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Pólya urn schemes with infinitely many colors

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In this work, we introduce a class of *balanced* urn schemes with infinitely many colors indexed by $\mathbb{Z}^d$, where the replacement schemes are given by the transition matrices associated with bounded increment random walks. We show that the color of the nth selected ball follows a Gaussian distribution on $\mathbb{R}^d$ after $\mathcal{O}(\log n)$ centering and $\mathcal{O}(\sqrt{\log n})$ scaling irrespective of whether the underlying walk is null recurrent or transient. We also provide finer asymptotic similar to local limit theorems for the expected configuration of the urn. The proofs are based on a novel representation of the color of the nth selected ball as "slowed down" version of the underlying random walk.

Keywords: central limit theorem; infinite color urn; local limit theorem; random walk; reinforcement processes; urn models

1. Introduction

1.1. Background and motivation

In recent years, there has been a wide variety of work on *random reinforcement models* of various kind [3,11,13,15,17–20,24,29,30,33,34,36]. In particular, there has been several work on different kinds of *urn models* and their generalizations [3,11,13,15,18,19,24,29–31]. The so called *urn schemes* was started by the seminal work of Pólya [38]. Since then various generalizations with finitely many colors have been studied in the literature [1–3,11–14,19,24–29,35]. See [36] for an extensive survey of the known results. However, other than the classical work by Blackwell and MacQueen [10], there has not been much development of infinite color generalization of the Pólya urn scheme. In this paper, we introduce and analyze a new Pólya type urn scheme with infinitely many colors indexed by $\mathbb{Z}^d$.

In this work, we will only consider *balanced* urn schemes. More precisely, if $R := ((R(i,j)))_{i,j\in\mathbb{Z}^d}$ denotes the *replacement matrix*, that is, $R(i,j) \geq 0$ is the *number of balls of color j to be placed in the urn when the color of the selected ball is i*, then for a balanced urn, all row sums of R are constant. In this case, we may assume R is a *stochastic matrix*. We will also assume that the starting configuration $U_0 := (U_{0,j})_{j\in\mathbb{Z}^d}$ is a probability distribution on the set of colors, namely, $\mathbb{Z}^d$. As we will see from the proofs of our main results, this apparent loss of generality can easily be removed if $\sum_{j\in\mathbb{Z}^d} U_{0,j} < \infty$. Since R is a stochastic matrix and U_0 a probability distribution on $\mathbb{Z}^d$, we can now consider a Markov chain on $\mathbb{Z}^d$ with transition matrix

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R and initial distribution U_0. We will call such a chain, a chain associated with the urn model and vice-versa. In this work, we are interested to study the process when R is the transition matrix of a bounded increment random walk on $\mathbb{Z}^d$. This is a novel generalization of the Pólya urn scheme, which we term as *urn model with infinitely many colors*.

Our main motivation comes from the work of Blackwell and MacQueen [10], where the authors introduced a possibly infinite color generalization of Pólya urn scheme, with only a diagonal replacement scheme. The model then described a process whose limiting distribution is the *Ferguson distribution* [9,10], also known as the *Dirichlet process prior* in the Bayesian statistics literature [23]. Our model complements this work where we consider replacement mechanisms with non-zero off-diagonal entries and observe that unlike in the Blackwell and MacQueen model [10], the asymptotic configuration is non-random and is always approximately Gaussian after appropriate centering and scaling.

It is worth mentioning here that due to the presence of the off-diagonal entries in the replacement matrix our models do not exhibit *exchangeability* of the observed sequence of colors. Hence, we use completely different techniques than [10] to study our model. For finite color case with off-diagonal entries, the typical approach is to use eigenvalue techniques and martingale methods. For infinite color case, one may try to use eigenvalue techniques by considering the replacement matrix R as a bounded linear operator on an infinite dimensional topological vector space. However, such an approach immediately leads to technical difficulties.

In this work, we formulate a suitable "*coupling*" of the observed sequence of color with the underlying Markov chain (see Proposition 7 in Section 3.1 for exact statement). This approach is entirely new for studying urn models. So far we have been only able to use this approach when R is transition matrix of a bounded increment random walk. However, we strongly believe that the method may be generalized for more general replacement matrices.

Finally, we would like to note here that our model is also a generalization of a subclass of models studied in [15], namely the class of *linearly reinforced models*. In [15] the authors prove that for such models cardinality of all the colors will grow to infinity, provided a color is observed. As we will see in Section 2, our results will not only show that the cardinality of all colors will grow to infinity, but will also provide the exact rates of their growths.

1.2. Model

Let $\{X_j\}_{j\geq 1}$ be i.i.d. random vectors taking values in $\mathbb{Z}^d$ with probability mass function $p(u) := \mathbb{P}(X_1 = u)$, $u \in \mathbb{Z}^d$. We assume that the distribution of X_1 is bounded, that is, there exists a non-empty finite subset $B \subseteq \mathbb{Z}^d$ such that $p(u) = 0$ for all $u \notin B$. It is worthwhile to note that the assumption that B is finite may be removed. Instead, if we assume X_1 has moment generating function on an open interval around 0, then all the results of this paper will hold. But for simplicity, we will assume B to be finite.

Throughout this paper, we take the convention of writing all vectors as row vectors. Thus, for a vector $x \in \mathbb{R}^d$ we will write x^T to denote it as a column vector. The notation $\langle \cdot, \cdot \rangle$ will denote the usual Euclidean inner product on $\mathbb{R}^d$ and $\| \cdot \|$ the Euclidean norm. We shall always write

$$\begin{aligned} \boldsymbol{\mu} &:= \mathbb{E}[X_1], \\ \varSigma &:= \mathbb{E}\big[X_1^T X_1\big], \\ \Upsilon(\lambda) &:= \mathbb{E}\big[\exp\big(\langle \lambda, X_1\rangle\big)\big], \qquad \lambda \in \mathbb{R}^d. \end{aligned} \tag{1}$$

We shall write $\varSigma := ((\sigma_{ij}))_{1\le i,j\le d}$ and assume that it is a positive definite matrix. Also $\varSigma^{\frac{1}{2}}$ will denote the unique *positive definite square root* of $\varSigma$, that is, $\varSigma^{\frac{1}{2}}$ is a positive definite matrix such that $\varSigma = \varSigma^{\frac{1}{2}}\varSigma^{\frac{1}{2}}$. When $d = 1$, we will denote the mean and the second moment (and not the variance) by μ and μ_2, respectively, and in that case we will assume $\mu_2 > 0$.

Now let U_0 be a probability distribution on $\mathbb{Z}^d$. Throughout this paper, we will assume there exists $r > 0$, with

$$\sum_{v\in\mathbb{Z}^d} \exp\big(\langle \lambda, v\rangle\big) U_{0,v} < \infty, \tag{2}$$

whenever $\|\lambda\| < r$. In particular, (2) is satisfied when $U_{0,v} = 0$, for all but finitely many $v \in \mathbb{Z}^d$.

Let $S_n := X_0 + X_1 + \cdots + X_n$, $n \ge 0$ be the random walk on $\mathbb{Z}^d$ starting at X_0 distributed according to U_0 and with increments $\{X_j\}_{j\ge1}$ which are independent. Needless to say that $\{S_n\}_{n\ge0}$ is a Markov chain with state space $\mathbb{Z}^d$, initial distribution given by the distribution of X_0 and the transition matrix

$$R := \big(\big(p(v-u)\big)\big)_{u,v\in\mathbb{Z}^d}. \tag{3}$$

In this work, we consider the following infinite color generalization of Pólya urn scheme where the colors are indexed by $\mathbb{Z}^d$. Denote by $U_n := (U_{n,v})_{v\in\mathbb{Z}^d} \in [0,\infty)^{\mathbb{Z}^d}$ the configuration of the urn at time n and define a random variable Z_n by

$$\mathbb{P}(Z_n = v \mid U_n, U_{n-1}, \ldots, U_0) \propto U_{n,v}, \qquad v \in \mathbb{Z}^d.$$

Note that Z_n represents the randomly chosen color at the $(n+1)$th draw. Starting with U_0 we define $(U_n)_{n\ge0}$ recursively as follows

$$U_{n+1} = U_n + R_{Z_n}, \tag{4}$$

where R_{Z_n} is the Z_nth row of the replacement matrix R. We will call the process $(U_n)_{n\ge0}$ as the *infinite color urn model* with initial configuration U_0 and replacement matrix R. We will also refer to it as the *infinite color urn model associated with the random walk* $\{S_n\}_{n\ge0}$ *on* $\mathbb{Z}^d$.

Random configuration of the urn: Observe that since R is a stochastic matrix

$$\sum_{v\in\mathbb{Z}^d} U_{n,v} = n+1, \tag{5}$$

for all $n \geq 0$. Thus the *random configuration of the urn*, namely, $\frac{U_n}{n+1}$ is a probability mass function on the set of colors. In fact,

$$\mathbb{P}(Z_n = v \mid U_n, U_{n-1}, \ldots, U_0) = \frac{U_{n,v}}{n+1}, \qquad v \in \mathbb{Z}^d. \tag{6}$$

In other words, the nth random configuration of the urn is the conditional distribution of the $(n+1)$th selected color, given $U_0, U_1, \ldots, U_n$.

