Game Theory - Midterm Examination

Date: September 19, 2023

Total marks: **30**

Duration: 3 PM to 5 PM

Note: Answer all questions clearly using pen. Please avoid unnecessary discussions.

1. Consider the game in Table 1.

	a	b	c	d
A	$(7,0)$	$(2,5)$	$(0,7)$	$(0,1)$
B	$(5,2)$	$(3,3)$	$(5,2)$	$(0,1)$
C	$(0,7)$	$(2,5)$	$(7,0)$	$(0,1)$
D	$(0,0)$	$(0,-2)$	$(0,0)$	$(9,-1)$

Table 1: A two player strategic form game

Answer the following questions for the game in Table 1.

(a) What is the profile of largest set of rationalizable strategies? **(4 marks)**

Answer. d is strictly dominated by $\frac{1}{2}a + \frac{1}{2}c$. Then, removing d, we see that D is strictly dominated. by B. Since each of A, B, and C are best responses to some strategy of Player 2, and each of a, b, and c are best responses to some strategy of Player 1, this implies that we cannot eliminate any further strategies of players. Since IESDS strategy set is same as largest set of rationalizable strategies, the answer is $\{\{A, B, C\}, \{a, b, c\}\}$. [1]

One can directly argue that this is rationalizable.

(b) Consider the following correlated strategy p:

$$p(A, a) = p(A, c) = p(C, a) = p(C, c) = \frac{1}{4}$$

Is p a correlated equilibrium? Justify your answer. **(2 marks)**

Answer. No. If Player 2 receives a, she believes Player 1 is either A or C with equal probability. Playing a gives 3.5 but playing b gives 5 for sure.

[1] You can also iteratively delete never best responses.

P_1	P_2	P_3
a	a	c
b	c	a
c	b	b

Table 2: Preference over objects

(c) Is there a correlated equilibrium $\tilde{p}$ such that $\tilde{p}(A,a) > 0$ and $\tilde{p}(C,a) = 0$. **(2 marks)**

Answer. No. If Player 2 receives a, she knows C is not played, then the maximum payoff she can get is less than 2 (since $p(A,a) > 0$). By playing c Player 2 gets higher expected payoff (given her belief).

(d) Find all pure strategy Nash equilibria of this game. **(2 marks)**

Answer. For Nash equilibrium, we can look at the game after IESDS. (B,b) is the unique pure strategy Nash equilibrium.

2. There are three objects $\{a,b,c\}$ to be allocated to three agents $\{1,2,3\}$. Each agent i has a preference ordering, which is a strict ranking, of the set of objects. Each agent wants at most one object from $\{a,b,c\}$. The preferences of the three agents are in Table 2.

The game is as follows: agent 1 comes and chooses an object (say $a_1 \in \{a,b,c\}$); agent 2 then comes and chooses an object (say $a_2 \in \{a,b,c\} \setminus \{a_1\}$); finally agent 3 chooses the remaining object (say a_3).

(a) What are the strategies for each player (i.e., define S_1, S_2, S_3 in this game)? **(4 marks)**

Answer. A strategy of Player 1 is an object $s_1 \in \{a,b,c\}$. A strategy of Player 2 chooses an object in every possible pair of objects:

$$s_2(\{a,b\}) \in \{a,b\}, s_2(\{b,c\}) \in \{b,c\}, s_2(\{a,c\}) \in \{a,c\}$$

These are the contingencies of Player 2.[2]

[2] Ishaan suggests that we can write the contingencies of Player 2 as what was chosen by Player 1 (which is the complement of what is available to Player 2). So, one can think of $s_2 : \{a,b,c\} \to \{a,b,c\}$ such that $s_2(x) \neq x$ for all x, i.e., for every choice of Player 1, Player 2 chooses a different object.

Strategy of Player 2 is a constant map: $s_3 : \{a, b, c\} \to \{a, b, c\}$ such that $s_3(x) = x$.

(b) Show that there is a weakly dominant strategy for each player such that the outcome of the game from this strategy profile is: agent 1 gets a, agent 2 gets c, and agent 3 gets b. **(3 marks)**

Answer. Player 1 is strictly better off choosing a in s_1 (in her preference a is highest ranked). Player 3 has only one strategy – so nothing to prove. Player 2 is the only non-trivial case. Consider the strategy s_2^*, where player 2 chooses the best object according to P_2:

$$s_2^*(\{a, b\}) = a; s_2^*(\{b, c\}) = c; s_2^*(\{a, c\}) = a$$

Since this gives the best object to Player 2 in every possible contingency, Player 2 is weakly better off than any other strategy. This is weakly because of the following. Suppose Player 1 plays the strategy we suggested. Then, she chooses a. Then, any strategy where player 2 chooses c from $\{b, c\}$ gives her the same utility (for instance, $s_2(\{a, b\}) = b; s_2(\{b, c\}) = c; s_2(\{a, c, \}) = a$).

