Adv. Appl. Prob. **50,** 1176–1192 (2018)
*doi:*10.1017/apr.2018.55

LINEAR DE-PREFERENTIAL URN MODELS

ANTAR BANDYOPADHYAY * ** AND
GURSHARN KAUR,* *** *Indian Statistical Institute*

Abstract

In this paper we consider a new type of urn scheme, where the selection probabilities are proportional to a weight function, which is linear but decreasing in the proportion of existing colours. We refer to it as the *de-preferential* urn scheme. We establish the almost-sure limit of the random configuration for any *balanced* replacement matrix R. In particular, we show that the limiting configuration is uniform on the set of colours if and only if R is a *doubly stochastic* matrix. We further establish the almost-sure limit of the vector of colour counts and prove central limit theorems for the random configuration as well as for the colour counts.

Keywords: Central limit theorem; de-preferential urn; strong law of large numbers; urn model

2010 Mathematics Subject Classification: Primary 60F05; 60F15
Secondary 60G57

1. Introduction

1.1. Background and motivation

Various kinds of *random reinforcement models* have attracted much interest in recent years [8], [16], [20]–[22], [24]–[26], [30], [36], [39], [41], [42], [47]. *Urn schemes*, which were first studied by Pólya [48], are perhaps the simplest reinforcement models. They have many applications and generalizations [6]–[8], [11]–[13], [16], [17], [20], [21], [24], [25], [30]–[33], [35], [36], [39], [40], [45], [46], [48]. In general, reinforcement models typically adhere to the structure of '*rich get richer*', which has also been termed *positive reinforcement*. However, there have been some studies on *negative reinforcement* models in the context of percolation, such as the *forest fire*-type models from the point of view of *self-destruction* [1], [2] [18], [19], [23], [27], [28], [49], [52]–[55], [57], and *frozen percolation*-type models from the point of view of *stagnation* [4], [10], [56], [58], [59]. For urn schemes, a type of negative reinforcement has been studied in which balls can be thrown away from as well as added to the urn [24], [29], [37], [38], [45], [60]. In such models, it is usually assumed that the model is *tenable*, that is, regardless of the stochastic path taken by the process, it is never required to remove a ball of a colour not currently present in the urn. Perhaps the most famous of such schemes is the *Ehrenfest urn* [29], [45], which models the diffusion of a gas between two chambers of a box. There are some models without tenability, such as the *OK Corral model* [37], [38], [60] or the *simple harmonic urn* [24] in two colours. Typically, these are used for modeling *destructive competition*.

Received 17 January 2018; revision received 19 October 2018.
* Postal address: Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Delhi Centre, 7 SJS Sansanwal Marg, New Delhi 110016, India.
** Email address: antar@isid.ac.in
*** Email address: gursharn.kaur24@gmail.com

Recently, there has been some interest in random graphs [9], [50], [51], where attachment probabilities of a new vertex are decreasing functions of the degree of the existing vertices. In most cases, such models also lead to negative reinforcement. They have been referred to as *de-preferential attachment* models [9] as opposed to usual 'preferential' attachment models [3], [14]. Motivated by this later set of works, in this paper we present a specific model of the de-preferential urn scheme where the selection probabilities are linear, but decreasing functions of the proportion of colours. As we will see in the sequel these models usually lead to negative reinforcement, provided the replacement scheme is 'positive' in nature!

De-preferential urn schemes are natural models for modelling problems with resource constraints, in particular, multiserver queueing systems with capacity constraints [43], [44]. For such cases, it is desirable that all agents have equal loads at the steady-state limit. In this work we show that, for a linear de-preferential urn scheme, such a limit is obtained under fairly general conditions on the replacement mechanism.

1.2. Model description

In this work we only consider *balanced* urn schemes with k colours, indexed by $S := \{0, 1, \dots, k-1\}$. More precisely, if $R := ((R_{i,j}))_{0\le i,j\le k-1}$ denotes the *replacement matrix*, that is, $R_{i,j} \ge 0$ denotes the number of balls of colour j to be placed in the urn when the colour of the selected ball is i, then, for a balanced urn, all row sums of R are constant. In this case, dividing all entries by the common row total, we may assume that R is a *stochastic matrix*. We will also assume that the starting configuration $U_0 := (U_{0,j})_{0\le j\le k-1}$ is a probability distribution on the set of colours S. As we will see from the proofs of our main results, this apparent loss of generality can be easily removed.

Denote by $U_n := (U_{n,j})_{0\le j\le k-1} \in [0,\infty)^k$ the random configuration of the urn at time n. Also, let $\mathcal{F}_n := \sigma(U_0, U_1, \dots, U_n)$ be the natural filtration. We define a random variable Z_n by

$$\mathbb{P}(Z_n = j \mid \mathcal{F}_n) \propto w_\theta\left(\frac{U_{n,j}}{n+1}\right), \qquad 0 \le j \le k-1,$$

where $w_\theta\colon [0,1] \to \mathbb{R}_+$ is given by

$$w_\theta(x) = \theta - x.$$

We will consider $\theta \ge 1$ as a parameter for the model. Note that Z_n represents the colour chosen at the $(n+1)$th draw. Starting with U_0, we define $(U_n)_{n\ge 0}$ recursively as

$$U_{n+1} = U_n + \chi_{n+1} R, \tag{1}$$

where $\chi_{n+1} := (\chi_{n+1,0}, \chi_{n+1,1}, \dots, \chi_{n+1,k-1})$ and $\chi_{n+1,j} = \mathbf{1}(Z_n = j)$ for $0 \le j \le k-1$.

We call the process $(U_n)_{n\ge 0}$ a *linear de-preferential urn scheme* with initial configuration U_0 and replacement matrix R. In this work we study the asymptotic properties of the following two processes.

Random configuration of the urn. Observe that, for all $n \ge 0$,

$$\sum_{j=0}^{k-1} U_{n,j} = n + 1. \tag{2}$$

This holds because R is a stochastic matrix and U_0 is a probability vector. Thus, the *random configuration of the urn*, namely, $U_n/(n+1)$ is a probability mass function. Furthermore,

$$\mathbb{P}(Z_n = j \mid \mathcal{F}_n) = \frac{\theta}{k\theta - 1} - \frac{1}{k\theta - 1}\frac{U_{n,j}}{n+1}, \qquad 0 \le j \le k-1. \tag{3}$$

Thus, $U_n A/(n+1)$ is the conditional distribution of the $(n+1)$th selected colour, namely Z_n, given $U_0, U_1, \dots, U_n$, where

$$A_{k\times k} = \frac{\theta}{k\theta - 1} J_k - \frac{1}{k\theta - 1} I_k, \tag{4}$$

and $J_k := \mathbf{1}^\top \mathbf{1}$ is the $k \times k$ matrix with all entries equal to 1 and I_k is the $k \times k$ identity matrix.