Expected configuration of the urn: By taking expectation in equation (5), we get

$$\sum_{v \in \mathbb{Z}^d} \mathbb{E}[U_{n,v}] = n + 1, \tag{7}$$

for all $n \geq 0$. Thus $(\frac{\mathbb{E}[U_{n,v}]}{n+1})_{v \in \mathbb{Z}^d}$ is also a probability mass function. In fact, it is the distribution of Z_n, the $(n+1)$th selected color. This follows by taking expectation on both sides of equation (6),

$$\mathbb{P}(Z_n = v) = \frac{\mathbb{E}[U_{n,v}]}{n+1}. \tag{8}$$

For the rest of this work, we will be interested in the asymptotic properties of the random and expected configurations.

1.3. Notations

Most of the notations used in this paper are consistent with the literature on finite color urn models. However, we use few specific notations as well, which are given below.

(i) As mentioned earlier, all vectors are written as row vectors unless otherwise stated. Column vectors are denoted by x^T, where x is a row vector.
(ii) For any vector x, x^2 will denote a vector with the coordinates squared.
(iii) The standard Gaussian measure on $\mathbb{R}^d$ will be denoted by Φ_d with its density given by

$$\phi_d(x) := \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{\|x\|^2}{2}\right), \qquad x \in \mathbb{R}^d.$$

For $d = 1$, we will simply write Φ for the standard Gaussian measure on $\mathbb{R}$ and ϕ for its density.
(iv) The symbol $\Rightarrow$ will denote convergence in distribution of random variables.
(v) The symbol $\xrightarrow{p}$ will denote convergence in probability.

1.4. Outline

In the following section, we state the main results, which we prove in Section 4. In Section 3, we state and prove two important results, which we use in the proofs of the main results. In Section 5, we further generalize our results for urns with infinitely many colors, where the color sets are indexed by other countable lattices on $\mathbb{R}^d$. An elementary technical result which is needed in the proofs of the main results is deferred to the Appendix.

2. Main results

Throughout this paper, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which all the random processes are defined.

2.1. Weak convergence of the expected configuration

We present in this subsection the central limit theorem for the randomly selected color. The centering and scaling of the central limit theorem are of the order $\mathcal{O}(\log n)$ and $\mathcal{O}(\sqrt{\log n})$, respectively. Such centering and scalings are available because the distribution of the randomly selected color behaves like that of a delayed random walk, where the delay is of the order $\mathcal{O}(\log n)$, see Proposition 7.

Theorem 1. *Consider an infinite color urn model with initial configuration U_0 and replacement matrix given by* (3). *Let Z_n be the $(n+1)$th selected color. Then*

$$\frac{Z_n - \boldsymbol{\mu} \log n}{\sqrt{\log n}} \Rightarrow N_d(0, \Sigma), \qquad \text{as } n \to \infty. \tag{9}$$

Recall that the probability mass function of Z_n is given by $(\frac{\mathbb{E}[U_{n,v}]}{n+1})_{v \in \mathbb{Z}^d}$. Thus the above result essentially gives an asymptotic weak limit of the *expected value of the configuration of the urn*, when viewed as a probability distribution on $\mathbb{R}^d$, after centering by $\boldsymbol{\mu} \log n$ and scaling by $\sqrt{\log n}$.

The following result is an immediate application of Theorem 1.

Corollary 2. *Consider the urn model associated with the simple symmetric random walk on $\mathbb{Z}^d$, $d \geq 1$. Then, as $n \to \infty$,*

$$\frac{Z_n}{\sqrt{\log n}} \Rightarrow N_d\big(0, d^{-1}\mathbb{I}_d\big),$$

where $\mathbb{I}_d$ is the $d \times d$ identity matrix.

The above result essentially shows that irrespective of the recurrent or transient behavior of the under lying random walk, the associated urn models have similar asymptotic behavior. In particular, the limiting distribution is always Gaussian with universal orders for centering and scaling, namely, $\mathcal{O}(\log n)$ and $\mathcal{O}(\sqrt{\log n})$, respectively.

2.2. Weak convergence of the random configuration

In this subsection, we will present an asymptotic result for the random configuration of the urn. Let $\mathcal{M}_1$ be the space of probability measures on $\mathbb{R}^d$, $d \geq 1$, endowed with the topology of weak convergence. Let Λ_n be the random probability measure on $\mathbb{Z}^d \subset \mathbb{R}^d$ corresponding to the random probability vector $\frac{U_n}{n+1}$. It is easy to see that the function Λ_n is measurable.

Theorem 3. *For any Borel subset $A \subseteq \mathbb{R}^d$, let*

$$\Lambda_n^{\mathrm{cs}}(A) = \Lambda_n\big(\sqrt{\log n} A \Sigma^{1/2} + \boldsymbol{\mu} \log n\big),$$

where we define

$$x A \Sigma^{1/2} := \big\{x y \Sigma^{1/2}: y \in A\big\}.$$

Then, as $n \to \infty$,

$$\Lambda_n^{\mathrm{cs}} \xrightarrow{p} \Phi_d \qquad \text{in } \mathcal{M}_1. \tag{10}$$

We note that Theorem 3 is a stronger version of Theorem 1.

2.3. Local limit theorem type results for the expected configuration

It turns out that under certain assumptions the expected configuration of the urn at time n, namely, $(\frac{\mathbb{E}[U_n]}{n+1})_{n\geq 0}$ satisfies a *local limit theorem*.

Note that X_1 is a lattice random vector taking values in $\mathbb{Z}^d$. Also as we assume that the $\Sigma = \mathbf{E}[X_1^T X_1]$ is positive definite, so X_1 is d-dimensional, that is, there is no sub-lattice $\mathcal{A} \subseteq \mathbb{Z}^d$ of dimension less or equal to $(d-1)$ such that $\mathbf{P}(X_1 \in \mathcal{A}) = 1$. Let $\mathcal{L}$ be its *minimal lattice*, that is,

$$\mathbb{P}(X_1 \in x + \mathcal{L}) = 1 \tag{11}$$

for every $x \in \mathbb{Z}^d$, such that, $\mathbb{P}(X_1 = x) > 0$, and if $\mathcal{L}'$ is any closed subgroup of $\mathbb{R}^d$, with $\mathbb{P}(X_1 \in y + \mathcal{L}') = 1$ for some $y \in \mathbb{Z}^d$, then $\mathcal{L} \subseteq \mathcal{L}'$ and the rank of $\mathcal{L}$ is d. We refer to pages 226–227 of [4] for formal definitions of minimal lattice of a d-dimensional lattice random variable and its rank. Let $\ell = \det(\mathcal{L})$ (see pages 228–229 of [4] for more details). Now let x_0 be such that $\mathbb{P}(X_1 \in x_0 + \mathcal{L}) = 1$ and we define

$$\mathcal{L}_n^{(d)} := \left\{x: x = \frac{n}{\sqrt{\log n}} x_0 \Sigma^{-1/2} - \sqrt{\log n}\, \boldsymbol{\mu} \Sigma^{-1/2} + \frac{1}{\sqrt{\log n}} z \Sigma^{-1/2}, z \in \mathcal{L}\right\}. \tag{12}$$

Theorem 4. *Assume that $\mathbb{P}(X_1 = 0) > 0$. Then, as $n \to \infty$*

$$\sup_{x \in \mathcal{L}_n^{(d)}} \left| \frac{\det(\Sigma^{1/2})(\sqrt{\log n})^d}{\ell} \mathbb{P}\left(\frac{Z_n - \boldsymbol{\mu} \log n}{\sqrt{\log n}} \Sigma^{-1/2} = x\right) - \phi_d(x)\right| \longrightarrow 0. \tag{13}$$

For $d=1$, it is well known that (11) can be explicitly written as

$$\mathbb{P}(X_1 \in a + h\mathbb{Z}) = 1, \tag{14}$$

where $a \in \mathbb{R}$, and $h > 0$, is the maximum value such that (14) holds. h is called the span for X_1 (see Section 3.5 of [22]). Then from (12) we obtain

$$\mathcal{L}_n^{(1)} := \left\{ x : x = \frac{n}{\sqrt{\mu_2 \log n}} a - \frac{\mu}{\sqrt{\mu_2}} \sqrt{\log n} + \frac{h}{\sqrt{\mu_2 \log n}} z, z \in \mathbb{Z} \right\}. \tag{15}$$

Therefore, in one dimension, (13) can be stated as

$$\sup_{x \in \mathcal{L}_n^{(1)}} \left| \frac{\sqrt{\mu_2 \log n}}{h} \mathbb{P}\left(\frac{Z_n - \mu \log n}{\sqrt{\mu_2 \log n}} = x \right) - \phi(x) \right| \longrightarrow 0. \tag{16}$$

Observe that Theorem 4 covers only the case when $\mathbb{P}(X_1 = 0) > 0$. The next theorem is for the special case when the urn is associated with the simple symmetric random walk on $\mathbb{Z}^d$, $d \geq 1$, which is not covered by Theorem 4.

