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Bayes–Nash equilibria of the generalized second-price auction [☆]

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ABSTRACT

We develop a Bayes–Nash analysis of the generalized second-price (GSP) auction, the multi-unit auction used by search engines to sell sponsored advertising positions. Our main result characterizes the efficient Bayes–Nash equilibrium of the GSP and provides a necessary and sufficient condition that guarantees existence of such an equilibrium. With only two positions, this condition requires that the click–through rate of the second position is sufficiently smaller than that of the first. When an efficient equilibrium exists, we provide a necessary and sufficient condition for the auction revenue to decrease as click–through rates increase. Interestingly, under optimal reserve prices, revenue increases with the click–through rates of all positions. Further, we prove that no inefficient equilibrium of the GSP can be symmetric. Our results are in sharp contrast with the previous literature that studied the GSP under complete information.

1. Introduction

Over the last few years a new multi-unit auction format known as the generalized second-price (GSP) auction has received much attention from economists. This auction format has been applied to diverse problems such as routing in fixed and wireless networks (see Su et al., 2010) and the allocation of capacity in electricity markets (see Gergen et al., 2008 and Schne, 2009). Most remarkably, search engines use the GSP to sell sponsored advertising links in the Internet, with revenues that exceeded 20 billion dollars industry-wide in 2009. According to the simplest version of this mechanism, each advertiser (bidder) submits one bid that represents his willingness to pay for a click in his sponsored link. Advertisers are then ranked in decreasing order of bids and sponsored links are assigned according to this ranking. Payments are determined according to a "next-bid" rule: Each advertiser pays for a click in his link the bid submitted by the advertiser immediately below him in the sponsored advertising list.

The GSP has been extensively studied in a complete information setting. Aggarwal et al. (2006), Varian (2007) and Edelman et al. (2007) are the first to derive complete information Nash equilibria of this auction. They show that, although its obvious similarities with the Vickrey auction, truthful bidding does not constitute an equilibrium of the GSP.

In an incomplete information setting, Edelman et al. (2007) modeled the GSP as an ascending auction for multiple goods (they call it the Generalized English auction). Their main result states that the unique perfect Bayesian equilibrium of this game implements the same payments as the Vickrey–Clark–Groves mechanism.

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Quite surprisingly, little is known about the Bayes–Nash equilibria of the GSP, where bids are submitted simultaneously and advertisers have private values per click which are position-independent. As Lahaie et al. stated in 2007, "To date nothing is known about the Bayesian equilibrium of the GSP auction".[1] In this paper, we develop a complete Bayes–Nash analysis of this mechanism.

Our main result characterizes the efficient Bayes–Nash equilibrium (simply "equilibrium" from now on) of the GSP and provides a necessary and sufficient condition that guarantees existence of such an equilibrium. The proof proceeds by using the integral-form envelope theorem to express the expected payment of a bidder with value per click v as a function of his probabilities of obtaining each position in an efficient equilibrium. Following the payment rule of the GSP, we derive an alternative expression for the expected payment of a bidder that depends on the symmetric bidding function employed by all bidders. In equilibrium, these expressions have to be equal for every valuation v. This condition leads to an integral equation (technically, a Volterra equation of the second kind) that any efficient equilibrium bidding function has to satisfy. We apply results from the theory of integral equations to show that there is a unique candidate bidding function that solves this integral equation, and derive its solution analytically. Using well-known results from Monotone Comparative statics, we show that an efficient equilibrium exists if and only if the candidate bidding function that solves the integral equation is strictly monotone at all possible valuations.

In the simple case where only two advertising positions are for sale, the candidate bidding function is strictly monotone (and an efficient equilibrium exists) if and only if the click–through rate of the second position is sufficiently smaller than that of the first. Intuitively, as the click–through rate of the second position approaches that of the first, obtaining the second position becomes a better deal for advertisers, since payments per click are lower (the third instead of the second highest bid) and click–through rates are similar. As a consequence, advertisers with high valuations have the incentive to shade their bids in order to be ranked second (rather than first) and obtain the second position. When click–through rates are close enough, bid shading is so intense that the bidding function is no longer monotonic, and the efficient equilibrium breaks down.

The bid shading phenomenon has interesting implications on how varying click–through rates affects the total revenue produced by an efficient equilibrium of the GSP. To obtain intuition, take the case of two positions and consider an increase in the click–through rate of the second position. There are two effects to consider: First, as payments are per click, revenue tends to increase as the number of clicks for sale increases (we call it the *supply effect*). Second, as click–through rates converge, advertisers with high values per click strategically shade their bids (we call it the *strategic effect*), and revenue tends to decrease. As it unfolds, we derive a necessary and sufficient condition on the distribution of values per click for the strategic effect to dominate the supply effect. Intuitively, this condition captures the notion that the distribution of advertisers' values per click is concentrated on high values, in which case the strategic effect is stronger than the supply effect; and total revenue goes down even though the number of clicks for sale is higher. This intuition readily generalizes to any number of positions.

Interestingly, with optimal reserve prices, we show that the search engine's revenue increases with the click–through rates of all positions (reversing the previous result on the GSP without reserve prices). Intuitively, reserve prices work to remove advertisers with low values per click from the auction. Hence, advertisers with high values per click can reduce their bid shading and still have a fair chance of getting lower positions. As a consequence, the supply effect unambiguously dominates the strategic effect; and the search engine's revenue can only grow with the number of clicks for sale in the auction.

Next, we turn to investigate the existence of inefficient equilibria of the GSP. We show that every equilibrium with symmetric strategies of the GSP is outcome equivalent to some monotone pure strategy equilibrium, and conclude that no inefficient equilibrium of the GSP can be symmetric. Our analysis reveals that the GSP is an interesting (yet simple) example of a game that satisfies the single-crossing condition applied by Athey (2001) and McAdams (2003) to discrete games, but still fails to admit (for a wide range of parameter values) a monotone Bayes–Nash equilibrium when action and type spaces are continuous.[2]

1.1. Related literature

Under complete information, Edelman et al. (2007) and Varian (2007) select among all equilibria that satisfy the envy-free criterion the one that produces the lowest revenues to the auctioneer. The envy-free criterion requires that no advertiser wishes to change positions and payments with some advertiser ranked above him. Cary et al. (2007) select the same equilibrium based on the convergence of myopic best responses. Edelman and Schwarz (2010) provide a different rationale for selecting this equilibrium based on the non-contradiction criterion (NCC). An equilibrium fails the NCC if it generates greater revenue (in expectation) than any equilibrium of an associated dynamic game.