Color count statistics. Let $N_n := (N_{n,0}, \dots, N_{n,k-1})$ be the vector of length k whose jth element is the number of times colour j was selected in the first n trials, that is,

$$N_{n,j} = \sum_{m=1}^{n} \chi_{m,j} = \sum_{m=0}^{n-1} \mathbb{I}(Z_m = j), \qquad 0 \le j \le k-1, \tag{5}$$

and

$$N_n = \sum_{m=1}^{n} \chi_m \quad \text{for all } n \ge 1.$$

It easily follows from (1) that

$$U_{n+1} = U_0 + N_{n+1} R. \tag{6}$$

1.3. Outline

In Section 2 we present the main results of the paper, with the proofs given in Section 3 and Section 4.

2. The main results

We define a new $k \times k$ stochastic matrix, namely,

$$\hat{R} := RA = \frac{1}{k\theta - 1}(\theta J_k - R), \tag{7}$$

where A is as defined in (4). As we state in the sequel, the asymptotic properties of $(U_n)_{n\ge 0}$ and $(N_n)_{n\ge 0}$ depend on whether the stochastic matrix $\hat{R}$ is *irreducible* or *reducible*. We begin by stating a necessary and sufficient condition for $\hat{R}$ to be irreducible.

2.1. A necessary and sufficient condition for $\hat{R}$ to be irreducible

We start with the following definitions, which are needed for stating our main results.

Definition 1. A directed graph $\mathcal{G} = (\mathcal{V}, \vec{\mathcal{E}})$ is called the graph associated with a $k \times k$ stochastic $R = ((R_{i,j}))_{0\le i,j\le k-1}$ if

$$\mathcal{V} = \{0, 1, \dots, k-1\} \quad \text{and} \quad \vec{\mathcal{E}} = \{\overrightarrow{(i, j)} \mid R_{i,j} > 0; i, j \in \mathcal{V}\}.$$

Definition 2. A stochastic matrix R is called a *star* if there exists a $j \in \{0, 1, \dots, k-1\}$ such that

$$R_{i,j} = 1 \quad \text{for all } i \neq j.$$

In this case we say that j is the central vertex.

By definition, for the graph associated with a star replacement matrix, there is a central vertex such that each vertex other than the central vertex has only one outgoing edge and that is towards the central vertex. We note that in the definition of a star we allow the central vertex to have a self loop. A graph associated with a star matrix with five vertices is given in Figure 1.

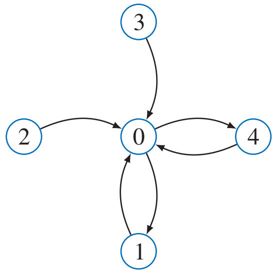

FIGURE 1: Graph of a star matrix, with 0 as the central vertex.

As we will see in the sequel, the asymptotic properties will depend on the irreducibility of the (new) stochastic matrix $\hat{R}$, as defined in (7). The following lemma provides a necessary and sufficient condition for $\hat{R}$ to be irreducible.

Proposition 1. *Let R be a $k \times k$ stochastic matrix with $k \geq 2$. Then $\hat{R}$ is irreducible if and only if either $\theta > 1$ or $\theta = 1$, but R is not a star.*

2.2. Asymptotics of the random configuration of the urn

2.2.1. *Case when $\hat{R}$ is irreducible.* Our first result concerns the almost-sure asymptotic property of the colour proportions.

Theorem 1. *Let $\hat{R}$ be irreducible. Then, for every starting configuration U_0,*

$$\frac{U_{n,j}}{n+1} \to \mu_j \quad \textit{a.s. for all } 0 \leq j \leq k-1,$$

where $\mu = (\mu_0, \mu_1, \ldots, \mu_{k-1})$ is the unique solution of the matrix equation

$$(\theta \mathbf{1} - \mu)R = (k\theta - 1)\mu. \tag{8}$$

Remark 1. (a) Note that if we define $\nu = \mu A$ then it follows from (4) and (8) that ν is the unique solution of the matrix equation $\nu \hat{R} = \nu$. Furthermore, from (8) we have $\mu = \nu R$.

(b) Since $U_{n,j}/(n+1)$ is a bounded random variable, we obtain

$$\frac{\mathbb{E}[U_{n,j}]}{n+1} \to \mu_j \quad \text{a.s. for all } 0 \leq j \leq k-1,$$

where μ satisfies (8).

(c) It is worth noting here that the uniform distribution of both stochastic matrices R and $\hat{R}$ is the unique stationary distribution if and only if R is doubly stochastic, that is, when $\mathbf{1}R = \mathbf{1}$.

Our next result is a central limit theorem for the colour proportions.

Theorem 2. *Suppose that $\hat{R}$ is irreducible. Then there exists a $k \times k$ variance–covariance matrix $\Sigma \equiv \Sigma(\theta, k)$ such that*

$$\frac{U_n - n\mu}{\sigma_n} \Rightarrow N_k(0, \Sigma),$$

where, for $k \geq 3$,

$$\sigma_n = \begin{cases} \sqrt{n \log n} & \textit{if } k = 3,\ \theta = 1, \textit{ and one of the eigenvalues of } R \textit{ is } -1, \\ \sqrt{n} & \textit{otherwise,} \end{cases} \tag{9}$$

and, for $k = 2$ *and* $\theta \in [1, \frac{3}{2}]$,

$$\sigma_n = \begin{cases} \sqrt{n \log n} & \text{if the eigenvalues of } R \text{ are } 1 \text{ and } \lambda = (1-2\theta)/2, \\ \sqrt{n} & \text{if the eigenvalues of } R \text{ are } 1 \text{ and } \lambda > (1-2\theta)/2. \end{cases} \tag{10}$$

Remark 2. (a) Note that Σ is necessarily a *positive semidefinite* matrix because of (2).

(b) It is worth noting here that the scaling is always by $\sqrt{n}$ for any parameter value $\theta \geq 1$ when $k \geq 4$. However, for a small number of colours, namely $k \in \{2, 3\}$, and certain specific parameter values, as given in (9) and (10), the scaling has an extra factor of $\sqrt{\log n}$.

2.2.2. *Case when* $\hat{R}$ *is reducible.* By Proposition 1 we know that $\hat{R}$ can be reducible if and only if R is a star and $\theta = 1$. Suppose that R is a star with $k \geq 2$ colours. Then, without loss of generality, we can write

$$R = \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{k-1} \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \quad \text{with} \sum_{j=0}^{k-1} \alpha_j = 1 \text{ and } \alpha_j \geq 0 \text{ for all } j, \tag{11}$$

by taking 0 as the central vertex. Taking $\theta = 1$, the matrix $\hat{R}$ is

$$\hat{R} = \frac{1}{k-1} \begin{pmatrix} 1-\alpha_0 & 1-\alpha_1 & \dots & 1-\alpha_{k-1} \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 1 \end{pmatrix}, \tag{12}$$

which is clearly reducible. In the next theorem we establish the limit of the urn configuration.