Theorem 5. *Assume that $\mathbb{P}(X_1 = \pm e_i) = \frac{1}{2d}$ for $1 \leq i \leq d$, where e_i is the ith unit vector in direction i. Then, as $n \to \infty$*

$$\sup_{x \in \mathcal{L}_n^{(d)}} \left| (d)^{\frac{d}{2}} (\sqrt{\log n})^d \mathbb{P}\left(\frac{\sqrt{d}}{\sqrt{\log n}} Z_n = x \right) - \phi_d(x) \right| \longrightarrow 0, \tag{17}$$

where $\mathcal{L}_n^{(d)}$ is as defined in (12) with $\mu = 0 = x_0$, $\Sigma = \mathbb{I}_d$ and $\mathcal{L} = \sqrt{d}\mathbb{Z}^d$.

The following result is immediate from the above theorem.

Corollary 6. *Assume that $\mathbb{P}(X_1 = \pm e_i) = \frac{1}{2d}$ for $1 \leq i \leq d$, where e_i is the ith unit vector in direction i. Then, as $n \to \infty$*

$$\mathbb{P}(Z_n = 0) \sim \frac{1}{(\sqrt{2\pi d \log n})^d}. \tag{18}$$

As we see in Theorem 5, the assumption $\mathbb{P}(X_1 = 0) > 0$ of Theorem 4 can be removed. For dimension $d = 1$, we will present later a somewhat technical result, namely, Theorem 10 in Section 4.3, which will give another such example. Unfortunately, we do not know the full generality under which the local limit theorem holds, though we conjecture that it holds for all cases.

3. Auxiliary results

In this section, we present two results which we need to prove our main results. These results are two very important tools for studying infinite color urn models associated with random walks on $\mathbb{Z}^d$ and hence presented separately.

For $z \in \mathbb{C}$, define $\Pi_n(z) = \prod_{j=1}^{n}(1 + \frac{z}{j})$, for $n \geq 1$, and set $\Pi_0(z) = 1$. It is known from Euler product formula for gamma function, which is also referred to as Gauss's formula (see page 178 of [16]), that

$$\lim_{n\to\infty} \frac{\Pi_n(z)}{n^z}\Gamma(z+1) = 1 \tag{19}$$

uniformly on compact subsets of $\mathbb{C} \setminus \{-1, -2, -3, \ldots\}$.

Recall $\Upsilon(\lambda) := \sum_{v\in B} \exp(\langle\lambda, v\rangle)p(v)$ is the moment generating function of X_1. It is easy to note that $\Upsilon(\lambda)$ is an eigenvalue of R corresponding to the right eigenvector $x(\lambda) = (\exp(\langle\lambda, v\rangle))^T_{v\in\mathbb{Z}^d}$. Let $\mathcal{F}_n = \sigma(U_j : 0 \leq j \leq n)$, $n \geq 0$ be the natural filtration. Define

$$\overline{M}_n(\lambda) = \frac{U_n x(\lambda)}{\Pi_n(\Upsilon(\lambda))}. \tag{20}$$

From the fundamental recursion (4) we get,

$$U_{n+1} = U_n + \chi_{n+1}R, \tag{21}$$

where $\chi_{n+1} = (\chi_{n+1,v})_{v\in\mathbb{Z}^d}$ is such that $\chi_{n+1,Z_n} = 1$ and $\chi_{n+1,u} = 0$ if $u \neq Z_n$, where Z_n is the random color chosen from the configuration U_n at the $(n+1)$th draw. Thus,

$$\begin{aligned} U_{n+1}x(\lambda) &= U_n x(\lambda) + \chi_{n+1}Rx(\lambda) \\ &= U_n x(\lambda) + \Upsilon(\lambda)\chi_{n+1}x(\lambda). \end{aligned} \tag{22}$$

Thus,

$$\mathbb{E}\big[U_{n+1}x(\lambda) \mid \mathcal{F}_n\big] = U_n x(\lambda) + \Upsilon(\lambda)\mathbb{E}\big[\chi_{n+1}x(\lambda) \mid \mathcal{F}_n\big] = \left(1 + \frac{\Upsilon(\lambda)}{n+1}\right)U_n x(\lambda).$$

Therefore, $\overline{M}_n(\lambda)$ is a non-negative martingale for every $\lambda \in \mathbb{R}^d$. In particular $\mathbb{E}[\overline{M}_n(\lambda)] = \overline{M}_0(\lambda)$.

3.1. A representation of Z_n

We now give a representation of the distribution of Z_n in terms of the increments $\{X_j\}_{j\geq 1}$. The following proposition shows that the distribution of Z_n is the same as a delayed random walk.

Proposition 7. *For each $n \geq 1$,*

$$Z_n \overset{d}{=} Z_0 + \sum_{j=1}^{n} I_j X_j, \tag{23}$$

where $\{I_j\}_{j\geq 1}$ are independent Bernoulli random variables such that $\mathbb{E}[I_j] = \frac{1}{j+1}$, $j \geq 1$ and are independent of $\{X_j\}_{j\geq 1}$; and Z_0 is a random vector taking values in $\mathbb{Z}^d$ distributed according to the probability vector U_0 and is independent of $(\{I_j\}_{j\geq 1}; \{X_j\}_{j\geq 1})$.

A formal proof of this result is given below, using the martingale sequence $(\overline{M}_n(\lambda))_{n\geq 0}$, as defined in (20). However, one can also establish the representation given in (23) intuitively. Let $\mathcal{T}_{n+1}$ be the *random recursive tree* on $n+1$ vertices, say, $\{v_0, v_1, v_2, \dots, v_n\}$ [21,39]. We put i.i.d. weights on each of the n edges of $\mathcal{T}_{n+1}$ given by $\{X_j\}_{j\geq 1}$, as defined in Proposition 7. Let Z_0 with distribution given by U_0 be as defined in Proposition 7. Then from the equations (21) and (5), it follows that the distribution of Z_n is given by

$$Z_0 + \textit{weighted length of the path} \qquad \text{from } v_0 \text{ to } v_n \text{ in } \mathcal{T}_{n+1}.$$

This shows (23) must hold.

It may be noted here that the above representation is similar to a discrete time *branching random walk* (*BRW*), with i.i.d. increments given by $\{X_j\}_{j\geq 1}$, but on the random recursive tree $(\mathcal{T}_n)_{n\geq 1}$. Unlike the usual BRW [5–7], in this case the underlying tree is not a *Galton–Watson branching process tree*. Also as described above, in this case we are interested in studying the position of a randomly chosen particle, instead of the maximum displacement.

Proof. As noted before, the probability mass function for the color of the $(n+1)$th selected ball, namely Z_n, is $(\frac{\mathbb{E}[U_{n,v}]}{n+1})_{v\in\mathbb{Z}^d}$. So for $\lambda \in \mathbb{R}^d$ with $\|\lambda\| < r$, where r is as in (2), the moment generating function of Z_n is given by

$$\begin{aligned}\frac{1}{n+1}\sum_{v\in\mathbb{Z}^d}\exp\bigl(\langle\lambda, v\rangle\bigr)\mathbb{E}[U_{n,v}] &= \frac{\Pi_n(\Upsilon(\lambda))}{n+1}\mathbb{E}\bigl[\overline{M}_n(\lambda)\bigr] \\ &= \frac{\Pi_n(\Upsilon(\lambda))}{n+1}\overline{M}_0(\lambda) \\ &= \overline{M}_0(\lambda)\prod_{j=1}^{n}\left(1 - \frac{1}{j+1} + \frac{\Upsilon(\lambda)}{j+1}\right).\end{aligned} \tag{24}$$

The equation (23) follows from (24). $\square$

Observe that in the proof of Proposition 7 we require the assumption (2), which guarantees that $\overline{M}_0(\lambda)$ is finite.

3.2. Uniform $\mathcal{L}_2$ boundedness

Our next theorem states that around a non-trivial closed neighborhood of 0 the martingales $(\overline{M}_n(\lambda))_{n\geq 0}$ are uniformly (in λ) $\mathcal{L}_2$-bounded.

