Homework # 10 *Statistics 134, Bandyopadhyay, Spring 2014*

5.1.6 a) $P(\text{Jack arrives at least two minutes before Jill}) = \frac{(1/2)13^2}{15^2} \approx 0.376$

b) Let $F = \{\text{first person arrives before 12:05}\}$ and $L = \{\text{last person arrives after 12:10}\}$ Then,

$$\begin{aligned} P(FL) &= 1 - P((FL)^c) = 1 - P(F^c \cup L^c) \\ &= 1 - P(F^c) - P(L^c) + P(F^c \cap L^c) \\ &= 1 - (\frac{10}{15})^{10} - (\frac{10}{15})^{10} + (\frac{5}{15})^{10} \\ &= 0.965 \end{aligned}$$

5.1.9 Standardise the length of the stick to be 1 (the solution clearly will not depend on the length of the stick!). Look at one end of the stick, and let X and Y denote the distances from that end of the stick to the break points. Then X and Y are independent uniform $(0,1)$ random variables. Let L denote the minimum of X and Y, and R the maximum. (L to suggest left, R to suggest right.) Then the lengths of the broken pieces can be expessed as $L, R-L, 1-R$. To form a triangle, the maximum of these three should be less than the sum of the rest. That is,

triangle $\iff \max\{L, R-L, 1-R\} < 1 - \max\{L, R-L, 1-R\}$

$\iff \max\{L, R-L, 1-R\} < 1/2$

$\iff L < 1/2$ and $R - L < 1/2$ and $1/2 < R$

$\iff$ ($X < 1/2$ and $Y - X < 1/2$ and $1/2 < Y$) or ($Y < 1/2$ and $X - Y < 1/2$ and $1/2 < X$).

Conclude: $P(\text{triangle}) = \frac{\text{shaded area}}{\text{total area}} = \frac{1}{4}$.

5.2.14 a) Let $V = \min(U_1, \dots, U_5)$ and $W = \max(U_1, \dots, U_5)$. Then $R = W - V$.

$$\begin{aligned}
P(W \geq w) &= 1 - P(W < w) = 1 - w^5 \\
E(W) &= \int_0^1 P(W \geq w)dw = \int_0^1 1 - w^5 dw = \frac{5}{6} \\
P(V \geq v) &= (1-v)^5 \\
E(V) &= \int_0^1 P(V \geq v)dv = \int_0^1 (1-v)^5 dv = \frac{1}{6} \\
E(R) &= E(W) - E(V) = \frac{2}{3}
\end{aligned}$$

b) For $0 < v < w < 1$,

$$\begin{aligned}
P(V \in dv, W \in dw) &= P(\text{no } U_i \text{ in } (0,v),\ 1 \text{ in } dv,\ 3 \text{ in } (v,w),\ 1 \text{ in } dw) \\
&= 20P(U_1 \in dv, v < U_2, U_3, U_4 < w, U_5 \in dw) \\
&= 20dv(w-v)^3 dw \\
f(v,w) &= 20(w-v)^3 \quad (0 < v < w < 1)
\end{aligned}$$

c)

$$\begin{aligned}
P(R > 0.5) &= \int_{0.5}^1 \int_0^{w-0.5} 20(w-v)^3 dv dw \\
&= 20 \int_{0.5}^1 \left[-\frac{(w-v)^4}{4} \right|_{v=0}^{w-0.5} dw \\
&= 5 \int_{0.5}^1 w^4 - (0.5)^4 dw \\
&= 5 \left[\frac{w^5}{5} - w(0.5)^4 \right|_{w=0.5}^1 \\
&= \frac{26}{32}
\end{aligned}$$

5.2.16 Considering the three-dimensional space which corresponds to triplets (X_1, X_2, X_3)

we wish to find the volume which gives us $X_1 < X_2 < X_3$. This volume is given by:

$$\begin{aligned}
P(X_1 < X_2 < X_3) &= \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty f(x_3)f(x_2)f(x_1)dx_3dx_2dx_1 \\
&= \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \lambda_3 e^{-\lambda_3 x_3}\lambda_2 e^{-\lambda_2 x_2}\lambda_1 e^{-\lambda_1 x_1}dx_3dx_2dx_1 \\
&= \int_0^\infty \int_{x_1}^\infty e^{-\lambda_3 x_2}\lambda_2 e^{-\lambda_2 x_2}\lambda_1 e^{-\lambda_1 x_1}dx_2dx_1 \\
&= \int_0^\infty \int_{x_1}^\infty \lambda_2 e^{-(\lambda_3+\lambda_2)x_2}\lambda_1 e^{-\lambda_1 x_1}dx_2dx_1 \\
&= \int_0^\infty \frac{\lambda_1\lambda_2}{\lambda_2+\lambda_3} e^{-(\lambda_3+\lambda_2)x_1} e^{-\lambda_1 x_1}dx_1 \\
&= \frac{\lambda_1\lambda_2}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)}
\end{aligned}$$

5.4.10

$$f_Y(y) = \begin{cases} 1/2 & \text{if } 0 < y < 1 \\ 1/(2y^2) & \text{if } y \geq 1 \\ 0 & \text{otherwise} \end{cases}$$