Still under complete information, Borgers et al. (2008) allow advertisers to have position-specific values per click.[3] They extend the notion of envy-free equilibrium to this more general setting and show that weak dominance does not select the

1 See the related literature for an account of recent work about the GSP under complete and incomplete information.

2 See also Reny (2009).

3 Milgrom (2010) considers an extension of the GSP where bidders are allowed to submit multiple bids, one for each position. Alternatively, Babaioff and Roughgarden (2009) measure the complexity of a payment rule by relating the profile of bids and the formation of prices for each position. Constantin et al. (2010) propose a variation of the GSP that allows each bidder to express how his value per click is affected by the identity of the other advertisers appearing in the sponsored list.

equilibrium studied by Edelman et al. (2007) and Varian (2007). Blumrosen et al. (2008) study complete information Nash equilibria of the GSP with non-uniform conversion rates. Thompson and Leyton-Brown (2009) introduce new computational techniques to select equilibria in different versions of the GSP.

Under incomplete information, the unique perfect Bayesian equilibrium of the Generalized English auction proposed by Edelman et al. (2007) is efficient. In contrast, our analysis shows that in a Bayes–Nash setting the GSP may not produce an efficient allocation of bidders to advertising positions. In this regard, Paes Leme and Tardos (2010) and Caragiannis et al. (2012) complement the analysis of the article by computing bounds on the price of anarchy of the GSP (that is, the welfare loss from inefficient equilibria) under complete and incomplete information.[4]

A recent literature endogenizes the click–through rates faced by advertisers by explicitly modeling the clicking behavior of browsers.[5] In a model where browsers take clicking decisions through a sequential search procedure, Aggarwal et al. (2008) and Kempe and Mahdian (2008) solve for the efficient allocation of positions to bidders, while Giotis and Karlin (2008) analyze complete information Nash equilibria of the GSP.[6] Athey and Ellison (2011) consider the Generalized English auction from Edelman et al. (2007) in a setting where clicking is costly and browsers perform optimal search. Gomes (2012) derives the revenue-maximizing auction in a two-sided model where searchers have correct expectations about the relevance of sponsored links. In a related setting, Rayo and Segal (2010) derive the optimal information disclosure rule by a revenue-maximizing platform.[7]

This paper is organized as follows: In the next section we characterize the efficient Bayes–Nash equilibrium of the GSP and provide a necessary and sufficient condition for such an equilibrium to exist. In Section 3, we analyze how the revenue produced by the efficient equilibrium is affected by changes in the click–through rates of different positions. In Section 4, we introduce reserve prices and show that, under the optimal reserve price, revenue increases with the click–through rates of all positions. In Section 5 we discuss other (inefficient) equilibria of the GSP. We conclude in Section 6.

2. The efficient equilibrium

A search engine sells S advertising positions to $N > S$ bidders through the generalized second-price (GSP) auction. In this mechanism, positions are assigned in decreasing order of bids, that is, the top position goes to highest bidder; the second highest position goes to the second highest bidder; and so on. Each bidder pays for a click in his position the bid immediately below his bid in the set of all bids submitted in the auction. As such, the s-th highest bidder's total payment is $c_s b^{(s+1)}$, where c_s denotes the click–through rate of the s-th position and $b^{(s+1)}$ denotes the $(s+1)$-th highest bid.

A bidder's value per click is the same across positions; and we denote it by v. We assume that bidders' values are independent draws from a distribution F (which density is f) with support on $[0, \bar{v}]$. Bidders privately observe their values and simultaneously submit their bids to the search engine. Accordingly, our solution concept is the Bayes–Nash equilibrium.

Our main result characterizes the unique efficient equilibrium of the GSP, and provides a condition that is both necessary and sufficient for an efficient equilibrium to exist.

Proposition 1. *Consider the generalized second-price auction (GSP) with N bidders, S positions ($N > S$) and click–through rates $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_S$. If an efficient Bayes–Nash equilibrium exists for this auction, then its symmetric bidding strategy is*

$$\beta(v) = v - \phi(v) - \sum_{n=1}^{\infty} \int_0^v K_n(v,t)\phi(t)\,dt, \tag{1}$$

where

$$\phi(v) = \frac{\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(s-1)(1-F(v))^{s-2} \int_0^v F^{N-s}(x)\,dx}{\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(1-F(v))^{s-1} F^{N-s-1}(v)}, \tag{2}$$

$$K_1(v,t) = \frac{\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(s-1)(1-F(v))^{s-2} F^{N-s-1}(t) f(t)}{\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(1-F(v))^{s-1} F^{N-s-1}(v)}, \tag{3}$$

$$K_n(v,t) = \int_0^v K_1(v,\varepsilon) K_{n-1}(\varepsilon,t)\,d\varepsilon \quad \text{for } n \geqslant 2, \tag{4}$$

and by assumption $\phi(v) \in L^2([0,\bar{v}])$ and $K_1(v,t) \in L^2([0,\bar{v}]^2)$.[8]

[4] Building a structural model of sponsored search, Athey and Nekipelov (2009) analyze equilibria in a setting where there is complete information about the value per click of each advertiser, but there is uncertainty about the score and the number of competing bidders faced by each advertiser.

[5] See Jerziorski and Segal (2009) for an empirical analysis of the demand for sponsored advertising.

[6] Abrams and Schwarz (2008) study in complete information a version of the GSP that incorporates the hidden costs of bad advertisements.

[7] See also Chen et al. (2010).

[8] That is, $\int_0^{\bar{v}} |\phi(v)|^2\,dv < \infty$ and $\int_0^{\bar{v}} \int_0^{\bar{v}} |K_1(v,t)|^2\,dv\,dt < \infty$. These integrability conditions are satisfied by the most common distributions with discrete support, including the uniform and the beta.

If $\beta(v)$ defined by (1) is strictly increasing, then an efficient Bayes–Nash equilibrium exists for the GSP. Otherwise, this auction does not admit an efficient equilibrium.

Proof. The proof proceeds in two steps. In the first step, we use the payoff equivalence formula to derive an integral equation that has to be satisfied by any bidding function that is consistent with an efficient equilibrium. Using results from the theory of Volterra equations, we show that this integral equation has a unique solution and derive this solution analytically. In the second step, we show that an efficient equilibrium exists if and only if the candidate bidding function identified in step 1 is strictly increasing.

Step 1. *If an efficient equilibrium exists, then the bidding function $\beta(v)$ satisfies the following Volterra equation of the second kind*:

$$v - \beta(v) = \phi(v) + \int_0^v K_1(v,t)\big(t - \beta(t)\big)\,dt, \tag{5}$$

where the functions $\phi(v)$ and $K_1(v,t)$ are defined by (2) and (3).