Proposition 8. *There exists $\delta > 0$ such that*

$$\sup_{\lambda\in[-\delta,\delta]^d}\sup_{n\geq 1}\mathbb{E}\bigl[\overline{M}_n^2(\lambda)\bigr] < \infty. \tag{25}$$

Proof. From (22), we obtain

$$\mathbb{E}\big[\big(U_{n+1}x(\lambda)\big)^2 \mid \mathcal{F}_n\big] = \big(U_n x(\lambda)\big)^2 + 2\Upsilon(\lambda) U_n x(\lambda) \mathbb{E}\big[\chi_{n+1}x(\lambda) \mid \mathcal{F}_n\big] + \Upsilon^2(\lambda)\mathbb{E}\big[\big(\chi_{n+1}x(\lambda)\big)^2 \mid \mathcal{F}_n\big].$$

It is easy to see that

$$\mathbb{E}\big[\chi_{n+1}x(\lambda) \mid \mathcal{F}_n\big] = \frac{1}{n+1} U_n x(\lambda) \quad \text{and} \quad \mathbb{E}\big[\big(\chi_{n+1}x(\lambda)\big)^2 \mid \mathcal{F}_n\big] = \frac{1}{n+1} U_n x(2\lambda). \tag{26}$$

Therefore, we get the recursion

$$\mathbb{E}\big[\big(U_{n+1}x(\lambda)\big)^2\big] = \left(1 + \frac{2\Upsilon(\lambda)}{n+1}\right)\mathbb{E}\big[\big(U_n x(\lambda)\big)^2\big] + \frac{\Upsilon^2(\lambda)}{n+1}\mathbb{E}\big[U_n x(2\lambda)\big]. \tag{27}$$

Dividing both sides of (27) by $\Pi_{n+1}^2(\Upsilon(\lambda))$,

$$\mathbb{E}\big[\overline{M}_{n+1}^2(\lambda)\big] = \frac{(1 + \frac{2\Upsilon(\lambda)}{n+1})}{(1 + \frac{\Upsilon(\lambda)}{n+1})^2}\mathbb{E}\big[\overline{M}_n^2(\lambda)\big] + \frac{\Upsilon^2(\lambda)}{n+1}\,\frac{\mathbb{E}[U_n x(2\lambda)]}{\Pi_{n+1}^2(\Upsilon(\lambda))}. \tag{28}$$

Recall that $\overline{M}_n(2\lambda)$ is a martingale, thus $\mathbb{E}[U_n x(2\lambda)] = \Pi_n(\Upsilon(2\lambda))\overline{M}_0(2\lambda)$. Therefore from (28), we get

$$\mathbb{E}\big[\overline{M}_n^2(\lambda)\big] = \frac{\Pi_n(2\Upsilon(\lambda))}{\Pi_n(\Upsilon(\lambda))^2}\overline{M}_0^2(\lambda) + \sum_{k=1}^{n} \frac{\Upsilon^2(\lambda)}{k}\left\{\prod_{j>k}^{n} \frac{(1 + \frac{2\Upsilon(\lambda)}{j})}{(1 + \frac{\Upsilon(\lambda)}{j})^2}\right\}\frac{\Pi_{k-1}(\Upsilon(2\lambda))}{\Pi_k^2(\Upsilon(\lambda))}\overline{M}_0(2\lambda). \tag{29}$$

We observe that as $\Upsilon(\lambda) > 0$, so $\frac{1+\frac{2\Upsilon(\lambda)}{j}}{(1+\frac{\Upsilon(\lambda)}{j})^2} < 1$ and hence $\frac{\Pi_n(2\Upsilon(\lambda))}{\Pi_n^2(\Upsilon(\lambda))} < 1$. Thus

$$\mathbb{E}\big[\overline{M}_n^2(\lambda)\big] \le \overline{M}_0^2(\lambda) + \Upsilon^2(\lambda)\overline{M}_0(2\lambda)\sum_{k=1}^{n} \frac{1}{k}\,\frac{\Pi_{k-1}(\Upsilon(2\lambda))}{\Pi_k^2(\Upsilon(\lambda))}. \tag{30}$$

Using (19), we know that

$$\Pi_n^2\big(\Upsilon(\lambda)\big) \sim \frac{n^{2\Upsilon(\lambda)}}{\Gamma^2(\Upsilon(\lambda)+1)}. \tag{31}$$

Note that $\Upsilon(0) = 1$ and $\Upsilon(\lambda)$ is continuous as a function of λ. So given $\eta > 0$, there exists $0 < K_1, K_2 < \infty$, such that for all $\lambda \in [-\eta, \eta]^d$, $K_1 \le \Upsilon(\lambda) \le K_2$. Since the convergence in (19) is uniform on compact subsets of $[0, \infty)$, given $\varepsilon > 0$ there exists $N_1 > 0$ such that for all $n \geq N_1$ and $\lambda \in [-\eta, \eta]^d$,

$$(1-\varepsilon)\frac{\Gamma^2(\Upsilon(\lambda)+1)}{\Gamma(\Upsilon(2\lambda)+1)}\sum_{k\geq N_1}^{n}\frac{1}{k^{1+2\Upsilon(\lambda)-\Upsilon(2\lambda)}}$$
$$\leq \sum_{k\geq N_1}^{n}\frac{1}{k}\frac{\Pi_{k-1}(\Upsilon(2\lambda))}{\Pi_k^2(\Upsilon(\lambda))}$$
$$\leq (1+\varepsilon)\frac{\Gamma^2(\Upsilon(\lambda)+1)}{\Gamma(\Upsilon(2\lambda)+1)}\sum_{k\geq N_1}^{n}\frac{1}{k^{1+2\Upsilon(\lambda)-\Upsilon(2\lambda)}}.$$

Recall that $\Upsilon(\lambda) = \sum_{v\in B}\exp(\langle\lambda, v\rangle)p(v)$. Since the cardinality of B is finite, we can choose a $\delta_0 > 0$ such that for every $\lambda \in [-\delta_0, \delta_0]^d$, $2\Upsilon(\lambda) - \Upsilon(2\lambda) > 0$. Choose $\delta = \min\{\eta, \delta_0\}$. Since $2\Upsilon(\lambda) - \Upsilon(2\lambda)$ is continuous as a function of λ, there exists a $\lambda_0 \in [-\delta, \delta]^d$ such that $\min_{\lambda\in[-\delta,\delta]^d}(2\Upsilon(\lambda) - \Upsilon(2\lambda)) = 2\Upsilon(\lambda_0) - \Upsilon(2\lambda_0) > 0$. Therefore,

$$\sum_{k=1}^{\infty}\frac{1}{k^{1+2\Upsilon(\lambda)-\Upsilon(2\lambda)}} \leq \sum_{k=1}^{\infty}\frac{1}{k^{1+2\Upsilon(\lambda_0)-\Upsilon(2\lambda_0)}}.$$

Therefore given $\varepsilon > 0$ there exists $N_2 > 0$ such that $\forall\lambda \in [-\delta, \delta]^d$

$$\sum_{k>N_2}^{\infty}\frac{1}{k^{1+2\Upsilon(\lambda)-\Upsilon(2\lambda)}} \leq \sum_{k>N_2}^{\infty}\frac{1}{k^{1+2\Upsilon(\lambda_0)-\Upsilon(2\lambda_0)}} < \varepsilon.$$

$\frac{\Gamma^2(\Upsilon(\lambda)+1)}{\Gamma(\Upsilon(2\lambda)+1)}$, $\Upsilon^2(\lambda)$ and $\overline{M}_0(2\lambda)$ being continuous as functions of λ, they are bounded for $\lambda \in [-\delta, \delta]^d$. Choose $N = \max\{N_1, N_2\}$. From (30) we obtain for all $n \geq N$,

$$\mathbb{E}\big[\overline{M}_n^2(\lambda)\big] \leq \overline{M}_0^2(\lambda) + C_1\sum_{k=1}^{N}\frac{1}{k}\frac{\Pi_{k-1}(\Upsilon(2\lambda))}{\Pi_k^2(\Upsilon(\lambda))} + C_2\varepsilon \tag{32}$$

for an appropriate positive constants C_1, C_2.

$\sum_{k=1}^{N}\frac{1}{k}\frac{\Pi_{k-1}(\Upsilon(2\lambda))}{\Pi_k^2(\Upsilon(\lambda))}$ and $\overline{M}_0^2(\lambda)$ being continuous as functions of λ, they are bounded for $\lambda \in [-\delta, \delta]^d$. Therefore, from (32) we obtain that there exists $C > 0$ such that for all $\lambda \in [-\delta, \delta]^d$ and for all $n \geq 1$

$$\mathbb{E}\big[\overline{M}_n^2(\lambda)\big] \leq C.$$

This proves (25). $\square$

4. Proofs of the main results

For proving the central and local limit theorems, we will use the representation (23), which after the appropriate centering and scaling can be re-written as

$$\frac{Z_n - \boldsymbol{\mu} \log n}{\sqrt{\log n}} \stackrel{d}{=} \frac{Z_0}{\sqrt{\log n}} + \frac{\sum_{j=1}^n I_j X_j - \boldsymbol{\mu} \log n}{\sqrt{\log n}}. \tag{33}$$

So, without loss of generality, we may assume that $Z_0 \equiv 0$, in other words, the initial configuration of the urn consists of only one ball of color 0.

