Homework # 5 *Statistics 134, Bandyopadhyay, Spring 2014*

3.1.6 There are 8 equally likely outcomes for three fair coin tosses:

outcome	probability	X	Y	$X+Y$
HHH	1/8	2	2	4
HHT	1/8	2	1	3
HTH	1/8	1	1	2
HTT	1/8	1	0	1
THH	1/8	1	2	3
THT	1/8	1	1	2
TTH	1/8	0	1	1
TTT	1/8	0	0	0

a) Joint distribution table for (X, Y): (the entries clearly sum to 1)

Y \ X	0	1	2
0	1/8	1/8	0
1	1/8	2/8	1/8
2	0	1/8	1/8

b) X and Y are not independent. For instance, $P(X = 2, Y = 0) = 0$, which is clearly not the same as $P(X = 2)P(Y = 0) = (1/4)(1/4)$.

c)

z	0	1	2	3	4
$P(X+Y=z)$	1/8	2/8	2/8	2/8	1/8

The terms clearly sum to 1.

3.1.14 a) For team A to win in g games, it must win 3 of the first $g-1$ games and also win the gth game. The chance is $\binom{g-1}{3}p^3q^{g-1-3} \cdot p = \binom{g-1}{3}p^4q^{g-4}$ for $g = 4, 5, 6, 7$.

b) The event is the disjoint union of "Team A wins in g games" as g ranges from 4 to 7. So the answer is $\sum_{g=4}^{7} \binom{g-1}{3}p^4q^{g-4}$.

c) $P(\text{A wins}) = 1808 \ / \ 2187 = 0.8267$

d) You have to compare the current World Series scheme of playing only until one team has one four games with an alternative scheme of playing all 7 games and declaring the winner to be the team that wins at least 4 out of the 7. The same team will be declared the winner under both schemes. For the details, let X be the number of times that A wins if all 7 games are played. Then X has binomial $(7, p)$ distribution. If $X \geq 4$ then A has won the World series, since B can have won at most 3 games. And if A won the World Series, then A won four games before B did, so $X \geq 4$. So $P(\text{A wins}) = P(X \geq 4)$.

To check algebraically, notice that

$$P(\text{A wins}) = p^4[1 + \binom{4}{3}q + \binom{5}{3}q^2 + \binom{6}{3}q^3] = p^4[1 + 4q + 10q^2 + 20q^3] = p^4 f(q)$$

where $f(q)$ is the given polynomial in q. And

$$P(X \geq 4) = p^4[\binom{7}{4}q^3 + \binom{7}{5}pq^2 + \binom{7}{6}p^2q + \binom{7}{7}p^3]$$

$$= p^4[35q^3 + 21(1-q)q^2 + 7(1-q)^2q + (1-q)^3] = p^4 g(q)$$

where $g(q)$ is also a polynomial in q. So all we need to do is to check that the coefficients are the same in the two polynomials:

term	1	q	q^2	q^3
coefficient in f	1	4	10	20
coefficient in g	1	7-3 = 4	21 - (2×7) + 3 = 10	35 -21 + 7 - 1 = 20

So the two polynomials are the same.

e) G has range $\{4, 5, 6, 7\}$. $P(G = g) = P(\text{A wins in } g \text{ games}) + P(\text{B wins in } g \text{ games}) = \binom{g-1}{3}p^4q^{g-4} + \binom{g-1}{3}q^4p^{g-4}$. If $p = 1/2$, then G and the winner are independent. Check that for $p = 1/2$ the answer in (b) reduces to $P(\text{A wins}) = 1/2$, and also for each g,

$$P(G = g) = 2P(\text{A wins in } g \text{ games})$$

which is the same as saying

$$P(G = g \text{ and A wins}) = P(G = g) \cdot \frac{1}{2}$$

Therefore when $p = 1/2$, for each g you have $P(G = g \text{ and A wins}) = P(G = g)P(\text{A wins})$. So G and the winner are independent.

