid}{\tau_1 + \sum_{m=1}^{k_n-1}(\tau_{m+1} - \tau_m) + n - \tau_{k_n}}. \tag{32}$$

For any $s \in S$, define

$$Y_i(s) = \begin{cases} \tau_1 I_{\{s_{X_{\tau_1}}=s\}} & i = 1 \\ (\tau_i - \tau_{i-1}) I_{\{s_{X_{\tau_i}}=s\}} & i > 1, \end{cases} \quad Z_i(s) = \begin{cases} \mid X_{\tau_1} \mid I_{\{s_{X_{\tau_1}}=s\}} & i = 1 \\ (\mid X_{\tau_i} \mid - \mid X_{\tau_{i-1}} \mid) I_{\{s_{X_{\tau_i}}=s\}} & i > 1. \end{cases} \tag{33}$$

Using Assumption (E1) we have for each $s \in S$, $\{Y_i(s)\}_{i\geq 1}$ and $\{Z_i(s)\}_{i\geq 1}$ are i.i.d. Using Proposition 3.4,

$$\mathbb{E}[Y_i(s)] = \mathbb{E}[Y_1(s)] = \mathbb{E}[\tau_1 I_{\{s_{X_{\tau_1}}=s\}}] \leq \mathbb{E}[\tau_1] < \infty,$$

and

$$\mathbb{E}[Z_i(s)] = \mathbb{E}[Z_1(s)] = \mathbb{E}[\mid X_{\tau_1} \mid I_{\{s_{X_{\tau_1}}=s\}}] \leq \mathbb{E}[\mid X_{\tau_1} \mid] \leq c_1 \mathbb{E}[\tau_1] < \infty.$$

By strong law of large numbers for each $s \in S$,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n Y_i(s)}{n} \to \mathbb{E}[Y_1(s)] \text{ and } \lim_{n\to\infty} \frac{\sum_{i=1}^n Z_i(s)}{n} \longrightarrow \mathbb{E}[Z_1(s)]$$

$\mathbb{P}$-almost surely, as $n \to \infty$. Consequently,

$$\frac{\sum_{l=1}^{k_n-1} \tau_{l+1} - \tau_l}{k_n - 1} = \sum_{s\in S} \sum_{i=1}^{k_n-1} \frac{Y_i(s)}{k_n - 1} \longrightarrow \sum_{s\in S} \mathbb{E}[Y_1(s)] = \mathbb{E}[\tau_1], \tag{34}$$

and

$$\frac{\sum_{l=1}^{k_n-1} \mid X_{\tau_{l+1}} \mid - \mid X_{\tau_l} \mid}{k_n - 1} = \sum_{s\in S} \sum_{i=1}^{k_n-1} \frac{Z_i(s)}{k_n - 1} \longrightarrow \sum_{s\in S} \mathbb{E}[Z_1(s)] = \mathbb{E}[\mid X_{\tau_1} \mid], \tag{35}$$

almost surely $\mathbb{P}$, as $n \to \infty$. Now,

$$0 \leq n - \tau_{k_n} \leq \tau_{k_n+1} - \tau_{k_n}, \tag{36}$$

$$\begin{aligned} &\mathbb{E}[\tau_{k_n} - \tau_{k_n-1}] = \sum_{s\in S} \mathbb{E}[\tau_{k_n} - \tau_{k_n-1} I_{\{s_{X_{\tau_{k_n}}}=s\}}] \\ &= \sum_{s\in S} \mathbb{E}[I_{\{s_{X_{\tau_{k_n}}}=s\}} \mathbb{E}[\tau_{k_n} - \tau_{k_n-1} \mid s_{X_{\tau_{k_n}}} = s]] = \sum_{s\in S} \mathbb{E}[I_{\{s_{X_{\tau_{k_n}}}=s\}} \mathbb{E}[\tau_2 - \tau_1 \mid s_{X_{\tau_1}} = s]] \\ &\leq \sum_{s\in S} \mathbb{E}[\mathbb{E}[\tau_2 - \tau_1 \mid s_{X_{\tau_1}} = s]] = \mid S \mid \mathbb{E}[\tau_2 - \tau_1], \end{aligned} \tag{37}$$

and

$$0 \leq \mid X_n \mid - \mid X_{\tau_{k_n}} \mid \leq \mid X_{\tau_{k_n+1}} \mid - \mid X_{\tau_{k_n}} \mid \text{ and } n - \mid X_{\tau_{k_n}} \mid \leq c_1(n - \tau_{k_n}) \tag{38}$$

for some $c_1 > 0$. Using (34)–(38) along with simple elementary algebra on (32) yields