4.1. Proofs for the expected configuration

Proof of Theorem 1. Observe that

$$\mathbb{E}\left[\sum_{j=1}^n I_j X_j\right] - \boldsymbol{\mu} \log n = \sum_{j=1}^n \frac{1}{j+1} \boldsymbol{\mu} - \boldsymbol{\mu} \log n \longrightarrow \boldsymbol{\mu}(\gamma - 1), \tag{34}$$

where γ is the Euler's constant.

Let $\Sigma_n = \operatorname{Var}(\sum_{j=1}^n I_j X_j)$. It is easy to see that

$$\Sigma_n = \sum_{j=1}^n \left(\frac{1}{j+1} \mathbb{E}\left[X_1^T X_1\right] - \frac{\boldsymbol{\mu}^T \boldsymbol{\mu}}{(j+1)^2} \right).$$

Thus as $n \to \infty$

$$\frac{1}{\log n} \Sigma_n \longrightarrow \Sigma, \tag{35}$$

where the matrix convergence is entry-wise.

Recall $\mathbb{P}(X_1 \in B) = 1$, where $B \subset \mathbb{Z}$ is a finite set, thus for any $\varepsilon > 0$, we have

$$\frac{1}{\log n} \sum_{j=1}^n \mathbb{E}\left[\| I_j X_j \|^2 1_{\{\| I_j X_j \| > \varepsilon \log n\}}\right] \longrightarrow 0, \tag{36}$$

as $n \to \infty$. Therefore, by the *Lindeberg–Feller Central Limit Theorem* (see Proposition 2.27 on page 20 of [40]) we conclude that as $n \to \infty$

$$\frac{Z_n - \boldsymbol{\mu} \log n}{\sqrt{\log n}} \Rightarrow N_d(0, \Sigma).$$

This completes the proof. □

4.2. Proofs for random configuration

In this subsection, we will present the proof of Theorem 3. We start with the following lemma.

Lemma 9. *Let δ be as in Proposition 8, then for every $\lambda \in [-\delta, \delta]^d$ as $n \to \infty$,*

$$\overline{M}_n\left(\frac{\lambda}{\sqrt{\log n}}\right) \xrightarrow{p} 1. \tag{37}$$

Proof. Observe that $\overline{M}_0(\lambda) \equiv 1$, since $Z_0 \equiv 0$. Therefore, from equation (29) we get

$$\mathbb{E}\big[\overline{M}_n^2(\lambda)\big] = \frac{\Pi_n(2\Upsilon(\lambda))}{\Pi_n^2(\Upsilon(\lambda))} + \frac{\Pi_n(2\Upsilon(\lambda))}{\Pi_n^2(\Upsilon(\lambda))} \sum_{k=1}^{n} \frac{\Upsilon^2(\lambda)}{k} \frac{\Pi_{k-1}(\Upsilon(2\lambda))}{\Pi_k(2\Upsilon(\lambda))}.$$

Replacing λ by $\lambda_n = \frac{\lambda}{\sqrt{\log n}}$, we obtain

$$\mathbb{E}\big[\overline{M}_n^2(\lambda_n)\big] = \frac{\Pi_n(2\Upsilon(\lambda_n))}{\Pi_n^2(\Upsilon(\lambda_n))} + \frac{\Pi_n(2\Upsilon(\lambda_n))}{\Pi_n^2(\Upsilon(\lambda_n))} \sum_{k=1}^{n} \frac{\Upsilon^2(\lambda_n)}{k} \frac{\Pi_{k-1}(\Upsilon(2\lambda_n))}{\Pi_k(2\Upsilon(\lambda_n))}. \tag{38}$$

We observe that

$$\lim_{n\to\infty} \Upsilon(\lambda_n) = 1. \tag{39}$$

Since the convergence in formula (19) is uniform on compact sets of $[0, \infty)$, we observe that for $\lambda \in [-\delta, \delta]^d$

$$\lim_{n\to\infty} \frac{\Pi_n(2\Upsilon(\lambda_n))}{\Pi_n^2(\Upsilon(\lambda_n))} = \frac{\Gamma^2(2)}{\Gamma(3)} = \frac{1}{2}.$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\Pi_n(2\Upsilon(\lambda_n))}{\Pi_n^2(\Upsilon(\lambda_n))} \frac{\Upsilon^2(\lambda_n)}{k} \frac{\Pi_{k-1}(\Upsilon(2\lambda_n))}{\Pi_k(2\Upsilon(\lambda_n))} &= \frac{1}{2}\frac{1}{k}\frac{\Pi_{k-1}(1)}{\Pi_k(2)} \\
&= \frac{1}{(k+2)(k+1)}.
\end{aligned}$$

Now using Proposition 8 and the dominated convergence theorem, we get

$$\lim_{n\to\infty} \frac{\Pi_n(2\Upsilon(\lambda_n))}{\Pi_n^2(\Upsilon(\lambda_n))} \sum_{k=1}^{n} \frac{\Upsilon^2(\lambda_n)}{k} \frac{\Pi_{k-1}(\Upsilon(2\lambda_n))}{\Pi_k(2\Upsilon(\lambda_n))} = \sum_{k=1}^{\infty} \frac{1}{(k+2)(k+1)} = \frac{1}{2}.$$

Therefore, from (38) we obtain

$$\mathbb{E}\big[\overline{M}_n^2(\lambda_n)\big] \longrightarrow 1 \qquad \text{as } n \to \infty. \tag{40}$$

Observing that $\mathbb{E}[\overline{M}_n(\lambda_n)] = 1$, we get

$$\operatorname{Var}\big(\overline{M}_n(\lambda_n)\big) \to 0, \tag{41}$$

as $n \to \infty$. This implies

$$\overline{M}_n(\lambda_n) \xrightarrow{p} 1 \qquad \text{as } n \to \infty,$$

completing the proof of the lemma. □

Proof of Theorem 3. Note that Λ_n is the random probability measure on $\mathbb{Z}^d \subset \mathbb{R}^d$, corresponding to the random probability vector $\frac{U_n}{n+1}$. That is, for any Borel subset A of $\mathbb{Z}^d$,

$$\Lambda_n(A) = \frac{1}{n+1} \sum_{v \in A} U_{n,v}.$$

For $\lambda \in \mathbb{R}^d$ the corresponding moment generating function is given by

$$\frac{1}{n+1} \sum_{v \in \mathbb{Z}^d} \exp\big(\langle \lambda, v\rangle\big) U_{n,v} = \frac{1}{n+1} U_n x(\lambda) = \frac{1}{n+1} \overline{M}_n(\lambda) \Pi_n\big(\Upsilon(\lambda)\big). \tag{42}$$

The moment generating function corresponding to the scaled and centered random measure Λ_n^{cs} is

$$\begin{aligned}
&\sum_{v \in \mathbb{Z}^d} \exp\left(\left\langle \lambda, \frac{v - \boldsymbol{\mu} \log n}{\sqrt{\log n}} \Sigma^{-1/2} \right\rangle\right) \frac{U_{n,v}}{n+1} \\
&\quad = \frac{1}{n+1} \exp\big(-\langle \lambda, \boldsymbol{\mu} \sqrt{\log n}\, \Sigma^{-1/2}\rangle\big) U_n x\left(\frac{\lambda \Sigma^{-1/2}}{\sqrt{\log n}}\right) \\
&\quad = \frac{1}{n+1} \exp\big(-\langle \lambda, \boldsymbol{\mu} \sqrt{\log n}\, \Sigma^{-1/2}\rangle\big) \overline{M}_n\left(\frac{\lambda \Sigma^{-1/2}}{\sqrt{\log n}}\right) \Pi_n\left(\Upsilon\left(\frac{\lambda \Sigma^{-1/2}}{\sqrt{\log n}}\right)\right).
\end{aligned}$$

To show (10) it is enough to show that for every subsequence $\{n_k\}_{k \geq 1}$, there exists a further subsequence $\{n_{k_j}\}_{j=1}^{\infty}$ such that as $j \to \infty$

$$\frac{\exp(-\langle \lambda, \boldsymbol{\mu} \sqrt{\log n_{k_j}}\rangle)}{n_{k_j} + 1} \overline{M}_{n_{k_j}}\left(\frac{\lambda}{\sqrt{\log n_{k_j}}}\right) \Pi_n\left(\Upsilon\left(\frac{\lambda}{\sqrt{\log n_{k_j}}}\right)\right) \longrightarrow \exp\left(\frac{\lambda \Sigma \lambda^T}{2}\right) \tag{43}$$

for all $\lambda \in [-\delta, \delta]^d$ almost surely, where δ is as in Proposition 8. From Theorem 1, we know that

$$\frac{Z_n - \boldsymbol{\mu} \log n}{\sqrt{\log n}} \Rightarrow N_d(0, \mathbb{I}_d).$$

