Solutions to Assignment 6
Stat 155: Game Theory

Question 1

We will write e_i for a vector with 1 at the i^{th} position and 0 everywhere else. p will be a mixed strategy for Player I and q will be a mixed strategy for Player II. Then, if you choose $p = e_i$ and $q = e_j$, we get $p^T A q = a_{ij}$, note that this corresponds to the situation where the strategy of Player I is to play pure strategy i and that of Player II is to play pure strategy j. The outcome should be and is a_{ij}. In particular, convince yourself that each pure strategy is in fact a mixed strategy. Also $p = \sum_i p_i e_i$ and $q = \sum_j q_j e_j$.

Then, it is clear that, for each fixed p,

$$\begin{aligned}
\min_{q\in\Delta_n} p^T A q = \min_{q\in\Delta_n} p^T A(\sum_j q_j e_j) &= \min_{q\in\Delta_n} \sum_j q_j (p^T A e_j) \\
&= \min_{q\in\Delta_n} \sum_j q_j (\sum_i p_i a_{ij}) \\
&\geq \min_{q\in\Delta_n} \sum_j q_j \min_{1\leq j\leq n} (\sum_i p_i a_{ij}) \\
&= \min_{1\leq j\leq n} (\sum_i p_i a_{ij}) \times \min_{q\in\Delta_n} \sum_j q_j \\
&= \min_{1\leq j\leq n} (\sum_i p_i a_{ij})
\end{aligned}$$

Taking max over p on the left hand, for any fixed p,

$$\max_{p\in\Delta_m} \min_{q\in\Delta_n} p^T A q \geq \min_{q\leq j\leq n} (\sum_i p_i a_{ij})$$

Notice that the expression on the left of the inequality above is V.

So, in particular, choosing $p = e_i$, the following is true for each i,

$$V \geq \min_{q\leq j\leq n} a_{ij}$$

Taking max over i on the right hand,

$$V \geq \max_{1 \leq i \leq m} \min_{q \leq j \leq n} a_{ij}$$

so we have proved the first inequality. The second inequality can be proven very similarly.

Question 2

The payoff matrix of this Two-Person Zero-Sum Game is as follows,

-1	2
2	-4

Writing $(p, 1-p)$ for the mixed strategy of Player I and $(q, 1-q)$ for the mixed strategy of Player II, the value of the game is $V = \max_p \min_q -pq + 2(1-p)q + 2p(1-q) - 4(1-p)(1-q)$.

$$\begin{aligned} V &= \max_p \min_q (6 - 9p)q - (6p - 4) \\ &= \max_p (2 - 3p)1_{p > 2/3} + (6p - 4)1_{p \leq 2/3} \\ &= 0 \end{aligned}$$

Thus, we see that the optimal strategy for Player I is $(2/3, 1/3)$ and any mixed strategy for Player II works as fine. The value is 0.