The welfare-maximizing (efficient) allocation assigns the bidder with s-highest value per click to the s-th highest position. Therefore, in an efficient equilibrium, a bidder with value v obtains the s-th highest position with probability:

$$z_s(v) = \binom{N-1}{s-1}\big(1 - F(v)\big)^{s-1} F^{N-s}(v). \tag{6}$$

Consider an efficient equilibrium of the GSP and denote by $E[P^{PE}(v)]$ the expected payment in equilibrium of a bidder with value per click v. By the Revelation Principle, $E[P^{PE}(v)]$ has to satisfy

$$v \in \arg\max_{\hat{v}} \sum_{s=1}^{S} c_s \cdot z_s(\hat{v}) \cdot v - E\big[P^{PE}(\hat{v})\big]. \tag{7}$$

By the Integral-form Envelope Theorem (see Milgrom, 2004, p. 67), it follows that (7) implies that

$$\sum_{s=1}^{S} c_s \cdot z_s(v) \cdot v - E\big[P^{PE}(v)\big] = -E\big[P^{PE}(0)\big] + \sum_{s=1}^{S} c_s \cdot \int_0^v z_s(t)\,dt.$$

Because $E[P^{PE}(0)] = 0$, the expected payment of a bidder with value per click v in any efficient equilibrium is given by

$$E\big[P^{PE}(v)\big] = \sum_{s=1}^{s=S} c_s \left\{ z_s(v)v - \int_0^v z_s(t)\,dt \right\} = \sum_{s=1}^{s=S} c_s \int_0^v \frac{dz_s(t)}{dt} t\,dt, \tag{8}$$

where the second equality follows from integration by parts.

We will now derive an alternative formula for the expected payment of a bidder with value v that explicitly depends on the equilibrium bidding function. It is clear that in any efficient equilibrium all bidders have to play symmetric bidding strategies, which we denote by $\beta(v)$.

Denote by $v^{j:k}$ the j-th highest realization among $k \geqslant j$ iid draws from the cdf $F(v)$ (with the convention that $v^{0:k} \equiv \bar{v}$). From the rules of the GSP, we know that the expected payment for a bidder with value per click v is given by

$$\begin{aligned}
E\big[P^{GSP}(v)\big] &= \sum_{s=1}^{s=S} c_s \cdot z_s(v) \cdot E\big[\beta\big(v^{s+1:N}\big) \mid v^{s+1:N} \leqslant v \leqslant v^{s-1:N}\big] \\
&= \sum_{s=1}^{s=S} c_s \cdot z_s(v) \cdot E\big[\beta\big(v^{1:N-s}\big) \mid v^{1:N-s} \leqslant v\big] \\
&= \sum_{s=1}^{s=S} c_s \cdot z_s(v) \cdot \int_0^v \beta(t) \underbrace{\frac{(N-s)F^{N-s-1}(t)f(t)}{F^{N-s}(v)}}_{\text{conditional density of } v^{1:N-s}} dt.
\end{aligned} \tag{9}$$

Since $E[P^{PE}(v)] = E[P^{GSP}(v)]$ for all v, we can equate (8) and (9) to obtain the following Volterra equation of the first kind:

$$\sum_{s=1}^{s=S} c_s \int_0^v \frac{dz_s(t)}{dt} t\,dt = \int_0^v \beta(t) \sum_{s=1}^{s=S} c_s \cdot \binom{N-1}{s-1}\big(1 - F(v)\big)^{s-1}(N-s)F^{N-s-1}(t) f(t)\,dt.$$

Differentiating both sides leads to

$$\sum_{s=1}^{s=S} c_s \frac{dz_s(v)}{dv} v = \beta(v) \sum_{s=1}^{s=S} c_s \cdot \binom{N-1}{s-1} \big(1-F(v)\big)^{s-1} (N-s) F^{N-s-1}(v) f(v)$$
$$- \int_0^v \beta(t) \sum_{s=1}^{s=S} c_s \cdot \binom{N-1}{s-1} (s-1) \big(1-F(v)\big)^{s-2} f(v) (N-s) F^{N-s-1}(t) f(t)\, dt.$$

The equation above can be rewritten as

$$v-\beta(v) = \frac{v\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(1-F(v))^{s-1}F^{N-s-1}(v) - \sum_{s=1}^{s=S} c_s \frac{dz_s(v)}{dv}\frac{v}{f(v)(N-1)}}{\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(1-F(v))^{s-1}F^{N-s-1}(v)}$$
$$\frac{-\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(s-1)(1-F(v))^{s-2}\int_0^v tF^{N-s-1}(t) f(t)\, dt}{\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(1-F(v))^{s-1}F^{N-s-1}(v)}$$
$$+\int_0^v \big(t-\beta(t)\big)\frac{\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(s-1)(1-F(v))^{s-2}F^{N-s-1}(t) f(t)}{\sum_{s=1}^{s=S} c_s \binom{N-2}{s-1}(1-F(v))^{s-1}F^{N-s-1}(v)}\, dt.$$

Plugging (6) into the expression above and using integration by parts in the second line leads to the Volterra equation of the second kind (5). Debnath and Mikusinski (1999) prove (on p. 235, Theorem 5.5.1) that the Volterra equation of second kind (5) has a unique solution given by (1) provided that the functions $\phi(v)$ and $K_1(v,t)$ are L^2-integrable (that is, $\phi(v) \in L^2([0,\bar{v}])$ and $K_1(v,t) \in L^2([0,\bar{v}]^2)$).

Step 2. *If the function* (1) *is strictly increasing for all* $v \in [0,\bar{v}]$*, then the GSP has a unique efficient equilibrium which symmetric bidding strategy is described by* (1)*. Otherwise, this auction does not admit an efficient equilibrium.*

By construction, the function (1) satisfies payoff equivalence (given by the envelope condition (8)). Since the bidders' payoff function satisfies strictly increasing differences, we can use the Constraint Simplification Theorem (Milgrom, 2004, p. 105) to conclude that $\beta(v)$ best responds all bids in the range $[0,\beta(\bar{v})]$ if and only if it is strictly increasing. As it is clearly not optimal to bid $b > \beta(\bar{v})$, we have that $\beta(v)$ indeed best responds all possible bids (and constitutes the unique efficient Bayes–Nash equilibrium) if and only if $\beta'(v) > 0$ for all $v \in [0,\bar{v}]$. Otherwise, we conclude there is no bidding function consistent with an efficient equilibrium. □

With only one position, the GSP boils down to a (single-unit) Vickrey auction, in which case truthful bidding constitutes the unique efficient equilibrium.[9] Indeed, Eq. (1) becomes $\beta(v) = v$ as we set $c_s = 0$ for $s \geqslant 2$. The next example derives the efficient equilibrium in the simple case where 3 bidders with uniformly distributed values per click compete for 2 advertising positions. Interestingly, an efficient equilibrium does not exist if the click–through rate of the second position is close enough to that of the first position.

