UNIVERSITY OF CALIFORNIA, BERKELEY

DEPARTMENT OF STATISTICS

STAT 134: Concepts of Probability

Spring 2014

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Practice Final Examination (II)

Date Given: April 25, 2014 **Duration: 180 minutes** **Total Points: 100**

Note: There are ten problems with a total of 100 points. Show all your works.

1. Suppose X is a random variable with the following density :

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

   (a) Find the CDF of $|X|$. [5]

   (b) Find the density of X^2. [5]

2. Let $T_1 \leq T_2$ be the times of 1^{st} and 2^{nd} arrivals in a Poisson arrival process of rate λ on $(0, \infty)$.

   (a) Find $\mathbf{E}\left[T_1 \middle| T_2 = 10\right]$. [5]

   (b) Find $\mathbf{E}[T_1 T_2]$. [5]

3. There are 90 students in a statistics class. Suppose each student has a standard deck of 52 cards of his/her own, and each of them selects 13 cards at random without replacement from his/her own deck independent of the others. What is the chance that there are at least 50 students who got 2 or more aces ? [10]

4. Suppose you and me are tossing two fair coins independently, and we will stop as soon as each one of us gets a head.

   (a) Find the chance that we stop simultaneously. [4]

   (b) Find the conditional distribution of the number of coin tosses given that we stop simultaneously. [6]

5. Suppose (X, Y) have the following joint density :

$$f(x, y) = \begin{cases} \frac{1}{2} & \text{if } |X| + |Y| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal distribution of X. [5]

(b) Find the conditional distribution of Y given $X = 1/2$. [5]

6. Suppose a box contains 10 green, 10 red and 10 black balls. We draw 10 balls from the box by sampling **with replacement**. Let X be the number of green balls, and Y be the number of black balls in the sample.

   - Find $\mathbf{E}[XY]$. [8]
   - Are X and Y independent ? Explain. [2]

7. Julia wants to catch a flight from Oakland Airport at 10:30 AM. In reality the flight leaves at a time uniformly distributed between 10:30 AM and 10:45 AM. Julia also knows that because of bad traffic, if she plans to reach to the airport by time t, then she can only be able to make it by a time which is uniformly distributed between t and $t + 15$ minutes. What time Julia should plan to reach the airport so that she will have exactly 90% chance of catching the flight ? [10]

8. Let X and Y be two independent random variables such that $X \sim \text{Normal}(\mu, 1)$ and $Y \sim \text{Normal}(0, 1)$.

   (a) Find the density of $Z = \min(X, Y)$. [5]

   (b) For each $t \in \mathbb{R}$, calculate $\mathbf{P}(\max(X, Y) - \min(X, Y) > t)$. [5]

9. There are n balls labeled $1, 2, 3, \ldots, n$, and n boxes also labeled $1, 2, 3, \ldots, n$. Balls are being placed in the boxes at random such that each box can contain **only one** ball. Say that there is a *matching* at the i^{th} position if the i^{th} ball goes into the i^{th} box. Let X be the number of matchings. Find $\mathbf{E}[X]$ and $\mathbf{Var}(X)$. [4 + 6]

10. Let Y be a random variable with a density f_Y given by :

$$f_Y(y) = \begin{cases} \frac{\alpha - 1}{y^\alpha} & y > 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 1$. Given $Y = y$, let X be a random variable which is Uniformly distributed on $(0, y)$.

   (a) Find the marginal distribution of X. [4]

   (b) Calculate $\mathbf{E}\left[Y \middle| X = x\right]$, for every $x > 0$. [6]