$$\frac{\mid X_n \mid}{n} \longrightarrow \frac{\mathbb{E}[\mid X_{\tau_1} \mid]}{\mathbb{E}[\tau_1]}$$

almost surely as $n \to \infty$. □

Proof of Theorem 2.2. Recall from (31) and (33), k_n, $\tau_\cdot$, $Z_\cdot(\cdot)$ and $Y_\cdot(\cdot)$. Let

$$v = \frac{\mathbb{E}[| X_{\tau_1} |]}{\mathbb{E}[\tau_1]}, \ W_k(s) = Z_k(s) - Y_k(s)v, \ S_n(s) = \sum_{k=1}^{n} W_k(s), \text{ and } S_n = \sum_{s \in S} S_n(s).$$

Observe that,

$$\sqrt{n}\left(\frac{|X_n|}{n} - v\right) = \sqrt{n}\left(\frac{|X_n|}{n} - S_{k_n} - v\right) + \frac{S_{k_n}}{\sqrt{n}}.$$

As,

$$\frac{1}{\sqrt{n}} | \, | X_n | - S_{k_n} - nv \, | \leq \max_{1 \leq i \leq k_n} \frac{\tau_i - \tau_{i-1}}{\sqrt{n}},$$

our result will follow if we establish that as $n \to \infty$

$$\frac{S_{k_n}}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2), \quad \text{for some } \sigma^2 > 0 \tag{39}$$

and for all $\delta > 0$

$$\mathbb{P}\left(\max_{0 \leq i \leq k_n} \frac{\tau_{i+1} - \tau_i}{\sqrt{n}} > \delta\right) \longrightarrow 0. \tag{40}$$

Proof of (39): Using Assumption (E1), it is easy to check that $s_{X_{\tau_i}}$ under the annealed law $\mathbb{P}$ is uniform on S (see [1, Section 2]) and thus the vector $(W_k(s))_{s \in S}$ forms an i.i.d sequence with mean

$$\mathbb{E}[W_1(s)] = \mathbb{E}[X_{\tau_1} I_{\{s_{X_{\tau_1}} = s\}}] - \mathbb{E}[\tau_1 I_{\{s_{X_{\tau_1}} = s\}}]v \text{ for each } s \in S,$$

and covariance

$$\sigma_{s_1 s_2} = \mathbb{E} \prod_{i=1}^{2} [\left(| \, X_{\tau_1} \, | \, I_{\{s_{X_{\tau_1}} = s_i\}} - \tau_1 I_{\{s_{X_{\tau_1}} = s_i\}} v - \mathbb{E}[W_1(s_i)]\right)] \text{ for } s_1, s_2 \in S.$$

Therefore by the multivariate central limit theorem, we have

$$\frac{(S_n(s) - n\mathbb{E}[W_1(s)])_{s \in S}}{\sqrt{n}} \xrightarrow{d} N(0, \Sigma)$$

where $\Sigma = (\sigma_{s_i s_j})_{s_i, s_j \in S}$. The continuous map theorem then implies that for each $s \in S$,

$$\frac{S_n(s) - n\mathbb{E}[W_1(s)]}{\sqrt{n}} \xrightarrow{d} N(0, \sigma_s^2)$$

with $\sigma_s^2 = \sigma_{ss}$. If $\sigma = 1^t \Sigma 1$ then it is immediate that as $n \to \infty$

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2). \tag{41}$$

Note that there is no centring because $\sum_{s \in S} \mathbb{E}[W_1(s)] = 0$. From proof of Theorem 2.1, we know that $\frac{k_n}{n} \to \frac{1}{\mathbb{E}[\tau_1]}$. Using this and (41) we are done.

Proof of (40): Using Proposition 3.4(c) the proof is standard (see [3, Proof of Theorem 2.3]). Since $k_n \leq n$, we have that for any $\delta > 0$,

$$\mathbb{P}\left(\max_{0 \leq i \leq k_n} \frac{\tau_{i+1} - \tau_i}{\sqrt{n}} > \delta\right) \leq \sum_{i=1}^{n} \mathbb{P}(\tau_1 > \delta\sqrt{n}). \tag{42}$$

Note that, since $\mathbb{E}\left[\tau_1^2\right] < \infty$, one has that

$$\sum_{i=1}^{\infty} \mathbb{P}(\tau_1 > \frac{\delta\sqrt{i}}{\sqrt{T}}) = \sum_{i=1}^{\infty} \mathbb{P}(\tau_1^2 > \frac{\delta^2 i}{T}) < \infty.$$