Example 1. Consider the GSP with $S = 2$ positions, $N = 3$ bidders and click–through rates normalized to $(1, c_2)$. The value per click of each bidder is an independent draw from the uniform distribution on the unit interval: $v \sim U[0,1]$. The Volterra equation (5) is then equivalent to the following first-order linear differential equation (its analytic solution is presented in Appendix A):

$$\beta'(v) = \frac{v\cdot(1-2c_2)+c_2}{v\cdot(1-c_2)+c_2} + \frac{1-2c_2}{v\cdot(1-c_2)+c_2}\cdot\big(v-\beta(v)\big). \tag{10}$$

It can be shown that $\beta(v)$ is strictly increasing for all $v \in [0,1]$ (and an efficient equilibrium exists) if an only if $c_2 \leqslant \frac{3}{4}$.

Fig. 1 plots $\beta(v)$ for $c_2 \in \{0, \frac{1}{2}, \frac{3}{4}, 1\}$. Intuitively, as c_2 approaches c_1, obtaining the second position becomes a better deal for bidders, since payments per click are lower (the third bid instead of the second) and click–through rates are similar. As a result, bidders strategically shade their bids and $\beta(v)$ becomes flatter. If c_2 is greater than $\frac{3}{4}$, bid shading is so intense that the monotonicity of $\beta(v)$ breaks down and an efficient equilibrium fails to exist.

The next proposition shows that the intuition from the example above is true for distributions other than the uniform.

[9] See Blume and Heidhues (2004) and Blume et al. (2009) for a complete Bayes–Nash analysis of the single and multi-unit Vickrey auction.

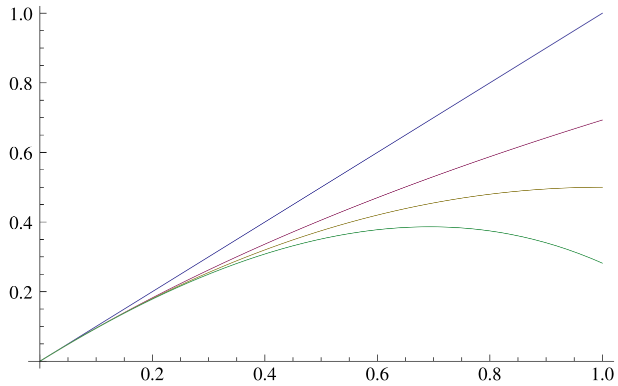

Fig. 1. The candidate bidding functions for $c_2 \in \{0, \frac{1}{2}, \frac{3}{4}, 1\}$ (from top to bottom, respectively) when three bidders with uniformly distributed values compete for two positions.

Proposition 2. *Consider the generalized second-price auction (GSP) with 2 positions, N bidders and click–through rates (c_1, c_2). There exists $\bar{c}_2 < c_1$ such that an efficient equilibrium exists if and only if $c_2 \leqslant \bar{c}_2$.*

Proof. For $S = 2$ and arbitrary N, Eq. (5) can be recast as

$$\beta(v) = v - \frac{(N-2)c_2}{F^{N-2}(v)c_1 + (N-2)(1-F(v))F^{N-3}(v)c_2} \int_0^v \big(v - \beta(x)\big) F^{N-3}(x) f(x)\, dx. \tag{11}$$

Therefore,

$$\big(v - \beta(v)\big) \cdot \left(\frac{F^{N-2}(v)c_1 + (N-2)(1-F(v))F^{N-3}(v)c_2}{(N-2)c_2}\right) = \int_0^v \big(v - \beta(x)\big) F^{N-3}(x) f(x)\, dx.$$

After totally differentiating (11), and using the expression above to substitute away the integral term $\int_0^v (v - \beta(x)) \times F^{N-3}(x) f(x)\, dx$, we obtain the following first-order differential equation:

$$\beta'(v) = \underbrace{\frac{F^{N-2}(v)[c_1 - c_2] + (N-2)(1-F(v))F^{N-3}(v)c_2}{F^{N-2}(v)c_1 + (N-2)(1-F(v))F^{N-3}(v)c_2}}_{A(v)} + \underbrace{\frac{\{(N-2)F^{N-3}(v) f(v)\big[c_1 + \big((N-3)\big(\frac{1}{F(v)} - 1\big) - 2\big)c_2\big]\}}{F^{N-2}(v)c_1 + (N-2)(1-F(v))F^{N-3}(v)c_2}}_{B(v)} \big(v - \beta(v)\big). \tag{12}$$

It is clear that if $c_2 \leqslant \frac{1}{2}$ then the two terms on the right-hand side of (12) are positive, in which case there exists an efficient equilibrium.

Now consider $\tilde{c}_2$ such that $\tilde{c}_2 > c_2 > \frac{1}{2}$ and denote by $\beta(v)$ and $\tilde{\beta}(v)$ the solutions of (12) associated to c_2 and $\tilde{c}_2$ (and analogously to $\tilde{A}(v)$ and $\tilde{B}(v)$). Denote by $\tilde{v}$ the value such that

$$c_1 + \left((N-3)\left(\frac{1}{F(\tilde{v})} - 1\right) - 2\right)\tilde{c}_2 = 0.$$

It is clear that $\tilde{B}(v) < 0$ if and only if $v > \tilde{v}$. Note that $\beta'(v), \tilde{\beta}'(v) > 0$ for all $v \leqslant \tilde{v}$. We will now show that if $\beta(v) \geqslant \tilde{\beta}(v)$ then $\beta'(v) \geqslant \tilde{\beta}'(v)$ for all $v > \tilde{v}$. Indeed:

$$\beta'(v) = A(v) + B(v) \cdot \big(v - \beta(v)\big) \geqslant \tilde{A}(v) + \tilde{B}(v) \cdot \big(v - \beta(v)\big) \geqslant \tilde{A}(v) + \tilde{B}(v) \cdot \big(v - \tilde{\beta}(v)\big) = \tilde{\beta}'(v),$$

where the first inequality follows from the fact that $\tilde{A}(v) \leqslant A(v)$ and $\tilde{B}(v) \leqslant B(v)$ and the second inequality follows from $\beta(v) \geqslant \tilde{\beta}(v)$ and $\tilde{B}(v) < 0$ for $v > \tilde{v}$.

