Solutions to Assignment 7
Stat 155: Game Theory

Question 1

This is very similar to the second question in the midterm paper. It is easy to guess that the uniform mixed strategy is going to be a Nash equilibrium and as such an optimal strategy. To prove this, we need to show that if we choose $p^* = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ and $q^* = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ then for any $p, q \in \Delta_n$,

$$p^T A q^* \leq p^{*T} A q^* \leq p^{*T} A q$$

Notice that, $p^{*T} A q^* = \sum_i \sum_j p_i^* q_j^* a_{ij} = \sum_i \sum_j \frac{a_{ij}}{n^2} = \frac{c}{n}$.

It is easy to see that

$$\begin{aligned} Aq^* &= (\sum_j a_{1j} q_j^*, \dots, \sum_j a_{nj} q_j^*)^T \\ &= (\sum_j \frac{a_{1j}}{n}, \dots, \sum_j \frac{a_{nj}}{n})^T \\ &= (\frac{c}{n}, \dots, \frac{c}{n})^T \end{aligned}$$

Similarly, $p^{*T} A = (\frac{c}{n}, \dots, \frac{c}{n})$.

Hence,

$$p^T A q^* = \sum_i p_i (Aq^*)_i = \frac{c}{n} \sum_i p_i = \frac{c}{n} = \frac{c}{n} \sum_j q_j = \sum_i (p^{*T} A)_j q_j = p^{*T} A q$$

So,

$$p^T A q^* \leq p^{*T} A q^* \leq p^{*T} A q$$

holds. Thus, $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ is optimal for both players and the value is $\frac{c}{n}$.

Question 2

In the payoff matrix A,

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & d_{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & d_n \end{pmatrix}$$

we can use the fact that $d_1, \ldots, d_k$ are all > 0 and $d_{k+1}, \ldots, d_n$ are all < 0, to conclude that eacch of the rows $k+1, \ldots, n$ are dominated by row 1.

In the reduced matrix,

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_k & 0 & \cdots & 0 \end{pmatrix}$$

the columns $1, \ldots, k$ are dominated by column $k+1$. Hence, the reduced payoff matrix now is,

$$\begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

that is, all zeros. Hence, any mixed strategy over the strategies $\{1, \ldots, k\}$ for Player I and over $\{k+1, \ldots, n\}$ for Player II is optimal and the value of the game is 0.

Try to solve this without using domination.