Solutions to Assignment 5
Stat 155: Game Theory

Question 1

For this problem, we will use induction. 0 is a terminal position and we have $h(0) = 0$. From 1, the only possible move is to 0, implies $h(1) = 1 = g(0) + 1$. So the base step holds.

For any other step, assume that, $h(x) = g(x-1) + 1 \quad \forall\, 1 \le x \le n$. We will show that $h(n+1) = g(n) + 1$. We will write, $F(x)$ for the followers of x in the original subtraction game. For a set S, $S + 1$ means $\{x + 1 | x \in S\}$. Notice that $F(n+1) = F(n) + 1$ for subtraction games.

From the definition, $h(n+1) = mex\{\{h(y) : y \in F(n+1)\} \cup \{g(0)\}\}$, as the possible moves from $n+1$ are all the moves in the subtraction set and a move to 0. By the induction step,

$$
\begin{aligned}
h(n+1) &= mex\{\{g(y-1) + 1 : y \in F(n+1)\} \cup \{0\}\} \\
&= mex\{\{g(y) + 1 : y \in F(n)\} \cup \{0\}\} \text{ as } F(n+1) = F(n) + 1 \\
&= g(n) + 1
\end{aligned}
$$

from the definition of mex. In general, notice that, $mex\{\{S + 1\} \cup \{0\}\} = mex\{S\} + 1$.

Question 2

Notice that in this game, 1 is the terminal position. Lets calculate the SG functions using backward induction

g(1)	0
g(2)	mex(g(1)) = 1
g(3)	mex(g(2)) = 0
g(4)	mex(g(3),g(2)) = 2
g(5)	mex(g(4)) = 0
g(6)	mex(g(5),g(4),g(3)) = 1
g(7)	mex(g(6)) = 0
g(8)	mex(g(7),g(6),g(4)) = 3
g(9)	mex(g(8),g(6)) = 0
g(10)	mex(g(9),g(8),g(5)) = 1
g(11)	mex(g(10)) = 0
g(12)	mex(g(11),g(10),g(9),g(8),g(6)) = 2
⋮	⋮

As we can see, a pattern emerges in the SG function, namely $g(x)$ = the largest power of 2 that divides x. The fact can be proven using induction.

Base step: $g(1) = g(2^0) = 0$ and $g(2) = g(2^1) = 1$. Suppose that the statement is true for $x \leq n-1$. We will show it for n.

Case 1: If n is odd, then all of its divisors are odd, which means that all the steps that can be accessed from n are even (odd minus odd). By the induction hypothesis, any even $x \leq n$ has $g(x) \geq 1$ as it is divided by 2 at least once. Thus, writing $g(n) = mex(S)$, where $S := \{g(y) : y \in F(n)\}$, then S never contains 0, hence $g(n) = 0$.

Case 2: If n is even, write $n = 2^p \times q$, where q is odd, and we will show that $g(n) = p$. Notice that for $0 \leq i < p$, $2^i q$ is a divisor of n, so $2^i(2^{(p-i)} - 1)q \in F(n)$, so, by induction hypothesis and the fact that q is odd, $i \in S$. So, $\{0, \ldots p-1\} \in S$.

Now, we will show that $p \notin S$. Suppose, contradictorily, $p \in S$, this implies $\exists q'$ odd such that $2^p q' \in F(n)$, then $2^p(q-q')$ is a divisor of n, but $q-q'$ is even. Then the largest power contained in this divisor of n is at least $p+1$ which is a contradiction as we have assumed that the largest power contained in n is p.

Thus, we have $\{0, \ldots p-1\} \in S$ and $p \notin S$. This implies $g(n) = mex(S) = p$.

In particular, $g(18) = g(2 \times 3^2) = 1$.