When $S = 2$, the functions $\phi(v)$ and $K_1(v, t)$ defined by (2) and (3) simplify to

$$\phi(v) = \frac{(N-2)c_2 \int_0^v F^{N-2}(x)\, dx}{F^{N-2}(v)c_1 + (N-2)(1-F(v))F^{N-3}(v)c_2}$$

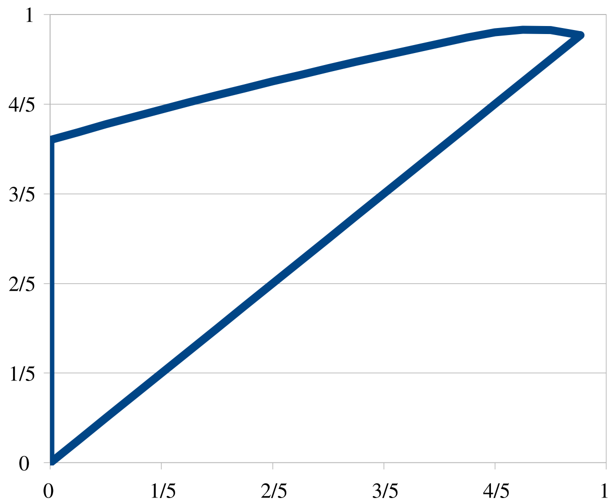

Fig. 2. Set of click–through rates (c_3, c_2) for which an efficient equilibrium exists when four bidders with uniformly distributed values compete for three positions.

and

$$K_1(v,t) = \frac{(N-2)F^{N-3}(t)f(t)c_2}{F^{N-2}(v)c_1 + (N-2)(1-F(v))F^{N-3}(v)c_2},$$

which are strictly increasing in c_2. It then follows from (4) that $K_n(v,t)$ is also strictly increasing in c_2 for all n.

As consequence, from Eq. (1) in Proposition 1 we can conclude that $\beta(v) \geqslant \tilde{\beta}(v)$ for all v. This in particular implies that $\beta(\tilde{v}) \geqslant \tilde{\beta}(\tilde{v})$, and therefore $\beta'(\tilde{v}) \geqslant \tilde{\beta}'(\tilde{v})$. We can then safely conclude that $\beta(v) \geqslant \tilde{\beta}(v)$ for $v > \tilde{v}$. As a consequence, $\tilde{c}_2 > c_2 > \frac{1}{2}$ implies that $\beta'(v) \geqslant \tilde{\beta}'(v)$ for $v > \tilde{v}$. Thus, if there exists an efficient equilibrium for $\tilde{c}_2$, then there exists an efficient equilibrium for any c_2 such that $c_2 \leqslant \tilde{c}_2$.

This shows that exists $\bar{c}_2 \leqslant c_1$ such that an efficient equilibrium exists if and only if $c_2 \leqslant \bar{c}_2$. To see that $\bar{c}_2 < c_1$ for any N, let's use Eq. (12) to evaluate $\beta'(\cdot)$ at the superior limit of the support of values $\bar{v}$:

$$\beta'(\bar{v}) = \frac{[c_1 - c_2]}{c_1} + (N-2)\frac{\{f(\bar{v})[c_1 - 2c_2]\}}{c_1}\bigl(\bar{v} - \beta(\bar{v})\bigr),$$

which is strictly negative when $c_1 = c_2 = 1$. □

For more than two positions, the existence of an efficient equilibrium depends on the ratios among all click–through rates. In such cases, the existence of an efficient equilibrium under click–through rates $\mathbf{c} \equiv (1, c_2, \ldots, c_S)$ can be easily verified by checking whether $\frac{\partial}{\partial v}\beta(v)|_{\mathbf{c}} > 0$, where $\beta(v)$ is given by (1). When an analytical solution to $\beta(v)$ is not available (e.g., because $\int_0^v K_n(v,t)\phi(t)\,dt$ has no analytical representation), it can be numerically approximated by

$$\beta_m(v) = v - \phi(v) - \sum_{n=1}^{m}\sum_{k=0}^{m} K_n\left(v, \frac{k}{m}\bar{v}\right)\phi\left(\frac{k}{m}\bar{v}\right),$$

which converges from above to $\beta(v)$ as $m \to \infty$. The next example applies this procedure to derive the set of click–through rates for which an efficient equilibrium exists in a GSP auction with 3 positions and 4 bidders whose valuations are uniformly distributed.

Example 2. Consider the GSP with $S = 3$ positions which click–through rates are $(1, c_2, c_3)$, and $N = 4$ bidders whose values per click are distributed uniformly on the unit interval: $v \sim U[0,1]$. Fig. 2 represents the set of click–through rates (c_3, c_2) for which an efficient equilibrium exists. When only two positions are available ($c_3 = 0$), Fig. 2 shows that an efficient equilibrium exists only if $c_2 \leqslant \bar{c}_2$.

Things get more interesting when $c_2 > \bar{c}_2$. There, an efficient equilibrium does not exist if c_3 is too low, but does exist if c_3 approaches c_2. As discussed in Example 1, if $c_3 \approx 0$ and $c_2 > \frac{3}{4}$, the bid shading from bidders with high values is enough to break down the monotonicity of the bidding function. Nevertheless, if $c_2 > \frac{3}{4}$ and $c_3 \approx c_2$, bidders with intermediate values shade their bids as well (trying to avoid the second position and targeting the third). Surprisingly, this works to restore the monotonicity of the bidding function and efficiency is once again implementable by the GSP!

3. Click–through rates and revenue

Search engines apply the GSP to sell sponsored links in millions of keywords, which greatly differ in their profiles of click–through rates $(c_1, c_2, \dots, c_S)$. In this section, we investigate how the GSP performs in terms of revenue under different click–through rate profiles.[10]

To do so, let's reconsider the expected payment of a bidder with value per click v, as given by Eq. (9). Differentiating this expression with respect to c_s leads to:

$$\frac{\partial}{\partial c_s} E\big[P^{GSP}(v)\big] = \underbrace{z_s^i(v) \cdot E\big[\beta\big(v^{s+1:N}\big) \mid v^{s+1:N} \leqslant v \leqslant v^{s-1:N}\big]}_{\text{supply effect}} + \underbrace{\sum_{t=1}^{t=S} z_t^i(v) c_t \left(\frac{\partial}{\partial c_s} E\big[\beta\big(v^{s+1:N}\big) \mid v^{s+1:N} \leqslant v \leqslant v^{s-1:N}\big]\right)}_{\text{strategic effect}}. \tag{13}$$

The term in the first line of (13) reflects the marginal effect of increasing the click–through rate c_s, holding constant equilibrium bids, on the expected payment of a bidder with value v. This term is clearly positive; and it reflects the increase in the number of clicks sold by the GSP. We call it the *supply effect* on the search engine's revenue.