Theorem 3. *Let* $\theta = 1$, *and let the replacement matrix* R *be the star matrix given in (11) with*

$$R \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\frac{U_{n,0}}{n+1} \to 1 \quad \text{a.s.}$$

Furthermore, there exists a random variable $W \geq 0$ *with* $\mathbb{E}[W] > 0$ *such that*

$$\frac{U_{n,j}}{n^{\gamma}} \to \frac{\alpha_j}{k-1} W \quad \text{a.s. for all } j = 1, 2, \dots, k-1,$$

where $\gamma = (1-\alpha_0)/(k-1) < 1$.

Remark 3. (a) In the trivial case when $\gamma = 0$ or $(\alpha_0 = 1)$ we have

$$U_{n,0} = U_{0,0} + n$$

and

$$U_{n,j} = U_{0,j} \quad \text{for all } j = 1, 2, \dots, k-1.$$

That is, only colour 1 is reinforced at every time n in the urn.

(b) When

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we obtain

$$\hat{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that then $\hat{R}$ is the reinforcement rule for the classical Pólya urn scheme. Now, using (1), we have

$$\mathbb{E}[U_{n+1} \mid \mathcal{F}_n] = U_n + \frac{U_n}{n+1} = (n+2)\frac{U_n}{n+1},$$

which implies that each coordinate of the vector $U_n/(n+1)$ is a positive martingale and, hence, converges. Moreover, by exchangeability and arguments similar to the classical Pólya urn, we can easily show that

$$\frac{U_{n,0}}{n+1} \to Z \quad \text{a.s.},$$

where $Z \sim \text{Beta}(U_{0,0}, U_{0,1})$.

2.3. Asymptotics of the colour count statistics

2.3.1. *$\hat{R}$ is irreducible.*

Theorem 4. *Suppose that $\hat{R}$ is irreducible. Then*

$$\frac{N_{n,j}}{n} \to \frac{1}{k\theta - 1}[\theta - \mu_j] \quad \textit{a.s. for all } 0 \le j \le k-1,$$

where $\mu = (\mu_0, \mu_1, \ldots, \mu_{k-1})$ satisfies (8).

Theorem 5. *Suppose that $\hat{R}$ is irreducible. Then there exists a* variance–covariance *matrix $\tilde{\Sigma}$ such that*

$$\frac{N_n - n(\theta\mathbf{1} - \mu)/(k\theta - 1)}{\sigma_n} \Rightarrow N(0, \tilde{\Sigma}),$$

where σ_n is given in (9) and (10). Moreover,

$$\Sigma = R^\top \tilde{\Sigma} R, \tag{13}$$

where Σ is as in Theorem 2.

Remark 4. It is worth noting here that it follows from (5) that $\sum_{j=0}^{k-1} N_{n,j} = n$; thus, $\tilde{\Sigma}$ is a *positive semidefinite* matrix. Furthermore, it follows from (13) that $\text{rank}(\Sigma) \le \text{rank}(\tilde{\Sigma})$ with equality holding if and only if the replacement matrix R is nonsingular.

2.3.2. *$\hat{R}$ is reducible.* Recall that $\hat{R}$ has the form (12) when it is reducible.

Theorem 6. *Let R be a star matrix with 0 as the central vertex and $\theta = 1$ such that*

$$R \ne \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\frac{N_{n,0}}{n} \to 0 \quad \textit{a.s.}$$

and

$$\frac{N_{n,j}}{n} \to \frac{1}{k-1} \quad \textit{a.s. for all } 1 \le j \le k-1.$$

Remark 5. For

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

using (6) and Remark 3(b), we obtain

$$\frac{N_{n,0}}{n+1} \to 1 - Z \quad \text{a.s.},$$

where, as before, $Z \sim \text{Beta}(U_{0,0}, U_{0,1})$.

Theorem 7. *Let R be a star matrix with 0 as the central vertex and $\theta = 1$ such that*

$$R \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then the following assertions hold.

(i) *If $\gamma = (1-\alpha_0)/(k-1) < \frac{1}{2}$,*

$$\frac{1}{\sqrt{n}}\bigg(\frac{n}{k-1}\mathbf{1} - N_{n,-}\bigg) \Rightarrow N\bigg(\mathbf{0}, \frac{1}{k-1}I - \frac{1}{(k-1)^2}J\bigg),$$

where $N_{n,-} = (N_{n,1}, \ldots, N_{n,k-1})$ and

$$\frac{N_{n,0}}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0.$$

(ii) *If $\gamma = (1-\alpha_0)/(k-1) > \frac{1}{2}$,*

$$\frac{1}{n^{\gamma}}\bigg(\frac{n}{k-1} - N_{n,j}\bigg) \xrightarrow{\mathbb{P}} \frac{\alpha_j}{\gamma(k-1)^2}W \quad \textit{for all } j \neq 0$$

and

$$\frac{N_{n,0}}{n^{\gamma}} \xrightarrow{\mathbb{P}} \frac{1}{k-1}W,$$

where W is as given in Theorem 3.

(iii) *if $\gamma = (1-\alpha_0)/(k-1) = \frac{1}{2}$,*

$$\frac{N_{n,-} - n\mathbf{1}/(k-1) + W\alpha_{-}/(k-1)(1-\alpha_0)}{\sqrt{n}} \Rightarrow N\bigg(0, \frac{1}{k-1}I - \frac{1}{(k-1)^2}J\bigg),$$

where $\alpha_{-} = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1})$ and

$$\frac{N_{n,0}}{n^{\gamma}} \xrightarrow{\mathbb{P}} \frac{1}{k-1}W.$$

Remark 6. Note that $\gamma < \frac{1}{2}$ if and only if $k \geq 4$ or $k = 3$ and $\alpha_0 > 0$, or $k = 2$ and $\alpha_0 > \frac{1}{2}$.

3. Proof of the necessary and sufficient condition for $\hat{R}$ to be irreducible

Suppose that G and $\hat{G}$ are the directed graphs associated with the matrices R and $\hat{R}$, respectively, as defined earlier. Observe that $\hat{R}$ is the product of two stochastic matrices, R and A. The underlying Markov chain of $\hat{R}$ can be seen as a two-step Markov chain, where the first step is taken according to R and the second step is taken according to A. Recall from (7) that

$$\hat{R} = \frac{1}{k\theta - 1}(\theta J - R).$$

FIGURE 2.

Now, to show that the Markov chain associated with $\hat{R}$ is irreducible, it is enough to show that there exists a directed path between any two fixed vertices, say u and v, in $\hat{G}$.

Clearly, for $\theta > 1$, $\hat{R}_{uv} > 0$ for all u and v, and, thus, $\hat{R}$ is irreducible. Therefore, we only have to verify irreducibility for the $\theta = 1$ case. For this, we first fix two vertices, say u and v. From (7) we obtain

$$\hat{R}_{uv} = \frac{1 - R_{u,v}}{k-1}. \tag{14}$$

To complete the proof, we will show that there is a path from u to v of length at most 2. We consider the following two cases.

Case 1: $R_{u,v} < 1$. In this case, from (14) we obtain $\hat{R}_{uv} > 0$. Therefore, (u, v) is an edge in $\hat{G}$ and trivially there is a path of length 1 from u to v in $\hat{G}$.