Strategy s_2^* is strictly better than an arbitrary strategy s_2 for *some* strategy of Player 1. Suppose s_2^* and s_2 differ in choice in some pair $\{x, y\}$. Consider a strategy of Player 1, where she chooses $\{a, b, c\} \setminus \{x, y\}$. Then, Player 2 is strictly better off playing s_2^* than s_2.

(c) Suppose we modify the game slightly. The game asks each player to submit a preference order over objects: (P_1', P_2', P_3'), where P_i' need not equal P_i. Once the submitted preferences are with us, we assign objects as follows: agent 1 is assigned the highest ranked object (say, a_1') according to P_1'; agent 2 is assigned the highest ranked object (say, a_2') from $\{a, b, c\} \setminus \{a_1'\}$ according to P_2'; agent 3 is assigned the remaining object P_3'.

Compare the strategies of each player in both the games. **(3 marks)**

Answer. PLAYER 1: The strategy is to choose an object (say, a_1) in original game. Any preference P_1' where a_1 is at top does the same job. But there are many such preferences. Hence, for every strategy in the original game, there are many strategies in the modified game that corresponds to the same outcome.

PLAYER 3: In the original game, she chooses the left-over object. In the modified game, he can choose whatever preference she wants, but gets assigned the left-over object. Hence, in original game, Player 3 has a unique strategy but in the modified game, he has 3! strategies (all resulting in the same outcome).

PLAYER 2: In the modified game, if player 2 submits P_2', this can be equivalently done by a strategy s_2 in the original game, where for every pair $x, y \in \{a, b, c\}$, we assign $s_2(\{x, y\})$ the higher ranked object between x and y according to P_2'.

However, not every s_2 can be constructed from P_2'. To see this, consider the following strategy s_2 in the original game:

$$s_2(\{a, b\}) = a; s_2(\{b, c\}) = b; s_2(\{a, c\}) = c$$

If Player 1 chose c, then according to this strtaegy, Player 2 chooses a. Similarly, if Player 1 choses a, Player 2 chooses b, and so on.

There is no preference P_2' that can generate these choices: if such P_2' existed, then to choose a when Player 1 chose c, we must have $aP_2'b$. Similarly, $bP_2'c$. But then $aP_2'c$ by transitivity which contradicts $s_2(\{a, c\}) = c$ (if Player 1 chose b, Player 2 chooses c). Hence, the original game contains strategies that are not "equivalently" present in the modified game.

3. State if the following are TRUE or FALSE in the mixed extension of a finite strategic form game (assume two players). In each case, either give a proof or a counterexample.

   (a) If a pair of pure strategies are strictly dominated then a mixed strategy involving these two strategies is also strictly dominated.

   Answer. TRUE. If s_1 and s_1' are strictly dominated by t_1 and t_1' respectively, then their mixture $ps_1 + (1-p)s_1'$ is strictly dominated by $pt_1 + (1-p)t_1'$.

   (b) If $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is a max-min strategy profile of a two-person zero-sum games and $\sigma_1^*(s_1) > 0$, then s_1 is not strictly dominated.

   Answer. TRUE. By min-max theorem σ^* is a Nash equilibrium. Hence, σ_1^* is a BR to σ_2^*. Since $\sigma_1^*(s_1) > 0$, by indifference lemma, s_1 is also a BR to σ_2^*. Hence,

s_1 is not strictly dominated.

(c) The mixed extension of a finite game has a finite number of Nash equilibria.

Answer. FALSE. Consider a game where each player is indifferent between all the outcomes: in other words, all utilities of all players are zero. Then, every pure strategy profile and mixed strategy profile is a Nash equilibrium.

(d) Suppose σ^* is a fixed point of the best response correspondence and $\sigma_1^*(s_1) > 0$, then s_1 is not strictly dominated.

Answer. TRUE. Since σ^* is a fixed point of the best response correspondence, it is a Nash equilibrium. Hence, by indifference lemma, s_1 is not strictly dominated.

(e) Suppose s_1 is weakly dominated by another strategy, then there is no Nash equilibrium where Player 1 plays s_1.

Answer. FALSE. In game in Table 3, B weakly dominates A for Player 1. However, (A, a) is a Nash equilibrium.

	a	b
A	$(5, 2)$	$(2, 0)$
B	$(5, 0)$	$(3, 1)$

Table 3: A two player strategic form game