Statistics 134 (Lecture - 2), Fall 2002

Practice Final (Time : 3 hours)

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NOTE : There are 10 problems, 10 points each. Solve as many as you can, the maximum you can score is 80. Show all your works, and write explanations when needed.

1. Suppose X is a random variable with the following density :

$$f(x) = \frac{1}{\pi(1+x^2)}\,, \quad -\infty < x < \infty.$$

(a) Find the CDF of $|X|$. **[5 points]**

(b) Find the density of X^2. **[5 points]**

2. Let $T_1 \leq T_2$ be the times of 1^{st} and 2^{nd} arrivals in a Poisson arrival process of rate λ on $(0, \infty)$.

(a) Find $\mathbf{E}\big[T_1 \big| T_2 = 10\big]$. **[5 points]**

(b) Find $\mathbf{E}[T_1 T_2]$. **[5 points]**

3. There are 90 students in a statistics class. Suppose each student has a standard deck of 52 cards of his/her own, and each of them selects 13 cards at random without replacement from his/her own deck independent of the others. What is the chance that there are at least 50 students who got 2 or more aces ? **[10 points]**

4. Suppose you and me are tossing two fair coins independently, and we will stop as soon as each one of us gets a head.

(a) Find the chance that we stop simultaneously. **[4 points]**

(b) Find the conditional distribution of the number of coin tosses given that we stop simultaneously. **[6 points]**

5. Suppose (X, Y) have the following joint density :

$$f(x, y) = \begin{cases} \frac{1}{2} & \text{if } |X| + |Y| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal distribution of X. **[5 points]**

(b) Find the conditional distribution of Y given $X = 1/2$. **[5 points]**

6. Suppose a box contains 10 green, 10 red and 10 black balls. We draw 10 balls from the box by sampling **with replacement**. Let X be the number of green balls, and Y be the number of black balls in the sample.

- Find $\mathbf{E}[XY]$. **[8 points]**
- Are X and Y independent ? Explain. **[2 points]**

7. Julia wants to catch a flight from Oakland Airport at 10:30 AM. In reality the flight leaves at a time uniformly distributed between 10:30 AM and 10:45 AM. Julia also knows that because of bad traffic, if she plans to reach to the airport by time t, then she can only be able to make it by a time which is uniformly distributed between t and $t + 15$ minutes. What time Julia should plan to reach the airport so that she will have exactly 90% chance of catching the flight ? **[10 points]**

8. Let X and Y be two independent random variables such that $X \sim \text{Normal}(\mu, 1)$ and $Y \sim \text{Normal}(0, 1)$.

(a) Find the density of $Z = \min(X, Y)$. **[5 points]**

(b) For each $t \in \mathbb{R}$, calculate $\mathbf{P}(\max(X, Y) - \min(X, Y) > t)$. **[5 points]**

9. There are n balls labeled $1, 2, 3, \ldots, n$, and n boxes also labeled $1, 2, 3, \ldots, n$. Balls are being placed in the boxes at random such that each box can contain **only one** ball. Say that there is a *matching* at the i^{th} position if the i^{th} ball goes into the i^{th} box. Let X be the number of matchings. Find $\mathbf{E}[X]$ and $\mathbf{Var}(X)$. **[4+6 points]**

10. Let Y be a random variable with a density f_Y given by :

$$f_Y(y) = \begin{cases} \frac{\alpha-1}{y^\alpha} & y > 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 1$. Given $Y = y$, let X be a random variable which is Uniformly distributed on $(0, y)$.

(a) Find the marginal distribution of X. **[4 points]**

(b) Calculate $\mathbf{E}\big[Y \big| X = x\big]$, for every $x > 0$. **[6 points]**