Case 2: $R_{u,v} = 1$. In this case u has no R neighbour other than v, that is, (u, v) is the only incoming edge to v in G and, from (14),

$$\hat{R}_{uv} = 0.$$

As mentioned earlier for $\theta = 1$ and $k = 2$, $\hat{R}$ is reducible only when R is the Friedman urn scheme, which is a star with two vertices. Thus, in the rest of the proof we take $k > 2$, and show that $\hat{R}^2_{uv} > 0$, i.e. there is a path of length 2.

If R is not a star then there must exist a vertex l such that it leads to a vertex other than the central vertex, say m, that is $R_{l,m} > 0$ $(m \neq v)$. See Figure 2. Now, according to the $\hat{R}$ chain, there is a positive probability of going from u to l in one step (first take an R-step from u to v, which happens with probability 1 in this case, as $R_{u,v} = 1$, and then take an A-step to l with probability $1/(k-1)$) and a positive probability of going from l to v in one step (first take an R-step from l to m with probability $R_{l,m}$, and then take an A-step to v with probability $1/(k-1)$). Therefore, there is path of length two in $\hat{G}$ from u to v and, thus, the chain is irreducible.

Remark 7. Note that from the proof it follows that, for a replacement matrix R with $k > 2$ such that $\hat{R}$ is irreducible, $\hat{R}$ is also aperiodic.

4. Proofs of the main results

We begin by observing the following fact. From (1), (3), and (4), we obtain

$$\mathbb{E}[U_{n+1} \mid \mathcal{F}_n] = U_n + \mathbb{E}[\chi_{n+1} \mid \mathcal{F}_n]R = U_n + \frac{U_n}{n+1}AR. \tag{15}$$

Thus,

$$\mathbb{E}[U_{n+1}A \mid \mathcal{F}_n] = U_nA + \frac{U_nA}{n+1}\hat{R}. \tag{16}$$

Let $\hat{U}_n := U_nA$, $n \geq 0$. Then

$$\hat{U}_{n+1} = \hat{U}_n + \chi_{n+1}\hat{R}.$$

From (16) we obtain

$$\mathbb{E}[\hat{U}_{n+1} \mid \mathcal{F}_n] = \hat{U}_n + \frac{\hat{U}_n}{n+1}\hat{R}.$$

Therefore, $(\hat{U}_n)_{n\geq 0}$ is a classical urn scheme (uniform selection) with replacement matrix $\hat{R}$.

The construction $(\hat{U}_n)_{n\geq 0}$ is essentially a coupling of a de-preferential urn $(U_n)_{n\geq 0}$ with replacement matrix R to a classical (positively reinforced) urn $(\hat{U}_n)_{n\geq 0}$ with replacement matrix $\hat{R}$. Note that we get a one-to-one correspondence, as A is always invertible.

Proof of Theorem 1. Recall that $\hat{U}_n = U_n A$ is the configuration of a classical urn model with replacement matrix $\hat{R}$. Since, by our assumption, $\hat{R}$ is irreducible, it follows from Theorem 2.2 of [8] that the limit of $\hat{U}_n/(n+1)$ is the normalized left eigenvector of $\hat{R}$ associated with the maximal eigenvalue 1. That is,

$$\frac{\hat{U}_n}{n+1} \to \nu \quad \text{a.s.},$$

where ν satisfies $\nu\hat{R} = \nu$. Since $U_n = \hat{U}_n A^{-1}$, we have

$$\frac{U_n}{n+1} \to \mu \quad \text{a.s.},$$

where $\mu = \nu A^{-1}$, and it satisfies the following matrix equation:

$$(\theta\mathbf{1} - \mu)R = (k\theta - 1)\mu.$$

This completes the proof. □

Proof of Theorem 2. Let $1, \lambda_1, \ldots, \lambda_s$ be the distinct eigenvalues of R such that $1 \geq \mathrm{Re}(\lambda_1) \geq \cdots \geq \mathrm{Re}(\lambda_s) \geq -1$, where $\mathrm{Re}(\lambda)$ denotes the real part of the eigenvalue λ. Recall from (7) that $\hat{R} = (\theta J_k - R)/(k\theta - 1)$. So the eigenvalues of $\hat{R}$ are $1, b\lambda_1, \ldots, b\lambda_s$, where $b = -1/(k\theta - 1)$. Let $\tau = \max\{0, b\,\mathrm{Re}(\lambda_s)\}$. Since $\hat{U}_n = U_n A$ is a classical urn scheme with replacement matrix $\hat{R}$, using Theorem 3.2 of [8], if $b\,\mathrm{Re}(\lambda_s) \leq \frac{1}{2}$ then there exists a variance–covariance matrix Σ' such that

$$\frac{\hat{U}_n - n\nu}{\sigma_n} \Rightarrow \mathcal{N}(0, \Sigma'),$$

where

$$\sigma_n = \begin{cases} \sqrt{n\log n} & \text{if } b\,\mathrm{Re}(\lambda_s) = \frac{1}{2}, \\ \sqrt{n} & \text{if } b\,\mathrm{Re}(\lambda_s) < \frac{1}{2}. \end{cases}$$

Note that

$$b\,\mathrm{Re}(\lambda_s) \leq \tfrac{1}{2} \quad \iff \quad \mathrm{Re}(\lambda_s) \geq -\tfrac{1}{2}(k\theta - 1). \tag{17}$$

Now since $\theta \geq 1$ and $\mathrm{Re}(\lambda_s) \geq -1$, (17) holds whenever $k \geq 3$. Furthermore, for $k \geq 3$, equality in (17) holds if and only if $\theta = 1$ and $k = 3$. Moreover, for $k = 2$, the condition is equivalent to $\mathrm{Re}(\lambda_s) \geq (1 - 2\theta)/2$. Thus, σ_n is as given in (9) and (10). Therefore,

$$\frac{U_n - n\mu}{\sigma_n} \Rightarrow \mathcal{N}(0, \Sigma),$$

where $\Sigma = A^\top \Sigma' A$. □

Proof of Theorem 3. Without loss of generality, we assume that $\gamma > 0$ (equivalently, $\alpha_0 < 1$), since, as noted in Remark 3, the result is trivial otherwise. Since the matrix $\hat{R}$ given in (12) is reducible without isolated blocks, using Proposition 4.3 of [33], we obtain

$$\frac{\hat{U}_{n,0}}{n+1} \to 0 \quad \text{and} \quad \frac{\hat{U}_{n,j}}{n+1} \to \frac{1}{k-1} \quad \text{for all } j \neq 0,$$

which implies that

$$\frac{U_{n,0}}{n+1} \to 1 \quad \text{and} \quad \frac{U_{n,j}}{n+1} \to 0 \quad \text{for all } j \neq 0.$$

Now recall that (15) provides the recursion

$$\mathbb{E}[U_{n+1} \mid \mathcal{F}_n] = U_n + \mathbb{E}[\chi_{n+1} \mid \mathcal{F}_n] R = U_n + \frac{U_n}{n+1} AR.$$