Hence, for each $\epsilon > 0$ there is a deterministic constant $N \equiv N(d, \delta, \epsilon)$ such that

$$\sum_{i=N}^{\infty} \mathbb{P}(\tau_1 > \frac{\delta\sqrt{i}}{\sqrt{T}}) < \epsilon.$$

Therefore,

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(\tau_1 > \delta\sqrt{n}) \leq \limsup_{n\to\infty} \left(\sum_{i=1}^{N} \mathbb{P}(\tau_1 > \delta\sqrt{n}) + \sum_{i=N+1}^{\infty} \mathbb{P}(\tau_1 > \frac{\delta\sqrt{i}}{\sqrt{T}}) \right) \leq \epsilon.$$

As $\epsilon > 0$ was arbitrary, one concludes from the last limit and (42) that (40) holds. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] Siva Athreya, Antar Bandyopadhyay, Amites Dasgupta, Random walks in i.i.d. random environment on Cayley trees, Statist. Probab. Lett. 92 (2014) 39–44.
[2] K.B. Athreya, P.E. Ney, Branching Processes, Dover Publications, Inc. Mineola, NY, 2004, Reprint of the 1972 original [Springer, New York; MR0373040].
[3] Antar Bandyopadhyay, Ofer Zeitouni, Random walk in dynamic Markovian random environment, ALEA Lat. Am. J. Probab. Math. Stat. 1 (2006) 205–224.
[4] Hans A. Bethe, Statistical theory of superlattices, Proc R Soc London. Ser A-Math Phys Sci 150 (871) (1935) 552–575.
[5] Leonid V. Bogachev, Utkir A. Rozikov, On the uniqueness of Gibbs measure in the Potts model on a Cayley tree with external field, J. Stat. Mech. Theory Exp. 76 (7) (2019) 073205.
[6] Andrea Collevecchio, Tom Schmitz, Bounds on the speed and on regeneration times for certain processes on regular trees, Ann. Appl. Probab. 21 (3) (2011) 1073–1101.
[7] Amir Dembo, Nina Gantert, Yuval Peres, Ofer Zeitouni, Large deviations for random walks on Galton-Watson trees: averaging and uncertainty, Probab. Theory Related Fields 122 (2) (2002) 241–288.
[8] Amir Dembo, Nike Sun, Central limit theorem for biased random walk on multi-type Galton-Watson trees, Electron. J. Probab. 17 (75) (2012) 40.
[9] Rick Durrett, Probability: Theory and Examples, Volume 31 of Cambridge Series in Statistical and Probabilistic Mathematics, fourth ed., Cambridge University Press, Cambridge, 2010.
[10] Yueyun Hu, Zhan Shi, Slow movement of random walk in random environment on a regular tree, Ann. Probab. 35 (5) (2007) 1978–1997.
[11] Russell Lyons, Random walks and percolation on trees, Ann. Probab. 18 (3) (1990) 931–958.
[12] Russell Lyons, Robin Pemantle, Random walk in a random environment and first-passage percolation on trees, Ann. Probab. 20 (1) (1992) 125–136.
[13] Russell Lyons, Robin Pemantle, Yuval Peres, Ergodic theory on Galton-Watson trees: speed of random walk and dimension of harmonic measure, Ergodic Theory Dynam. Systems 15 (3) (1995) 593–619.

[14] Russell Lyons, Robin Pemantle, Yuval Peres, Biased random walks on Galton-Watson trees, Probab. Theory Related Fields 106 (2) (1996) 249–264.
[15] Yuval Peres, Ofer Zeitouni, A central limit theorem for biased random walks on Galton-Watson trees, Probab. Theory Related Fields 140 (3–4) (2008) 595–629.
[16] U.A. Rozikov, Random walks in random environments on the Cayley tree, UkraïN. Mat. Zh. 53 (10) (2001) 1391–1401.
[17] Fred Solomon, Random walks in a random environment, Ann. Probability 3 (1975) 1–31.
[18] Alain-Sol Sznitman, Topics in random walks in random environment, in: School and Conference on Probability Theory, in: ICTP Lect. Notes, XVII, Abdus Salam Int. Cent. Theoret. Phys. Trieste, 2004, pp. 203–266.
[19] Simon Tavaré, Ofer Zeitouni, Lectures on Probability Theory and Statistics, Volume 1837 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2004, Lectures from the 31st Summer School on Probability Theory held in Saint-Flour, July (2001) 8–25, Edited by Jean Picard.
[20] Wolfgang Woess, Random Walks on Infinite Graphs and Groups, Volume 138 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2000.