The term in the second line of (13) accounts for changes in equilibrium bids that follow from different click–through rates. Intuitively, as we know from Fig. 1, equilibrium bids might decrease in response to higher click–through rates from lower positions; as bidders engage in bid shading. Accordingly, we call it the *strategic effect* on the search engine's revenue.

Which effect dominates: The supply of the strategic effect? Our next proposition derives a condition on the bidders' distribution of values that determines how an increase in the click–through rate of position s affects the search engine's revenue. By the Revenue Equivalence Principle, the conclusions of Proposition 3 apply to any auction format that implements the efficient allocation of bidders to positions.

Proposition 3. *Assume the GSP with N bidders and S positions possesses an efficient Bayes–Nash equilibrium. In this equilibrium, the search engine's expected revenue*:

1. *Increases with the click–through rate of the top position, c_1,*
2. *Increases with the click–through rate of position s, c_s, if and only if*:

$$\int_0^{\bar{v}} \big(1 - F(v)\big)^{s-1} F^{N-s}(v) \left(v - \frac{(1 - F(v))}{f(v)}\right) f(v)\, dv \geqslant 0. \tag{14}$$

Proof. Using the expected payment of a bidder with value per click v, given by Eq. (8), we can compute the search engine's expected revenue as:

$$E[R] = N \int_0^{\bar{v}} E\big[P^{PE}(v)\big] f(v)\, dv = \int_0^{\bar{v}} \sum_{s=1}^{S} \binom{N-1}{s-1} \big(1 - F(v)\big)^{s-1} F^{N-s}(v) c_s \left(v - \frac{(1 - F(v))}{f(v)}\right) f(v)\, dv. \tag{15}$$

Differentiating with respect to c_1 leads to:

$$\begin{aligned} \frac{\partial E[R]}{\partial c_1} &= N \int_0^{\bar{v}} F^{N-1}(v) \left(v - \frac{(1 - F(v))}{f(v)}\right) f(v)\, dv \\ &= N \int_0^{\bar{v}} \left\{ F(v)^{N-1} v - \int_0^{v} F(s)^{N-1}\, ds \right\} f(v)\, dv = N \int_0^{\bar{v}} A(v) f(v)\, dv, \end{aligned}$$

where $A(v) = F(v)^{N-1} v - \int_0^{\bar{v}} F(s)^{N-1}\, ds$. Since $A(0) = 0$ and

$$A'(v) = (N-1) F(v)^{N-2} f(v) v \geqslant 0 \quad \text{for all } v,$$

it follows that $A(v) \geqslant 0$. This implies that $\frac{\partial E[R]}{\partial c_1} \geqslant 0$.

[10] See Chen et al. (2009) for an analysis of the revenue-maximizing distribution of click–through rates.

Moreover, from (15) one can easily compute that:

$$\frac{\partial E[R]}{\partial c_s} = \int_0^{\bar{v}} \binom{N-1}{s-1} \left(1-F(v)\right)^{s-1} F^{N-s}(v) \left(v - \frac{(1-F(v))}{f(v)}\right) f(v)\,dv,$$

proving our claim. □

Bidders with low values per click are less prone than bidders with high values to shade their bids as the click–through rates of the lower positions increase, since low-value bidders expect to obtain the lowest positions at best. As a consequence, one might expect that the strategic effect is stronger when the distribution of bidders' values is concentrated around high values. In this case, the expression (14) is negative and revenue decreases as the click–through rates of lower positions increase. The next example confirms this intuition.

Example 3. Consider the GSP with $S = 2$ positions, $N = 3$ bidders and click–through rates normalized to $(1, c_2)$. Consider the parametric family of distributions of bidders' values with densities given by:

$$f_a(v) = av + \left(1 - \frac{a}{2}\right) \quad \text{with } -2 \leqslant a \leqslant 2$$

and support on $[0, 1]$.

Clearly, if $a = 0$ the distribution of bidders' values is uniform and:

$$\frac{\partial E[R]}{\partial c_2} = 2\int_0^1 v(1-v)[2v-1]\,dv = 0.$$

If $a > 0$, F_a is concentrated on high values of v and if $a < 0$, F_a is concentrated on low values of v. It is a matter of algebra to show that revenue decreases with the click–through rate of the bottom position if and only if F_a is concentrated on high values, that is, $\frac{\partial E[R]}{\partial c_2} \geqslant 0$ if and only if $a \leqslant 0$.

In the spirit of Bulow and Roberts (1989), Proposition 3 can be interpreted in terms of monopolistic price theory. Consider a monopoly offering a menu of S products of heterogeneous qualities $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_S$. Proposition 3 states that, depending on demand conditions captured by condition (14), the monopoly's profits can go down as the quality of some product $s \in \{2, \ldots, S\}$ increases. Intuitively, an increase in c_s for $s \geqslant 2$ is associated to a "demand reduction" for higher quality products. In the GSP auction, the demand reduction takes the form of the bid shading phenomenon illustrated in Fig. 1.

Incidentally, Proposition 3 delivers a procedure to compute the optimal number of sponsored links that search engines should sell in each keyword. The key, yet simple, observation behind the next corollary is that if $\frac{\partial E[R]}{\partial c_s} \geqslant 0$ then $\frac{\partial E[R]}{\partial c_t} > 0$ for all $t < s$. As such, it is profitable to offer at least s advertising positions in a particular keyword if and only if the distribution of bidders' values in that keyword satisfies condition (14) at position s. Once this condition fails, expanding the sponsored list is detrimental to the search engine's profits. This is summarized in the next corollary:

Corollary 1. *Assume the GSP with N bidders and S positions possesses an efficient Bayes–Nash equilibrium. Then, offering S positions maximizes the search engine's revenue if and only if:*

$$\frac{\partial E[R]}{\partial c_S} \geqslant 0 \quad \textit{and} \quad \frac{\partial E[R]}{\partial c_{S+1}} < 0, \quad \textit{or} \quad \frac{\partial E[R]}{\partial c_S} > 0 \quad \textit{and} \quad \frac{\partial E[R]}{\partial c_{S+1}} = 0.$$

4. The optimal reserve price

Interestingly, setting optimal reserve prices reverses the unintuitive result that increasing the click–through rate of lower positions might decrease revenue.[11] In the GSP with reserve price r, all bidders who bid less than r are eliminated and no bidder pays less than r for a click in his link. Formally, the total payment of the s-th highest bidder, with bid $b^{(s)} \geqslant r$ and $s \leqslant S$, is $c_s \cdot \max\{b^{(s+1)}, r\}$.