Note that in this case the matrix AR is given by

$$AR = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1-\gamma & \dfrac{\alpha_1}{k-1} & \dfrac{\alpha_2}{k-1} & \cdots & \dfrac{\alpha_{k-1}}{k-1} \\ 1-\gamma & \dfrac{\alpha_1}{k-1} & \dfrac{\alpha_2}{k-1} & \cdots & \dfrac{\alpha_{k-1}}{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-\gamma & \dfrac{\alpha_1}{k-1} & \dfrac{\alpha_2}{k-1} & \cdots & \dfrac{\alpha_{k-1}}{k-1} \end{pmatrix},$$

where $\gamma = (1-\alpha_0)/(k-1)$. Thus, the eigenvalues of AR are 1, γ, and $0, 0, \ldots, 0$ ($k-2$ times), and the eigenvector corresponding to the nonprincipal eigenvalue γ is

$$\xi = \big(0, 1, 1, \ldots, 1\big)^{\top}.$$

Therefore, again using (15) we obtain

$$\mathbb{E}[U_{n+1}\xi \mid \mathcal{F}_n] = U_n\left[I + \frac{AR}{n+1}\right]\xi = U_n\xi\left[1 + \frac{\gamma}{(n+1)}\right].$$

Let $\Pi_n(\gamma) = \prod_{i=1}^{n}(1+\gamma/i)$. Then $W_n := U_n\xi/\Pi_n(\gamma)$ is a nonnegative martingale and, using Euler's product, for large n,

$$\Pi_n(\gamma) \sim \frac{n^{\gamma}}{\Gamma(\gamma+1)}.$$

We now show that this martingale is $\mathcal{L}^2$-bounded, which will then imply that

$$\frac{U_n\xi}{n^{\gamma}} \to W, \tag{18}$$

where W is a nondegenerate random variable. More precisely, W is nonzero with positive probability. We can write

$$\mathbb{E}[W_{n+1}^2 \mid \mathcal{F}_n] = W_n^2 + \mathbb{E}[(W_{n+1}-W_n)^2 \mid \mathcal{F}_n]$$

and

$$\begin{aligned} W_{n+1} - W_n &= \frac{1}{\Pi_{n+1}(\gamma)}\left(U_{n+1}\xi - U_n\xi\left(1 + \frac{\gamma}{n+1}\right)\right) \\ &= \frac{1}{\Pi_{n+1}(\gamma)}\left(\chi_{n+1}R\xi - \frac{\gamma}{n+1}U_n\xi\right) \\ &= \frac{1}{\Pi_{n+1}(\gamma)}\left((1-\alpha_0)\chi_{n+1,0} - \gamma\frac{(n+1) - U_{n,0}}{n+1}\right) \\ &= \frac{(1-\alpha_0)}{\Pi_{n+1}(\gamma)}(\chi_{n+1,0} - \mathbb{E}[\chi_{n+1,0} \mid \mathcal{F}_n]). \end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[W_{n+1}^2 \mid \mathcal{F}_n] &= W_n^2 + \frac{(1-\alpha_0)^2}{\Pi_{n+1}^2(\gamma)}(\mathbb{E}[\chi_{n+1,0} \mid \mathcal{F}_n] - \mathbb{E}[\chi_{n+1,0} \mid \mathcal{F}_n]^2) \\
&\le W_n^2 + \frac{(1-\alpha_0)^2}{\Pi_{n+1}^2(\gamma)}\mathbb{E}[\chi_{n+1,0} \mid \mathcal{F}_n] \\
&= W_n^2 + \frac{(1-\alpha_0)^2}{\Pi_{n+1}^2(\gamma)}\frac{1}{k-1}\frac{n+1-U_{n,0}}{n+1} \\
&= W_n^2 + \frac{\gamma(1-\alpha_0)}{(n+1)\Pi_{n+1}(\gamma)}\frac{U_n\xi}{\Pi_{n+1}(\gamma)} \\
&\le W_n^2 + \frac{\gamma(1-\alpha_0)}{(n+1)\Pi_{n+1}(\gamma)}W_n \\
&\le W_n^2 + \frac{\gamma(1-\alpha_0)\Gamma(\gamma+1)}{2(n+1)^{\gamma+1}}(1+W_n^2).
\end{aligned}$$

The last inequality holds because $2W_n \le 1 + W_n^2$. Let $c := \frac{1}{2}\gamma(1-\alpha_0)\Gamma(\gamma+1)$. Then

$$\begin{aligned}
\mathbb{E}[W_{n+1}^2 + 1 \mid \mathcal{F}_n] &\le \bigg(1 + \frac{c}{(n+1)^{\gamma+1}}\bigg)(1+W_n^2) \\
&\le (1+W_0^2)\prod_{j=1}^{n}\bigg(1 + \frac{c}{(j+1)^{\gamma+1}}\bigg) \\
&\le (1+W_0^2)\exp\bigg(\sum_{j=1}^{n}\frac{c}{(j+1)^{\gamma+1}}\bigg) \\
&< \infty \quad (\text{since } \gamma > 0).
\end{aligned}$$

Thus, W_n is $\mathcal{L}^2$-bounded and, hence, converges to a nondegenerate random variable, say W. For a star matrix R (as given in (11)), recursion (1) reduces to

$$U_{n+1,0} = U_{n,0} + \alpha_0\chi_{n+1,0} + (1 - \chi_{n+1,0})$$

and

$$U_{n+1,h} = U_{n,h} + \alpha_h\chi_{n+1,0} \quad \text{for all } h \ne 0.$$

When $\alpha_h = 0$, there is nothing to prove. When $\alpha_h > 0$, dividing both sides by α_h yields

$$\frac{U_{n+1,h}}{\alpha_h} = \frac{U_{0,h}}{\alpha_h} + \sum_{j=1}^{n+1}\chi_{j,0}.$$

Since the above relation holds for every choice of $h > 0$, we obtain

$$\frac{U_{n+1,h}}{\alpha_h} - \frac{U_{n+1,l}}{\alpha_l} = \frac{U_{0,h}}{\alpha_h} - \frac{U_{0,l}}{\alpha_l}$$

for any $h, l \in \{1, 2, \dots, k-1\}$. Multiplying the above equation by $\alpha_l/(1-\alpha_0)$ and taking the sum over $l \ne 0$,

$$\frac{U_{n,h}}{\alpha_h} - \frac{1}{1-\alpha_0}\sum_{l\ne 0}U_{n,l} = \frac{U_{0,h}}{\alpha_h} - \frac{1}{1-\alpha_0}\sum_{l\ne 0}U_{0,l},$$

which can be written as

$$\frac{U_{n,h}}{\alpha_h} - \frac{1}{k-1} U_n \xi = \frac{U_{0,h}}{\alpha_h} - \frac{1}{k-1} U_0 \xi.$$

Now dividing both sides by n^γ,

$$\frac{1}{n^\gamma} \frac{U_{n,h}}{\alpha_h} - \frac{1}{k-1} \frac{U_n \xi}{n^\gamma} = \frac{1}{n^\gamma} \bigg(\frac{U_{0,h}}{\alpha_h} - \frac{1}{k-1} U_0 \xi \bigg).$$