Therefore using (24) as $n \to \infty$ we obtain,

$$\exp\bigl(-\langle\lambda, \boldsymbol{\mu}\sqrt{\log n}\rangle\bigr)\mathbb{E}\biggl[\exp\biggl(\biggl\langle\lambda, \frac{Z_n}{\sqrt{\log n}}\biggr\rangle\biggr)\biggr]$$
$$= \frac{1}{n+1}\exp\bigl(-\langle\lambda, \boldsymbol{\mu}\sqrt{\log n}\rangle\bigr)\Pi_n\biggl(\Upsilon\biggl(\frac{\lambda}{\sqrt{\log n}}\biggr)\biggr) \longrightarrow \exp\biggl(\frac{\lambda\Sigma\lambda^T}{2}\biggr).$$

Now using Theorem 14 from the Appendix it is enough to show (43) only for $\lambda \in \mathbb{Q}^d \cap [-\delta, \delta]^d$ which is equivalent to proving that for every $\lambda \in \mathbb{Q}^d \cap [-\delta, \delta]^d$ as $j \to \infty$

$$\overline{M}_{n_{k_j}}\biggl(\frac{\lambda}{\sqrt{\log n_{k_j}}}\biggr) \longrightarrow 1 \qquad \text{almost surely.}$$

From Lemma 9, we know that for all $\lambda \in [-\delta, \delta]^d$

$$\overline{M}_n\biggl(\frac{\lambda}{\sqrt{\log n}}\biggr) \xrightarrow{p} 1 \qquad \text{as } n \to \infty.$$

Therefore using the standard diagonalization argument, we can say that given a subsequence $\{n_k\}_{k\geq 1}$ there exists a further subsequence $\{n_{k_j}\}_{j=1}^{\infty}$ such that for every $\lambda \in \mathbb{Q}^d \cap [-\delta, \delta]^d$

$$\overline{M}_{n_{k_j}}\biggl(\frac{\lambda}{\sqrt{\log n_{k_j}}}\biggr) \longrightarrow 1 \qquad \text{almost surely.}$$

This completes the proof. □

4.3. Proofs of the local limit type results

In this section, we present the proofs for the local limit theorems.

Proof of Theorem 4. Since without loss we have assumed $U_0 = \delta_0$, so from Proposition 7, we obtain $Z_n \stackrel{d}{=} \sum_{j=1}^{n} I_j X_j$. The random vector X_j is a lattice random vector. Therefore, $I_j X_j$ is also a lattice random vector. By our assumption $\mathbb{P}(X_1 = 0) > 0$, so $0 \in B$, therefore, X_j and $I_j X_j$ are supported on the same lattice.

Observe that Z_n is a lattice random vector, for every $n \in \mathbb{N}$. By Fourier inversion formula (see (21.28) on page 230 of [4]), we get for $x \in \mathcal{L}_n^{(d)}$,

$$\mathbb{P}\biggl(\frac{Z_n - \boldsymbol{\mu}\log n}{\sqrt{\log n}}\Sigma^{-1/2} = x\biggr)$$
$$= \frac{\ell}{(2\pi\sqrt{\log n})^d \det(\Sigma^{1/2})}\int_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})} \psi_n(t)\exp\bigl(-i\langle t, x\rangle\bigr)\,\mathrm{d}t,$$

where $\psi_n(t) = \mathbb{E}[\exp(i\langle t, \frac{Z_n - \boldsymbol{\mu}\log n}{\sqrt{\log n}}\Sigma^{-1/2}\rangle)]$, $\ell = |\det(\mathcal{L})|$ and $\mathcal{F}^*$ is the fundamental domain for X_1 as defined in equation (21.22) on page 229 of [4]. Also, by Fourier inversion formula we have

$$\phi_d(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \exp\bigl(-i\langle t, x\rangle\bigr)\exp\Bigl(-\frac{\|t\|^2}{2}\Bigr)\,\mathrm{d}t.$$

Thus, given $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$, we get the estimate

$$\begin{aligned}
&\left|\frac{\det(\Sigma^{1/2})(\sqrt{\log n})^d}{\ell}\mathbb{P}\Bigl(\frac{Z_n - \boldsymbol{\mu}\log n}{\sqrt{\log n}}\Sigma^{-1/2} = x\Bigr) - \phi_d(x)\right| \\
&\quad \leq \frac{1}{(2\pi)^d}\int_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})}\left|\psi_n(t) - \exp\Bigl(-\frac{\|t\|^2}{2}\Bigr)\right|\mathrm{d}t + \varepsilon.
\end{aligned}$$

Therefore, it is enough to prove that as $n \to \infty$

$$\int_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})}\left|\psi_n(t) - \exp\Bigl(-\frac{\|t\|^2}{2}\Bigr)\right|\mathrm{d}t \longrightarrow 0. \tag{44}$$

Now we follow an argument similar to the proof of a local limit theorem for sum of i.i.d. lattice random variables in one dimension, as given on page 190 of [37]. We observe that to prove (44), it is then enough to show that for any $\varepsilon > 0$, we can choose a compact subset $A \subseteq \mathbb{R}^d$, such that for all n large enough

$$\mathcal{I}(n, A) := \int_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})\setminus A}\bigl|\psi_n(t)\bigr|\,\mathrm{d}t < \varepsilon. \tag{45}$$

Fix $\varepsilon > 0$, we will show that (45) holds for a suitable choice of a compact subset A of $\mathbb{R}^d$. Note that, for every $t \in \mathbb{R}^d$, and each n

$$\bigl|\psi_n(t)\bigr| \leq \bigl|g_n(t)\bigr|,$$

where

$$g_n(t) := \mathbb{E}\Bigl[\exp\Bigl(\Bigl\langle it, \frac{\sum_{j=1}^n I_j X_j}{\sqrt{\log n}}\Sigma^{-1/2}\Bigr\rangle\Bigr)\Bigr] = \frac{1}{n+1}\Pi_n\Bigl(\Upsilon\Bigl(\frac{1}{\sqrt{\log n}}it\Sigma^{-1/2}\Bigr)\Bigr), \tag{46}$$

where the last equality follows from argument similar to that of (24). This implies that

$$\begin{aligned}
\mathcal{I}(n, A) &\leq \int_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})\setminus A}\bigl|g_n(t)\bigr|\,\mathrm{d}t \\
&= (\sqrt{\log n})^d\int_{\mathcal{F}^*\Sigma^{1/2}\setminus\frac{1}{\sqrt{\log n}}A}\bigl|g_n(\sqrt{\log n}\,w)\bigr|\,\mathrm{d}w.
\end{aligned} \tag{47}$$

We can choose $\delta > 0$, such that for all $w \in B(0,\delta) \setminus \{0\}$, there exists $b > 0$, such that

$$\left|\Upsilon(iw)\right| \le 1 - \frac{b\|w\|^2}{2} \tag{48}$$

(see Lemma 2.3.2(a) of [32] for a proof). From (46), we obtain for some positive constant C_1,

$$\begin{aligned}\left|g_n(\sqrt{\log n}w)\right| &\le \exp\left(-\sum_{j=1}^{n} \frac{b}{j+1}\frac{w\Sigma w^T}{2}\right)\\ &\le C_1 \exp\left(-b\frac{w\Sigma w^T}{2}\log n\right),\end{aligned} \tag{49}$$

where the first inequality follows from $1-x \le \exp(-x)$, and the second inequality is implied by (48). We write

$$(\sqrt{\log n})^d \int_{\mathcal{F}^*\Sigma^{1/2}\setminus\frac{1}{\sqrt{\log n}}A} \left|g_n(\sqrt{\log n}w)\right| \mathrm{d}w = \mathcal{I}_1(n,A) + \mathcal{I}_2(n),$$

where

$$\mathcal{I}_1(n,A) := (\sqrt{\log n})^d \int_{(B(0,\delta)\Sigma^{1/2}\setminus\frac{1}{\sqrt{\log n}}A)\cap\mathcal{F}^*\Sigma^{1/2}} \left|g_n(\sqrt{\log n}w)\right| \mathrm{d}w$$

and

$$\mathcal{I}_2(n) = (\sqrt{\log n})^d \int_{\mathcal{F}^*\Sigma^{1/2}\setminus B(0,\delta)\Sigma^{1/2}} \left|g_n(\sqrt{\log n}w)\right| \mathrm{d}w.$$

Now it is enough to show that for a suitable choice of a compact subset A of $\mathbb{R}^d$ and n sufficiently large, $\mathcal{I}_1(n,A) < \varepsilon$ and $\mathcal{I}_2(n) \longrightarrow 0$ as $n \to \infty$.

$$\mathcal{I}_1(n,A) \le C_1(\sqrt{\log n})^d \int_{B(0,\delta)\Sigma^{1/2}\setminus\frac{1}{\sqrt{\log n}}A} \exp\left(-b\frac{w\Sigma w^T}{2}\log n\right) \mathrm{d}w \tag{50}$$

$$\le C_1 \int_{B(0,\delta\sqrt{\log n})\Sigma^{1/2}\setminus A} \exp\left(-b\frac{t\Sigma t^T}{2}\right) \mathrm{d}t < \varepsilon, \tag{51}$$

where in the above equation we use (49) to obtain the first inequality, and for the last inequality we note that for a given $\varepsilon > 0$, we choose a compact $A \subseteq \mathbb{R}^d$, such that

$$C_1 \int_{\mathbb{R}^d\setminus A} \exp\left(-b\frac{t\Sigma t^T}{2}\right) \mathrm{d}t < \varepsilon. \tag{52}$$

Since the lattices for X_1 and I_1X_1 are same, for all $w \in \mathcal{F}^* \setminus B(0,\delta)$, we get $|\Upsilon(iw)| < 1$, so there exists an $0 < \eta < 1$, such that, $|\Upsilon(iw\Sigma^{-1/2})| \le \eta$, for all $w \in \mathcal{F}^*\Sigma^{1/2} \setminus B(0,\delta)\Sigma^{1/2}$.