We say that an allocation of positions to bidders is *quasi-efficient* if it assigns higher positions to bidders with higher values, and no position is left vacant when there is an advertiser with value $v \geqslant r$ who was not assigned any position. We refer to an equilibrium that implements a quasi-efficient allocation as a quasi-efficient equilibrium.

We start the analysis of the GSP with reserve price r by deriving its unique quasi-efficient Bayes–Nash equilibrium.

[11] See Ostrovsky and Schwarz (2010) for a field experiment that quantifies the effect of reserve prices on the search engine's revenue.

Proposition 4. *Consider the generalized second-price auction (GSP) with N bidders, S positions ($N > S$) and a uniform reserve price $r \geqslant 0$. If a quasi-efficient Bayes–Nash equilibrium exists for this auction, then its symmetric bidding strategy is*

$$\hat{\beta}(v,r) = v - \hat{\phi}(v,r) - \sum_{n=1}^{\infty} \int_r^v K_n(v,t)\hat{\phi}(t,r)\,dt, \tag{16}$$

where

$$\hat{\phi}(v,r) = \frac{\sum_{s=1}^{s=S} \binom{N-2}{s-1}(s-1)(1-F(v))^{s-2} \int_r^v F^{N-s}(x)\,dx\, c_s}{\sum_{t=1}^{t=S} \binom{N-2}{t-1}(1-F(v))^{t-1} F^{N-t-1}(v) c_t}, \tag{17}$$

$K_1(v,t)$ is given by (3), $K_n(v,t)$ is defined by (4) and $\hat{\phi}(v,\cdot) \in L^2([0,\bar{v}])$ and $K_1(v,t) \in L^2([0,\bar{v}]^2)$.

If $\hat{\beta}(v,r)$ defined by (16) is strictly increasing in v, then a quasi-efficient Bayes–Nash equilibrium exists for the GSP. Otherwise, this auction does not admit a quasi-efficient equilibrium.

The revenue generated by the GSP in the quasi-efficient equilibrium is[12]*:*

$$\sum_{s=1}^{S} \int_r^{\bar{v}} c_s \cdot z_s(t) \cdot \left(t - \frac{(1-F(t))}{f(t)}\right) f(t)\,dt, \tag{18}$$

where $z_s(\cdot)$ is given by (6).

If the distribution F satisfies the increasing hazard rate condition, the revenue-maximizing reserve prices r^* satisfies[13]*:*

$$r^* = \frac{(1-F(r^*))}{f(r^*)} \quad \text{for all } s = 1, \ldots, S. \tag{19}$$

If the GSP with reserve price r^* admits a quasi-efficient equilibrium, then it constitutes the revenue-maximizing mechanism for selling S positions to N bidders.

Proof. See Appendix A. □

We can now state our main result about reserve prices:

Proposition 5. *Consider the GSP with N bidders, S positions and reserve price r^* given by (19). Assume that the distribution F satisfies the increasing hazard rate condition. In the quasi-efficient equilibrium of this auction, the search engine's expected revenue increases with the click–through rates of all positions $s = 1, \ldots, S$.*

Proof. Differentiating (18) leads to

$$\frac{\partial E[R]}{\partial c_s} = \int_{r^*}^{\bar{v}} \binom{N-1}{s-1} \big(1-F(v)\big)^{s-1} F^{N-s}(v) \left(v - \frac{(1-F(v))}{f(v)}\right) f(v)\,dv.$$

The reserve price r^* as in (19) assures that virtual valuations $v - \frac{(1-F(v))}{f(v)}$ are strictly positive for all $v \geqslant r^*$. As a consequence, $\frac{\partial E[R]}{\partial c_s} > 0$ for all s. □

Intuitively, reserve prices simply remove bidders with low values from the auction. This implies that bidders with high values per click don't need to bid too low to have a chance of obtaining lower positions. As it turns out, this reduces equilibrium bid shading to a point such that the supply effect unambiguously dominates the strategic effect, and revenue increases with all click–through rates.

[12] Lahaie and Pennock (2007) derive a similar formula for the ex-ante revenue (before values are realized) generated by the complete information Nash equilibrium studied by Edelman et al. (2007) and Varian (2007). See also Ulku (2009) for revenue equivalence results in the context of combinatorial auctions.

[13] Iyengar and Kumar (2006) and Roughgarden and Sundararajan (2007) extend Myerson (1981) to a multi-unit setting, and derive the optimal reserve price to any auction that possesses a quasi-efficient equilibrium. The same formula is obtained by Edelman and Schwarz (2010) in the context of the Generalized English auction, which is revenue-equivalent to the Bayes–Nash GSP.

5. Nonexistence of symmetric inefficient equilibrium

As we have highlighted in previous sections, the GSP may fail to possess an efficient equilibrium for a wide range of parameter values. In such cases, one might conjecture that the GSP possesses a *symmetric inefficient* equilibrium; i.e. all bidders are using the same bid function, but either randomize or pool on certain bids. The main result of this section shows this conjecture to be false. Namely, all symmetric equilibria are essentially pure strategy equilibria, and pooling cannot be sustained in equilibrium. As a consequence, if there is no efficient equilibrium, then there are no symmetric equilibria.

Our first step toward ruling out symmetric inefficient equilibria is simply to note that all symmetric equilibria are essentially pure, monotone strategy equilibria. Both the purity and monotonicity follow from a straightforward and standard single-crossing argument.[14] We call the strategies "essentially" pure because the single-crossing argument does not preclude a bidder from randomizing at the jumps.[15] We record the above mentioned results in the following lemma.

Lemma 1. *All symmetric equilibria of the GSP are outcome equivalent to a pure strategy equilibria.*

Now the only possible source of inefficiency – when bidders use symmetric, monotone increasing bid functions – is pooling; i.e. every type in an interval, say $[\underline{w}, \overline{w}] \subseteq [0, \bar{v}]$, submits the same bid $\hat{\beta}$. Ties will be broken at random, and this introduces inefficiency. Before getting to the main proposition, we provide an example showing why pooling fails to be a viable equilibrium candidate in the simple case of 3 bidders and 2 positions.