Note that the right-hand side of the above expression goes to 0 as n tends to ∞. Therefore,

$$\lim_{n\to\infty} \frac{1}{n^\gamma} \frac{U_{n,h}}{\alpha_h} - \frac{1}{k-1} \frac{U_n \xi}{n^\gamma} = 0.$$

Using the limit from (18), we obtain

$$\frac{U_{n,h}}{n^\gamma} \to \frac{\alpha_h}{k-1} W. \qquad \square$$

Proof of Theorem 4. Note that from (1), (5), and (15) we can write

$$\begin{aligned} N_n &= \sum_{i=1}^{n} (\chi_i - \mathbb{E}[\chi_i \mid \mathcal{F}_{i-1}]) + \sum_{i=1}^{n} \mathbb{E}[\chi_i \mid \mathcal{F}_{i-1}] \\ &= \sum_{i=1}^{n} (\chi_i - \mathbb{E}[\chi_i \mid \mathcal{F}_{i-1}]) + \frac{1}{k\theta - 1} \sum_{i=1}^{n} \bigg[\theta \mathbf{1} - \frac{U_{i-1}}{i} \bigg]. \end{aligned} \tag{19}$$

Since $(\chi_i - \mathbb{E}[\chi_i \mid \mathcal{F}_{i-1}])_{i\geq 1}$ is a bounded martingale difference sequence, using Azuma's inequality (see [15]), we obtain

$$\frac{1}{n} \sum_{i=1}^{n} (\chi_i - \mathbb{E}[\chi_i \mid \mathcal{F}_{i-1}]) \to 0 \quad \text{a.s.} \tag{20}$$

Using Theorem 1 and Cesaro's lemma (see [5]), we obtain

$$\frac{N_{n,j}}{n} \to \frac{1}{k\theta - 1} (\theta - \mu_j) \quad \text{a.s. for all } 0 \leq j \leq k-1. \qquad \square$$

Proof of Theorem 5. Note that, under our coupling, N_n remains the same for the two processes, namely, $(U_n)_{n\geq 0}$ and $(\hat{U}_n)_{n\geq 0}$. Thus, applying Theorem 4.1 of [8] to the urn process $(\hat{U}_n)_{n\geq 0}$, we conclude that there exists a matrix $\tilde{\Sigma}$ such that

$$\frac{N_n - n\mu A}{\sigma_n} \Rightarrow \mathcal{N}(0, \tilde{\Sigma}).$$

Finally, (13) follows from (6). This completes the proof. $\square$

Proof of Theorem 6. The proof follows from (19) and (20). $\square$

Proof of Theorem 7. Let $M_{n,j} := \sum_{i=1}^{n} (\chi_{i,j} - \mathbb{E}[\chi_{i,j} \mid \mathcal{F}_{i-1}])$ and $M_n = (M_{n,0}, M_{n,1}, \ldots, M_{n,k-1})$. Then $\{M_n, \mathcal{F}_n\}$ is a martingale. Define $X_i = (X_{i,0}, X_{i,1}, \ldots, X_{i,k-1})$, where

$$X_{i,j} := \frac{1}{\sqrt{n}} (\chi_{i,j} - \mathbb{E}[\chi_{i,j} \mid \mathcal{F}_{i-1}])$$

are the martingale differences and $(M_n)_{n\geq 1}$ is a k-dimensional bounded increment martingale.

Let $M_{n,-} := (M_{n,1}, \dots, M_{n,k-1})$ and $X_{n,-} := (X_{n,1}, \dots, X_{n,k-1})$. In this proof, we first provide a central limit theorem for $M_{n,-}$, and then for N_n. Observe that the (l, m)th entry of the matrix $\mathbb{E}[X_{i,-}^\top X_{i,-} \mid \mathcal{F}_{i-1}]$ is

$$\begin{aligned}
&\frac{1}{n}(\mathbb{E}[\chi_{i,l}\chi_{i,m} \mid \mathcal{F}_{i-1}] - \mathbb{E}[\chi_{i,l} \mid \mathcal{F}_{i-1}]\mathbb{E}[\chi_{i,m} \mid \mathcal{F}_{i-1}]) \\
&\quad = \begin{cases} \frac{1}{n}\mathbb{E}[\chi_{i,l} \mid \mathcal{F}_{i-1}](1 - \mathbb{E}[\chi_{i,l} \mid \mathcal{F}_{i-1}]) & \text{if } l = m, \\ -\frac{1}{n}\mathbb{E}[\chi_{i,l} \mid \mathcal{F}_{i-1}]\mathbb{E}[\chi_{i,m} \mid \mathcal{F}_{i-1}] & \text{if } l \neq m, \end{cases} \\
&\quad = \begin{cases} \dfrac{1}{n(k-1)}\left(1 - \dfrac{U_{i-1,l}}{i}\right)\left(1 - \dfrac{1}{k-1}\left(1 - \dfrac{U_{i-1,l}}{i}\right)\right) & \text{if } l = m, \\ -\dfrac{1}{n(k-1)^2}\left(1 - \dfrac{U_{i-1,l}}{i}\right)\left(1 - \dfrac{U_{i-1,m}}{i}\right) & \text{if } l \neq m. \end{cases}
\end{aligned}$$

So, as $n \to \infty$ (using Theorem 3), we have

$$\sum_{i=1}^{n} \mathbb{E}[X_{i,-}^\top X_{i,-} \mid \mathcal{F}_{i-1}]_{(l,m)} \to \begin{cases} \dfrac{(k-2)}{(k-1)^2} & \text{if } l = m, \\ -\dfrac{1}{(k-1)^2} & \text{if } l \neq m. \end{cases}$$

Therefore,

$$\sum_{i=1}^{n} \mathbb{E}[X_{i,-}^\top X_{i,-} \mid \mathcal{F}_{i-1}] \to \frac{1}{k-1}I - \frac{1}{(k-1)^2}J,$$

and, by the martingale central limit theorem [34],

$$\frac{1}{\sqrt{n}}M_{n,-} \Rightarrow N\left(0, \frac{1}{k-1}I - \frac{1}{(k-1)^2}J\right). \tag{21}$$

Now, for colour 0, we have

$$\frac{1}{\sqrt{n}}M_{n,0} = -\frac{1}{\sqrt{n}}\sum_{j=1}^{k-1} M_{n,-},$$

which implies that

$$\frac{1}{\sqrt{n}}M_{n,0} \xrightarrow{\mathbb{P}} 0.$$

We now prove the central limit theorem for N_n. By (19) we have

$$N_n = M_n + \frac{1}{k-1}\sum_{i=1}^{n}\left[\mathbf{1} - \frac{U_{i-1}}{i}\right].$$

Therefore,

$$\frac{n}{k-1}\mathbf{1} - N_{n,-} = -M_{n,-} + \frac{1}{k-1}\sum_{i=1}^{n}\frac{U_{i-1,-}}{i}. \tag{22}$$