Therefore, using the inequality $1 - x \le \exp(-x)$, we obtain

$$\left|g_n(\sqrt{\log n}w)\right| \le \exp\left(-\sum_{j=i}^{n} \frac{1}{j+1}(1-\eta)\right) \le C_2 \exp\bigl(-(1-\eta)\log n\bigr) \tag{53}$$

for some positive constant C_2. Therefore, using equation (21.25) on page 230 of [4], we obtain

$$\mathcal{I}_2(n) \le C_2'(\sqrt{\log n})^d \exp\bigl(-(1-\eta)\log n\bigr) \longrightarrow 0 \qquad \text{as } n \to \infty, \tag{54}$$

where C_2' is an appropriate positive constant. This completes the proof. □

Proof of the Theorem 5. In this case $\mathbb{P}(X_1 = \pm e_i) = \frac{1}{2d}$ for $1 \le i \le d$, where e_i is the ith unit vector in direction i. Thus $\mu = 0$ and $\Sigma = \frac{1}{d}\mathbb{I}_d$.

For notational simplicity, we consider the case $d = 2$, the general case can be written similarly.

Now for each $j \in \mathbb{N}$, $I_j X_j$ is a lattice random vector with the minimal lattice $\mathbb{Z}^2$. It is easy to note that $2\pi\mathbb{Z} \times 2\pi\mathbb{Z}$ is the set of all periods for $I_j X_j$ and its fundamental domain is given by $(-\pi, \pi)^2$. Similar to the proof of Theorem 4, it is enough to show that for any $\varepsilon > 0$, we can choose a compact subset A of $\mathbb{R}^d$, such that for all n large enough

$$\mathcal{I}(n, A) < \varepsilon, \tag{55}$$

where

$$\mathcal{I}(n, A) := \int_{H_n \setminus A} \left|\psi_n(t)\right| \mathrm{d}t,$$

with $H_n := (-\sqrt{\log n}\pi, \sqrt{\log n}\pi)^2$.

As before, we can write

$$\mathcal{I}(n, A) = \mathcal{I}_1(n, A) + \mathcal{I}_2(n).$$

Using arguments similar to (51), one can easily show $\mathcal{I}_1(n, A) < \varepsilon$. It remains to show that

$$\mathcal{I}_2(n) \longrightarrow 0 \qquad \text{as } n \to \infty.$$

To do so, we first observe that for $t = (t^{(1)}, t^{(2)}) \in \mathbb{R}^2$, the characteristic function for X_1 is given by $\Upsilon(it) = \frac{1}{2}(\cos t^{(1)} + \cos t^{(2)})$. If $t \in [-\pi, \pi]^2$ be such that $|\Upsilon(it)| = 1$, then $t \in \{(\pi, \pi), (-\pi, \pi), (\pi, -\pi), (-\pi, -\pi)\}$. The function $\cos\theta$ is continuous and decreasing as a function of θ for $t \in [\frac{\pi}{2}, \pi]$. Choose $0 < \eta < \frac{\pi}{2}$ such that for $t \in A_1 = (-\pi, \pi)^2 \cap B^c(0, \delta) \cap D^c$, we have $|\Upsilon(it)| < 1$, where $D = [\pi - \eta, \pi)^2 \cup [-\pi + \eta, -\pi) \times [\pi - \eta, \pi) \cup [-\pi + \eta, -\pi)^2 \cup [\pi - \eta, \pi) \times [-\pi + \eta, -\pi)$, and δ is as in (48). Let us write

$$\mathcal{I}_2(n) := \mathcal{J}_1(n) + \mathcal{J}_2(n),$$

where

$$\mathcal{J}_1(n) := \log n \int_{A_1} \left|\psi_n(\sqrt{\log n}w)\right| \mathrm{d}w,$$

and

$$\mathcal{J}_2(n) := \log n \int_D \left|\psi_n(\sqrt{\log n}w)\right| \mathrm{d}w.$$

It is easy to note that

$$\mathcal{J}_1(n) \le \log n \int_{\overline{A}_1} \left|\psi_n(\sqrt{\log n}w)\right| \mathrm{d}w,$$

where $\overline{A}_1$ denotes the closure of A_1. For $w \in \overline{A}_1$, there exists $0 < \alpha < 1$, such that, $|\Upsilon(it)| \le \alpha$. Therefore, using bounds similar to that in (53) we can show that

$$\mathcal{J}_1(n) \longrightarrow 0 \qquad \text{as } n \to \infty.$$

We observe that

$$\mathcal{J}_2(n) \le 4 \log n \int_{[\pi-\eta,\pi]^2} \left|\psi_n(\sqrt{\log n}w)\right| \mathrm{d}w.$$

Hence, it is enough to show that as $n \to \infty$

$$\log n \int_{[\pi-\eta,\pi]^2} \left|\psi_n(\sqrt{\log n}w)\right| \mathrm{d}w \longrightarrow 0.$$

For $w \in [\pi - \eta, \pi]^2$, we have $0 < |(1 + \frac{\Upsilon(iw)}{j})| \le (1 + \frac{\cos(\pi-\eta)}{j}) \le 1$. Therefore,

$$\left|\psi_n(w)\right| = \frac{1}{n+1} \prod_{j=1}^{n} \left| \left(1 + \frac{\Upsilon(iw)}{j}\right)\right| \le \frac{1}{n+1}.$$

So,

$$\log n \int_{[\pi-\eta,\pi]^2} \left|\psi_n(\sqrt{\log n}w)\right| \mathrm{d}w \le \frac{\eta^2}{n+1} \log n \longrightarrow 0 \qquad \text{as } n \to \infty. \qquad \square$$

We now assume that $d = 1$ and $\mathbb{P}(X_1 = 0) = 0$. Let $\tilde{h}$ be the span for X_1. We can now write $\mathbb{P}(I_1 X_1 \in a + h\mathbb{Z}) = 1$, where $a \in \mathbb{R}$ and $h > 0$ is the span for $I_1 X_1$. It is easy to note that $h \le \tilde{h}$. The following result gives a local limit theorem under the assumption that $\tilde{h} < 2h$. This assumption is non-trivial, and a concrete case is, when $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = 2) = 1/2$. Then, $\tilde{h} = 1$. The support for $I_1 X_1$ is $\{0, 1, 2\}$ and $h = 1$, satisfying the assumption $\tilde{h} < 2h$.

Theorem 10. *Assume that $\tilde{h} < 2h$, then, as $n \to \infty$*

$$\sup_{x \in \mathcal{L}_n^{(1)}} \left| \frac{\sqrt{\mu_2 \log n}}{h} \mathbb{P}\left(\frac{Z_n - \mu \log n}{\sqrt{\mu_2 \log n}} = x \right) - \phi(x) \right| \longrightarrow 0, \tag{56}$$

where $\mathcal{L}_n^{(1)}$ is as defined in (15).

Proof. This proof is similar to the proof of Theorem 4, except we now have $d = 1$, which is simpler. So we omit most of the details and only point out the key differences, where the assumption $\tilde{h} < 2h$ is crucial.

We begin by observing that the bounds for $|\frac{\sqrt{\mu_2 \log n}}{h}\mathbb{P}(\frac{Z_n - \mu \log n}{\sqrt{\mu_2 \log n}} = x) - \phi(x)|$ are similar to those in the proof of Theorem 4, except for that of $\mathcal{I}_2(n)$, where in this case,

$$\mathcal{I}_2(n) := \sqrt{\log n}\int_{\sqrt{\mu_2}\delta}^{\frac{\sqrt{\mu_2}\pi}{h}} \left|g_n(w\sqrt{\log n})\right| \mathrm{d}w$$

and δ is chosen as in (48) and $g_n(\cdot)$ is as in (46). We have to show

$$\mathcal{I}_2(n) \longrightarrow 0 \qquad \text{as } n \to \infty.$$

The span of X_1 being $\tilde{h}$, for all $t \in [\delta, \frac{2\pi}{\tilde{h}})$, $|\Upsilon(it)| < 1$. Note that here we use the assumption $\tilde{h} < 2h$. The characteristic function being continuous in t, there exists $0 < \eta < 1$, such that, $|\Upsilon(\frac{it}{\sqrt{\mu_2}})| \le \eta$ for all $t \in [\delta\sqrt{\mu_2}, \frac{\pi\sqrt{\mu_2}}{h}] \subset [\delta\sqrt{\mu_2}, \frac{2\pi\sqrt{\mu_2}}{\tilde{h}})$. So, as $n \to \infty$,

$$\mathcal{I}_2(n) \le C_2\sigma \exp\bigl(-(1-\eta)\log n\bigr)(\pi - \delta)\sqrt{\log n} \longrightarrow 0. \tag{57}$$

□

5. Urns with colors indexed by other lattices on $\mathbb{R}^d$

We can further generalize the urn models with colors indexed by certain countable lattices in $\mathbb{R}^d$. Such a model will be associated with the corresponding random walk on the lattice. To state the results rigorously we consider the following notations.

