Statistics - 134 (Lecture - 2), Fall 2002

Solutions of the Midterm Exam Problems

1. Let $A :=$ Other coin is **gold**, and $B :=$ you got a **gold** coin. Then from definition,

$$\mathbf{P}\left(A \,\middle|\, B\right) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}.$$

Now,

$$\begin{aligned}
\mathbf{P}(B) &= \mathbf{P}(\text{ you got a } \textbf{gold} \text{ coin }) \\
&= 1 \times \frac{2}{10} + \frac{1}{2} \times \frac{3}{10} + 0 \times \frac{5}{10} \\
&= \frac{7}{20},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{P}(A \cap B) &= \mathbf{P}(\text{ you selected either drawer 1 or drawer 2 }) \\
&= \frac{2}{10}.
\end{aligned}$$

Thus

$$\mathbf{P}\left(A \,\middle|\, B\right) = \frac{2/10}{7/20} = \frac{4}{7}.$$

2. $X :=$ number of green balls in the sample, so the values of X are $\{1, 2\}$. $Y :=$ sum total of the two numbers selected, so the values of Y are $\{3, 4, 5\}$. Also note that the two selected tickets could only be $\{1, 2\}, \{1, 3\}$, or $\{2, 3\}$, and these three pairs are equally likely.

(a) The joint distribution of (X, Y) is given by

	$X = 1$	$X = 2$	Marginal of Y
$Y = 3$	1/3	0	1/3
$Y = 4$	0	1/3	1/3
$Y = 5$	1/3	0	1/3
Marginal of X	2/3	1/3	1

Note that $\mathbf{P}(X = 2, Y = 3) = 0$, while $\mathbf{P}(X = 2) = 1/3$ and $\mathbf{P}(Y = 3) = 1/3$, and hence X and Y are **not** independent.

(b) From definition $X \sim \text{Hypergeometric}(n = 2; N = 3, G = 2)$.

(c)
$$\mathbf{E}[Y] = \tfrac{1}{3} \times (3 + 4 + 5) = 4$$
$$\mathbf{E}[X] = 1 \times \tfrac{2}{3} + 2 \times \tfrac{1}{3} = \tfrac{4}{3}$$
$$\mathbf{E}\left[X^2\right] = 1^2 \times \tfrac{2}{3} + 2^2 \times \tfrac{1}{3} = 2$$
$$\mathbf{Var}(X) = \mathbf{E}\left[X^2\right] - \left(\mathbf{E}[X]\right)^2 = 2 - \left(\tfrac{4}{3}\right)^2 = \tfrac{2}{9}$$

3. Note that balls are being placed at random so the total number of possibilities is n^n.

(a) Fix $1 \le i \le n$,

$$\begin{aligned} \mathbf{P}(A_i) &= \frac{\text{Number of ways to keep } i^{\text{th}} \text{ box empty}}{\text{Total number of ways to place the balls}} \\ &= \frac{(n-1)^n}{n^n} \\ &= \left(1 - \frac{1}{n}\right)^n. \end{aligned}$$

(b) Fix $1 \le i \ne j \le n$,

$$\begin{aligned} \mathbf{P}(A_i \cap A_j) &= \frac{\text{Number of ways to keep } i^{\text{th}} \text{ and } j^{\text{th}} \text{ empty}}{\text{Total number of ways to place the balls}} \\ &= \frac{(n-2)^n}{n^n} \\ &= \left(1 - \frac{2}{n}\right)^n. \end{aligned}$$

Clearly, A_i and A_j are **not** independent.

(c) Write $X = \sum_{i=1}^{n} \mathbf{I}_{A_i}$, and hence $\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{P}(A_i) = n \left(1 - \frac{1}{n}\right)^n$.

(d) Note that if $X = n - 1$, then all the n balls goes into only one box. There are exactly n possible such arrangements, thus

$$\mathbf{P}(X = n - 1) = \frac{n}{n^n} = \frac{1}{n^{n-1}}.$$