From Theorem 3 we know that, for each $j \neq 0$,

$$\frac{U_{i-1,j}}{i^\gamma} \to \frac{\alpha_j}{k-1} W \quad \text{a.s.},$$

$$\sum_{i=1}^{n} \frac{U_{i-1,j}}{i} \asymp \frac{\alpha_j}{k-1} W \sum_{i=1}^{n} i^{\gamma-1} \sim \frac{\alpha_j}{k-1} \frac{n^\gamma}{\gamma} W.$$

Therefore,

$$\frac{1}{n^\gamma} \sum_{i=1}^{n} \frac{U_{i-1,j}}{i} \to \frac{\alpha_j}{\gamma(k-1)} W \quad \text{a.s.} \tag{23}$$

Therefore, for $\gamma < \frac{1}{2}$, using (21), (22), and (23),

$$\frac{1}{\sqrt{n}}\left(\frac{n}{k-1}\mathbf{1} - N_{n,-}\right) \Rightarrow N\left(0, \frac{1}{k-1} I - \frac{1}{(k-1)^2} J\right),$$

and, for $\gamma \geq \frac{1}{2}$,

$$\frac{1}{n^\gamma}\left(\frac{n}{k-1} - N_{n,j}\right) \xrightarrow{\mathbb{P}} \frac{\alpha_j}{\gamma(k-1)^2} W \quad \text{for all } j \neq 0,$$

since then $M_{n,j}/n^\gamma \xrightarrow{\mathbb{P}} \mathbf{0}$. For $j = 0$, we have

$$N_{n,0} = n - \sum_{j=1}^{k-1} N_{n,j} = \sum_{j=1}^{k-1}\left(\frac{n}{k-1} - N_{n,j}\right).$$

Therefore, for $\gamma < \frac{1}{2}$, we have

$$\frac{N_{n,0}}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0,$$

and, for $\gamma > \frac{1}{2}$, we have

$$\begin{aligned} \frac{1}{n^\gamma} N_{n,0} &= \sum_{j=1}^{k-1} \frac{1}{n^\gamma}\left(\frac{n}{k-1} - N_{n,j}\right) \\ &\xrightarrow{\mathbb{P}} \frac{W}{(k-1)(1-\alpha_0)} \sum_{j=1}^{k-1} \alpha_j \\ &= \frac{W}{k-1}. \end{aligned}$$

□

From (22) we have

$$N_{n,-} - \frac{n}{k-1}\mathbf{1} + \frac{1}{k-1}\sum_{i=1}^{n} \frac{U_{i-1,-}}{i} = M_{n,-},$$

$$\frac{N_{n,-} - n\mathbf{1}/(k-1) + (1/(k-1)) \sum_{i=1}^{n} U_{i-1,-}/i}{\sqrt{n}} = \frac{M_{n,-}}{\sqrt{n}};$$

therefore, for $\gamma = \frac{1}{2}$, we obtain

$$\frac{N_{n,-} - n\mathbf{1}/(k-1) + W\alpha_{-}/(k-1)(1-\alpha_1)}{\sqrt{n}} \Rightarrow N\left(0, \frac{1}{k-1} I - \frac{1}{(k-1)^2} J\right),$$

where $\alpha_- = (\alpha_1, \alpha_2, \dots, \alpha_{k-1})$ and

$$N_{n,0} = \sum_{j=1}^{k-1}\left(\frac{n}{k-1} - N_{n,j}\right)$$

$$= \frac{1}{k-1}W + \sum_{j=1}^{k-1}\left(\frac{n}{k-1} - N_{n,j} - \frac{\alpha_j}{(k-1)(1-\alpha_0)}W\right)$$

$$\Rightarrow \frac{N_{n,0}}{\sqrt{n}}$$

$$\xrightarrow{\mathbb{P}} \frac{1}{k-1}W.$$

Acknowledgements

The authors are grateful to the anonymous referees for their valuable remarks, which have helped to much improve the exposition of the article.

References

[1] Ahlberg, D., Sidoravicius, V. and Tykesson, J. (2014). Bernoulli and self-destructive percolation on non-amenable graphs. *Electron. Commun. Prob.* **19,** 6pp.
[2] Ahlberg, D., Duminil-Copin, H., Kozma, G. and Sidoravicius, V. (2015). Seven-dimensional forest fires. *Ann. Inst. H. Poincaré Prob. Statist.* **51,** 862–866.
[3] Albert, R. and Barabási, A.-L. (2002). Statistical mechanics of complex networks. *Rev. Modern Phys.* **74,** 47–97.
[4] Aldous, D. J. (2000). The percolation process on a tree where infinite clusters are frozen. *Math. Proc. Camb. Phil. Soc.* **128,** 465–477.
[5] Apostol, T. M. (1974). *Mathematical Analysis*, 2nd edn. Addison-Wesley, Reading, MA.
[6] Athreya, K. B. and Karlin, S. (1968). Embedding of urn schemes into continuous time Markov branching processes and related limit theorems. *Ann. Math. Statist.* **39,** 1801–1817.
[7] Bagchi, A. and Pal, A. K. (1985). Asymptotic normality in the generalized Pólya-Eggenberger urn model, with an application to computer data structures. *SIAM J. Algebraic Discrete Methods* **6,** 394–405.
[8] Bai, Z.-D. and Hu, F. (2005). Asymptotics in randomized URN models. *Ann. Appl. Prob.* **15,** 914–940.
[9] Bandyopadhyay, A. and Sen, S. (2017). De-preferential attachment random graphs. Preprint.
[10] Bandyopadhyay, A. (2006). A necessary and sufficient condition for the tail-triviality of a recursive tree process. *Sankhyā* **68,** 1–23.
[11] Bandyopadhyay, A. and Thacker, D. (2014). Rate of convergence and large deviation for the infinite color Pólya urn schemes. *Statist. Prob. Lett.* **92,** 232–240.
[12] Bandyopadhyay, A. and Thacker, D. (2016). A new approach to Pólya urn schemes and its infinite color generalization. Preprint. Available at https://arxiv.org/abs/1606.05317v1.
[13] Bandyopadhyay, A. and Thacker, D. (2017). Pólya urn schemes with infinitely many colors. *Bernoulli* **23,** 3243–3267.
[14] Barabási, A.-L. and Albert, R. (1999). Emergence of scaling in random networks. *Science* **286,** 509–512.
[15] Billingsley, P. (1995). *Probability and Measure*, 3rd edn. John Wiley, New York.
[16] Bose, A., Dasgupta, A. and Maulik, K. (2009). Multicolor urn models with reducible replacement matrices. *Bernoulli* **15,** 279–295.
[17] Bose, A., Dasgupta, A. and Maulik, K. (2009). Strong laws for balanced triangular urns. *J. Appl. Prob.* **46,** 571–584.
[18] Bressaud, X. and Fournier, N. (2010). Asymptotics of one-dimensional forest fire processes. *Ann. Prob.* **38,** 1783–1816.
[19] Bressaud, X. and Fournier, N. (2014). A mean-field forest-fire model. *ALEA* **11,** 589–614.
[20] Chen, M.-R., Hsiau, S.-R. and Yang, T.-H. (2014). A new two-urn model. *J. Appl. Prob.* **51,** 590–597.
[21] Collevecchio, A., Cotar, C. and LiCalzi, M. (2013). On a preferential attachment and generalized Pólya's urn model. *Ann. Appl. Prob.* **23,** 1219–1253.
[22] Cotar, C. and Limic, V. (2009). Attraction time for strongly reinforced walks. *Ann. Appl. Prob.* **19,** 1972–2007.