Let $\{X_j\}_{j\ge 1}$ be a sequence of random d-dimensional i.i.d. vectors with non-empty support set $B \subseteq \mathbb{R}^d$ and probability mass function p. We assume that B is finite. Consider the countable subset

$$\mathbb{S}^d := \left\{\sum_{i=1}^{k} n_i b_i : n_1, n_2, \dots, n_k \in \mathbb{N}, b_1, b_2, \dots, b_k \in B\right\}$$

of $\mathbb{R}^d$ which will index the set of colors.

As before we consider $S_n := X_0 + X_1 + \cdots + X_n$, $n \ge 0$, the random walk starting at X_0 which is distributed as U_0. We say a process $(U_n)_{n\ge 0}$ is a urn scheme with colors indexed by $\mathbb{S}^d$ and replacement matrix R and starting configuration U_0, if $(U_n)_{n\ge 1}$ is defined recursively by (21), where now R is given by

$$R := \bigl(\bigl(p(v-u)\bigr)\bigr)_{u,v\in S^d}. \tag{58}$$

Following the same nomenclature as done earlier, we will call this process the *infinite color urn model associated with the random walk* $\{S_n\}_{n\ge 0}$ *on* $\mathbb{S}^d$. Naturally, when $\mathbb{S}^d = \mathbb{Z}^d$, this process is exactly the one discussed earlier.

We will use same notations as earlier for the mean, non-centered second moment matrix and moment generating function for the increment X_1 (see (1) for the definitions). As before, we denote by Z_n the $(n+1)$th selected color. Just like in the previous case, the expected proportion of colors in the urn at time n will be given by the distribution of Z_n but now on $\mathbb{S}^d$.

From the proof of Proposition 7, it follows that the result holds also for this generalization. This enable us to generalize Theorem 1 and Theorem 3 as follows.

Theorem 11. *Consider an infinite color urn model with initial configuration U_0 and replacement matrix given by* (58). *Let Z_n be the $(n+1)$th selected color. Then*

$$\frac{Z_n - \boldsymbol{\mu} \log n}{\sqrt{\log n}} \Rightarrow N_d(0, \Sigma), \qquad \text{as } n \to \infty. \tag{59}$$

Theorem 12. *Let $\Lambda_n \in \mathcal{M}_1$ be the random probability measure corresponding to the random probability vector $\frac{U_n}{n+1}$. Let*

$$\Lambda_n^{\mathrm{cs}}(A) = \Lambda_n\big(\sqrt{\log n} A \Sigma^{1/2} + \boldsymbol{\mu} \log n\big),$$

where A is a Borel subset of $\mathbb{R}^d$. Then, as $n \to \infty$,

$$\Lambda_n^{\mathrm{cs}} \xrightarrow{p} \Phi_d \qquad \text{in } \mathcal{M}_1. \tag{60}$$

The proofs of these two theorems are similar to Theorem 1 and Theorem 3, respectively, and hence omitted.

As an application we now consider a specific example, namely, the triangular lattice in two dimensions. For this the support, set for the i.i.d. increment vectors is given by

$$B = \{(1,0), (-1,0), \omega, -\omega, \omega^2, -\omega^2\},$$

where ω, ω^2 are the complex cube roots of unity. The law of X_1 is uniform on B. This gives the random walk on the triangular lattice in two dimensions. The following is an immediate corollary of Theorem 11.

Corollary 13. *Consider the urn model associated with the random walk on two dimensional triangular lattice then as $n \to \infty$*

$$\frac{Z_n}{\sqrt{\log n}} \Rightarrow N_2\left(0, \frac{1}{2}\mathbb{I}_2\right). \tag{61}$$

Proof. Since $1 + \omega + \omega^2 = 0$, therefore it is immediate that $\boldsymbol{\mu} = 0$. Also we know that $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Writing $\omega = (\mathrm{Re}(\omega) + i\,\mathrm{Im}(\omega))$, we get

$$\mathbb{E}\big[\big(X_1^{(1)}\big)^2\big] = \frac{2}{6}\big(1 + \big(\mathrm{Re}(\omega)\big)^2 + \big(\mathrm{Re}\big(\omega^2\big)\big)^2\big).$$

Since $\mathrm{Re}(\omega) = \mathrm{Re}(\omega^2)$, therefore

$$\mathbb{E}\big[\big(X_1^{(1)}\big)^2\big] = \frac{2}{6}\big(1 + 2\big(\mathrm{Re}(\omega)\big)^2\big) = \frac{1}{2}.$$

Similarly, $\mathrm{Im}(\omega) = -\mathrm{Im}(\omega^2)$, and hence $\mathbb{E}[(X_1^2)^2] = \frac{2}{6}((\mathrm{Im}(\omega))^2 + (\mathrm{Im}(\omega^2))^2) = \frac{1}{2}$. Finally

$$\mathbb{E}\big[X_1^{(1)} X_1^{(2)}\big] = -\frac{2}{6}\,\mathrm{Im}\big(1 + \omega + \omega^2\big) = 0.$$

So $\Sigma = \frac{1}{2}\mathbb{I}_2$. The rest is just an application of Theorem 11. □

Appendix

We present here an elementary but technical result which we have used in the proof of Theorem 3. It is really a generalization of the classical result for Laplace transform, namely, Theorem 22.2 of [8].

Theorem 14. *Let ν_n be a sequence of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and let $m_n(\cdot)$ be the corresponding moment generating functions. Suppose there exists $\delta > 0$ such that $m_n(\lambda) \longrightarrow \exp(\frac{\|\lambda\|^2}{2})$ as $n \to \infty$ for every $\lambda \in [-\delta, \delta]^d \cap \mathbb{Q}^d$, then as $n \to \infty$*

$$\nu_n \Rightarrow \Phi_d. \tag{62}$$

Proof. Choose a $\delta' \in \mathbb{Q}$ such that $0 < \delta' < \delta$, and observe that for every $a > 0$

$$\nu_n\big(\big([-a, a]^d\big)^c\big) \le \sum_{i=1}^{d} \exp\big(-\delta' a\big)\big(m_n\big(-\delta' e_i\big) + m_n\big(\delta' e_i\big)\big),$$

where $\{e_i\}_{i=1}^d$ are the d-unit vectors. Now for our assumption we get $m_n(\delta' e_i) \longrightarrow \exp(\frac{\delta'^2}{2})$ and $m_n(-\delta' e_i) \longrightarrow \exp(\frac{\delta'^2}{2})$ as $n \to \infty$ for every $1 \le i \le d$. Thus, we get

$$\sup_{n \ge 1} \nu_n\big(\big([-a, a]^d\big)^c\big) \longrightarrow 0 \qquad \text{as } a \to \infty.$$

So the sequence of probability measures $(\nu_n)_{n\ge1}$ is tight. Therefore, for every subsequence $\{n_k\}_{k\ge1}$ there exists a further subsequence $\{n_{k_j}\}_{j\ge1}$ and a probability measure ν such that as $n \to \infty$,

$$\nu_{n_{k_j}} \Rightarrow \nu.$$

Then by dominated convergence theorem

$$m_{n_{k_j}}(\lambda) \longrightarrow m_\infty(\lambda), \qquad \forall \lambda \in (-\delta, \delta)^d \cap \mathbb{Q}^d,$$

where m_∞ is the moment generating function of ν. But from our assumption

$$m_{n_{k_j}}(\lambda) \longrightarrow \exp\left(\frac{\|\lambda\|^2}{2}\right), \qquad \forall \lambda \in [-\delta, \delta]^d \cap \mathbb{Q}^d.$$

So we conclude that

$$m_\infty(\lambda) = \exp\left(\frac{\|\lambda\|^2}{2}\right), \qquad \forall \lambda \in (-\delta, \delta)^d \cap \mathbb{Q}^d.$$

Since both sides of the above equation are continuous functions on their respective domains, we get that $m_\infty(\lambda) = \exp(\frac{\|\lambda\|^2}{2})$ for every $\lambda \in (-\delta, \delta)^d$. But the standard Gaussian distribution is characterize by the values of its moment generating function in a open neighborhood of 0, so we conclude that every sub-sequential limit is standard Gaussian. This proves (62). □

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