Homework # 9 *Statistics 134, Bandyopadhyay, Spring 2014*

4.4.4 The density of X is $f_X(x) = 1/2$ for $x \in (-1, 1)$, so by the *change of variable formula* the density of Y is

$$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} = \frac{\frac{1}{2} + \frac{1}{2}}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}, y \in (0, 1).$$

Note: This distribution is called the beta (1/2, 1) distribution.

4.4.6 Notice that $Y = \tan \Phi$. Its range is $(-\infty, \infty)$. The function $y = \tan \phi$ is strictly increasing with derivative $\sec^2 \phi$ for $\phi \in (-\pi/2, \pi/2)$. Also note that $\phi = tan^{-1}(y)$, and (draw the right-angled triangle!) $\sec \phi = \sqrt{1 + y^2}$. By the one to one change of variable formula for densities, the density of Y is, for all y,

$$\begin{aligned} f_Y(y) &= f_\Phi(\phi) \Big/ \left|\frac{dy}{d\phi}\right| \\ &= \frac{1}{\pi} / \sec^2 \phi \\ &= \frac{1}{\pi(1 + y^2)} \end{aligned}$$

The Cauchy distribution is symmetric since $f_Y(y) = f_Y(-y)$.

Even though the density is symmetric about 0, the expectation of a Cauchy random variable is undefined. To check whether the integral is absolutely convergent:

$$\begin{aligned} \int_{-\infty}^{\infty} |y f_Y(y)| dy &= \int_{-\infty}^{\infty} \frac{|y|}{\pi(1 + y^2)} dy \\ &= \int_0^{\infty} \frac{2y}{\pi(1 + y^2)} dy \\ &= \frac{1}{\pi} \log(1 + y^2)|_0^{\infty} \ = \ \infty \end{aligned}$$

so the required integral does not converge absolutely. Thus the expectation is undefined.

4.5.2 a)

x	0	1	2	3
$F_X(x)$	1/8	1/2	7/8	1

b) Since $P(k) = (1/2)^k$ for $k = 1, 2, 3, \ldots$, we have:

If $x \geq 1$ then $F(x) = \sum_{k \leq x} P(k) = \sum_{k=1}^{int\ (x)} (1/2)^k = 1 - (1/2)^{int\ (x)}$;

If $x < 1$ then $F(x) = 0$.

4.5.6 a) $P(X \geq 1/2) = 1 - F(1/2) = 7/8$.

b) $f(x) = \frac{d}{dx} F(x) = \begin{cases} 0 & x \leq 0 \\ 3x^2 & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$.

c) $E(X) = \int x f(x) dx = \int_0^1 x 3x^2 dx = \int_0^1 3x^3 dx = 3/4$.

d) Let Y_1, Y_2, Y_3 be independent uniform $(0, 1)$ random variables. Then for $i = 1, 2, 3$

$$P(Y_i \leq x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$$

so if $X = max(Y_1, Y_2, Y_3)$, then

$$\begin{aligned} P(X \leq x) &= P(Y_1 \leq x, Y_2 \leq x, Y_3 \leq x) \\ &= [P(Y_1 \leq x)]^3 \\ &= \left\{ \begin{array}{ll} 0 & x \leq 0 \\ x^3 & 0 \leq xl \\ 1 & x \geq 1 \end{array} \right\} \\ &= F(x). \end{aligned}$$