[23] Crane, E., Freeman, N. and Tóth, B. (2015). Cluster growth in the dynamical Erdős-Rényi process with forest fires. *Electron. J. Prob.* **20,** 33 pp.
[24] Crane, E. *et al.* (2011). The simple harmonic urn. *Ann. Prob.* **39,** 2119–2177.
[25] Dasgupta, A. and Maulik, K. (2011). Strong laws for urn models with balanced replacement matrices. *Electron. J. Prob.* **16,** 1723–1749.
[26] Davis, B. (1990). Reinforced random walk. *Prob. Theory Relat. Fields* **84,** 203–229.
[27] Dürre, M. (2006). Existence of multi-dimensional infinite volume self-organized critical forest-fire models. *Electron. J. Prob.* **11,** 513–539.
[28] Dürre, M. (2006). Uniqueness of multi-dimensional infinite volume self-organized critical forest-fire models. *Electron. Commun. Prob.* **11,** 304–315.
[29] Ehrenfest, P. and Ehrenfest, T. (1990). *The Conceptual Foundations of the Statistical Approach in Mechanics*. Dover Publications, New York.
[30] Flajolet, P., Dumas, P. and Puyhaubert, V. (2006). Some exactly solvable models of urn process theory. In *4th Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities* (Discrete Math. Theoret. Comput. Sci. Proc. **AG**), Assoc. Discrete Math. Theoret. Comput. Sci., Nancy, pp. 59–118.
[31] Freedman, D. A. (1965). Bernard Friedman's urn. *Ann. Math. Statist.* **36,** 956–970.
[32] Friedman, B. (1949). A simple urn model. *Commun. Pure Appl. Math.* **2,** 59–70.
[33] Gouet, R. (1997). Strong convergence of proportions in a multicolor Pólya urn. *J. Appl. Prob.* **34,** 426–435.
[34] Hall, P. and Heyde, C. C. (1980). *Martingale Limit Theory and Its Application*. Academic Press, New York.
[35] Janson, S. (2004). Functional limit theorems for multitype branching processes and generalized Pólya urns. *Stoch. Process. Appl.* **110,** 177–245.
[36] Janson, S. (2006). Limit theorems for triangular urn schemes. *Prob. Theory Relat. Fields* **134,** 417–452.
[37] Kingman, J. F. C. (1999). Martingales in the OK Corral. *Bull. London Math. Soc.* **31,** 601–606.
[38] Kingman, J. F. C. and Volkov, S. E. (2003). Solution to the OK Corral model via decoupling of Friedman's urn. *J. Theoret. Prob.* **16,** 267–276.
[39] Laruelle, S. and Pagès, G. (2013). Randomized urn models revisited using stochastic approximation. *Ann. Appl. Prob.* **23,** 1409–1436.
[40] Launay, M. and Limic, V. (2012). Generalized interacting urn models. Preprint. Available at https://arxiv.org/abs/1207.5635v1.
[41] Limic, V. (2003). Attracting edge property for a class of reinforced random walks. *Ann. Prob.* **31,** 1615–1654.
[42] Limic, V. and Tarrès, P. (2007). Attracting edge and strongly edge reinforced walks. *Ann. Prob.* **35,** 1783–1806.
[43] Luczak, M. J. and McDiarmid, C. (2005). On the power of two choices: balls and bins in continuous time. *Ann. Appl. Prob.* **15,** 1733–1764.
[44] Luczak, M. J. and McDiarmid, C. (2006). On the maximum queue length in the supermarket model. *Ann. Prob.* **34,** 493–527.
[45] Mahmoud, H. M. (2009). *Pólya Urn Models*. CRC Press, Boca Raton, FL.
[46] Pemantle, R. (1990). A time-dependent version of Pólya's urn. *J. Theoret. Prob.* **3,** 627–637.
[47] Pemantle, R. (2007). A survey of random processes with reinforcement. *Prob. Surveys* **4,** 1–79.
[48] Pólya, G. (1930). Sur quelques points de la théorie des probabilités. *Ann. Inst. H. Poincaré Prob. Statist.* **1,** 117–161 (in French).
[49] Ráth, B. and Tóth, B. (2009). Erdős-Rényi random graphs + forest fires = self-organized criticality. *Electron. J. Prob.* **14,** 1290–1327.
[50] Sevim, V. and Rikvold, P. A. (2006). Effects of preference for attachment to low-degree nodes on the degree distributions of a growing directed network and a simple food-web model. *Phys. Rev. E* **73,** 056115.
[51] Sevim, V. and Rikvold, P. A. (2008). Network growth with preferential attachment for high indegree and low outdegree. *Physica A* **387,** 2631–2636.
[52] Van den Berg, J. and Brouwer, R. (2004). Self-destructive percolation. *Random Structures Algorithms* **24,** 480–501.
[53] Van den Berg, J. and Brouwer, R. (2006). Self-organized forest-fires near the critical time. *Commun. Math. Phys.* **267,** 265–277.
[54] Van den Berg, J. and de Lima, B. N. B. (2009). Linear lower bounds for $\delta_c(p)$ for a class of 2D self-destructive percolation models. *Random Structures Algorithms* **34,** 520–526.
[55] Van den Berg, J. and Járai, A. A. (2005). On the asymptotic density in a one-dimensional self-organized critical forest-fire model. *Commun. Math. Phys.* **253,** 633–644.
[56] Van den Berg, J. and Nolin, P. (2017). Two-dimensional volume-frozen percolation: exceptional scales. *Ann. Appl. Prob.* **27,** 91–108.
[57] Van den Berg, J., Brouwer, R. and Vágvölgyi, B. (2008). Box-crossings and continuity results for self-destructive percolation in the plane. In *In and Out of Equilibrium. 2* (Progress Prob. **60**), Birkhäuser, Basel, pp. 117–135.

[58] Van den Berg, J., de Lima, B. N. B. and Nolin, P. (2012). A percolation process on the square lattice where large finite clusters are frozen. *Random Structures Algorithms* **40,** 220–226.
[59] Van den Berg, J., Kiss, D. and Nolin, P. (2012). A percolation process on the binary tree where large finite clusters are frozen. *Electron. Commun. Prob.* **17,** 11pp.
[60] Williams, D. and McIlroy, P. (1998). The OK Corral and the power of the law (a curious Poisson-kernel formula for a parabolic equation). *Bull. London Math. Soc.* **30,** 166–170.