3.4.12 Write $q_1 = 1 - p_1, q_2 = 1 - p_2$.

a) $P(W_1 = W_2) = \sum_{k=1}^{\infty} P(W_1 = k, W_2 = k)$

$$= \sum_{k=1}^{\infty} P(W_1 = k)P(W_2 = k) = \sum_{k=1}^{\infty} q_1^{k-1}p_1q_2^{k-1}p_2 = \frac{p_1p_2}{1 - q_1q_2}$$

b) $P(W_1 < W_2) = \sum_{k=1}^{\infty} P(W_1 = k, W_1 < W_2)$
$= \sum_{k=1}^{\infty} P(W_1 = k, W_2 > k) = \sum_{k=1}^{\infty} q_1^{k-1}p_1q_2^k = \frac{p_1q_2}{1-q_1q_2}$

c) By symmetry, it's $\frac{p_2q_1}{1-q_2q_1}$. Check: (a) + (b) + (c) = 1.

d) There are many ways to find and to express the answers in (c) and (d). Here are some of the ways. Put $X = \min(W_1, W_2)$. For $k = 0, 1, 2, ...$ we have

$$P(X > k) = P(W_1 > k \text{ and } W_2 > k) = P(W_1 > k)P(W_2 > k) = q_1^k q_2^k = (q_1q_2)^k.$$

So X is geometric with parameter $1 - q_1 q_2 = p_1 + p_2 - p_1 p_2$.

You can also do this as follows:

$$\begin{aligned} P(X = k) &= P(W_1 = k, W_2 > k) + P(W_1 > k, W_2 = k) + P(W_1 = k, W_2 = k) \\ &= q_1^{k-1} p_1 q_2^k + q_1^k q_2^{k-1} p_2 + q_1^{k-1} p_1 q_2^{k-1} p_2 \end{aligned}$$

Check by algebra that this is equal to $(q_1 q_2)^{k-1}(1 - q_1 q_2)$.

e) Put $Y = \max(W_1, W_2)$. Y has range $\{1, 2, 3, ...\}$. For $n = 0, 1, 2, ...$ we have

$$P(Y \le n) = P(W_1 \le n \text{ and } W_2 \le n) = P(W_1 \le n) P(W_2 \le n)$$

$$= [1 - P(W_1 > n)][1 - P(W_2 > n)] = (1 - q_1^n)(1 - q_2^n).$$

For $n = 1, 2, 3, ...$ we then have

$$P(Y = n) = P(Y \le n) - P(Y \le n - 1) = (1 - q_1^n)(1 - q_2^n) - (1 - q_1^{n-1})(1 - q_2^{n-1}).$$

3.4.14 Let $n \ge 1$. V_n is a random variable having range $\{n, \ldots, 2n-1\}$. For $k = n, \ldots, 2n-1$ we have

$(V_n = k) = (V_n = k, k\text{th trial is success}) \cup (V_n = k\ ,\ k\text{th trial is failure})$

The two events on the right are mutually exclusive. The first event is

$(V_n = k, k\text{th trial is success}) = (\text{exactly } n - 1 \text{ successes in first } k - 1 \text{ trials}, k\text{th trial is success})$

and has probability

$P(\text{exactly } n - 1 \text{ successes in first } k - 1 \text{ rials}, k\text{th trial is success})$

$= P(\text{exactly } n - 1 \text{ successes in first } k - 1 \text{ trials}) P(k\text{th trial is success})$

$= \binom{k-1}{n-1} p^{n-1} q^{k-n} \cdot p = \binom{k-1}{n-1} p^n q^{k-n}$. Similarly the second event has probability $\binom{k-1}{n-1} q^n p^{k-n}$. Hence

$P(V_n = k) = \binom{k-1}{n-1}(p^n q^{k-n} + q^n p^{k-n})\ ,\ k = n, \ldots, 2n - 1.$

3.6.2 a) 1/13 b) 1/4 c) $(13 \times 12 \times 11 \times 10 \times 9)/(52 \times 51 \times 50 \times 49 \times 48)$ d) $(48 \times 47 \times 46 \times 45 \times 4)/(52 \times 51 \times 50 \times 49 \times 48)$