Example 4. Consider the GSP with $S = 2$ positions, $N = 3$ bidders and click–through rates normalized to $(1, c_2)$. Assume there exists a symmetric equilibrium with pooling at some interval and consider the lowest interval of pooling $[\underline{w}, \overline{w}]$, with associated bid $\hat{\beta}$. We proceed in two steps: (1) if c_2 is too small ($< \frac{1}{2}$), then $\hat{\beta}$ is dominated by a convex combination of slightly overbidding $\hat{\beta} + \varepsilon$ and slightly underbidding $\hat{\beta} - \varepsilon$, and (2) if c_2 is too large ($\geqslant \frac{1}{2}$), then $\hat{\beta}$ is too small relative to the rest of the $\beta(\cdot)$ and the bid function is necessarily non-monotone. We rule out pooling by piecing these together.

(1) It is clear there cannot be any pooling at 0 since any type in the pooling interval would strictly prefer to bid $\varepsilon > 0$. Thus, there will necessarily be separation at the bottom, that is $\underline{w} > 0$. Each bidder with type $v \in [\underline{w}, \overline{w}]$ must weakly prefer bidding $\hat{\beta}$ to bidding $\hat{\beta} + \varepsilon$ and $\hat{\beta} - \varepsilon$. That is, the following constraints must hold.

The condition $u(\hat{\beta}) - u(\hat{\beta} + \varepsilon) \geqslant 0$ implies

$$\begin{aligned}
&\Pr\{v^{1:2}, v^{2:2} \in [\underline{w}, \overline{w}]\} \cdot \left[\frac{1+c_2}{3}(v - \hat{\beta}) - (v - \hat{\beta})\right] \\
&\quad + \Pr\{v^{1:2} \in [\underline{w}, \overline{w}], v^{2:2} < \underline{w}\} \cdot \left[\frac{1}{2}(v - \hat{\beta}) + \frac{c_2}{2}\big(v - E\big[\beta(x) \,\big|\, x \leqslant \underline{w}\big]\big) - (v - \hat{\beta})\right] \\
&\quad + \Pr\{v^{2:2} \in [\underline{w}, \overline{w}], v^{1:2} > \overline{w}\} \cdot \left[\frac{c_2}{2}(v - \hat{\beta}) - c_2(v - \hat{\beta})\right] \geqslant 0,
\end{aligned}$$

and $u(\hat{\beta}) - u(\hat{\beta} - \varepsilon) \geqslant 0$ implies

$$\begin{aligned}
&\Pr\{v^{1:2}, v^{2:2} \in [\underline{w}, \overline{w}]\} \cdot \left[\frac{1+c_2}{3}(v - \hat{\beta})\right] + \Pr\{v^{1:2} \in [\underline{w}, \overline{w}], v^{2:2} < \underline{w}\} \cdot \left[\frac{1}{2}(v - \hat{\beta}) - \frac{c_2}{2}\big(v - E\big[\beta(x) \,\big|\, x \leqslant \underline{w}\big]\big)\right] \\
&\quad + \Pr\{v^{2:2} \in [\underline{w}, \overline{w}], v^{1:2} > \overline{w}\} \cdot \left[\frac{c_2}{2}(v - \hat{\beta})\right] \geqslant 0,
\end{aligned}$$

which when summed yield

$$\Pr\{v^{1:2}, v^{2:2} \in [\underline{w}, \overline{w}]\} \cdot \left[\frac{2c_2 - 1}{3}(v - \hat{\beta})\right] \geqslant 0,$$

which implies the necessary condition of $c_2 \geqslant \frac{1}{2}$.

(2) The integral equation identified in Proposition 1 pins down the equilibrium bidding function whenever it is strictly increasing. Therefore, evaluated at $\underline{w}$ we have that $\beta(\underline{w}) = \underline{w} - \frac{c_2}{F(\underline{w}) + (1 - F(\underline{w}))c_2} \int_0^{\underline{w}} (\underline{w} - \beta(x)) f(x)\,dx$. Optimality requires

[14] A function $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies single-crossing if for $b' > b$ and $v' > v$ we have $h(b', v) - h(b, v) \geqslant (>) 0 \Rightarrow h(b', v') - h(b, v') \geqslant (>) 0$. That is, if we consider the $g(v|b', b) = h(b', v) - h(b, v)$ as the incremental change as function v (given b' and b), then g crosses 0 at most once and from below. In our GSP setting, interpret a bid as b, the value-per-click as v and h as the expected payoff from bidding b with value v. It is straightforward to verify that if a type v prefers a higher bid b' to a lower bid b, then so do all higher types v'; i.e. single-crossing is satisfied in our GSP setting.

[15] The set of points where an increasing function jumps upward is countable and of (Lebesgue) measure zero.

that the lowest type $\underline{w}$ must be indifferent between $\beta(\underline{w})$ and $\hat{\beta}$. This pins down $\hat{\beta}$: $\hat{\beta} = \underline{w} - \frac{c_2}{F(\underline{w})+(F(\overline{w})-F(\underline{w}))\frac{1+c_2}{3}} \times \int_0^{\underline{w}} (\underline{w} - \beta(x)) f(x)\,dx$. As a consequence:

$$\hat{\beta} - \beta(\underline{w}) = c_2 \int_0^{\underline{w}} \big(\underline{w} - \beta(x)\big) f(x)\,dx \left[\frac{1}{F(\underline{w}) + (1 - F(\underline{w}))c_2} - \frac{1}{F(\underline{w}) + (F(\overline{w}) - F(\underline{w}))\frac{1+c_2}{3}}\right].$$

Therefore, $\hat{\beta} - \beta(\underline{w}) > 0$ if and only if

$$F(\underline{w}) + \big(F(\overline{w}) - F(\underline{w})\big)\frac{1+c_2}{3} > F(\underline{w}) + \big(1 - F(\underline{w})\big)c_2,$$

which is equivalent to

$$\frac{3c_2}{1+c_2} < \frac{(F(\overline{w}) - F(\underline{w}))}{(1 - F(\underline{w}))} < 1,$$

what implies that $c_2 < \frac{1}{2}$. This contradicts our previous necessary condition, and allows us to conclude that the GSP with 2 positions and 3 bidders possesses no pure strategy inefficient equilibrium.

The general insight exhibited in the above example is generalized in the following proposition.

Proposition 6. *Consider the generalized second-price auction (GSP) with N bidders, S positions (N > S) and click–through rates $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_S$. If this auction does not possess an efficient equilibrium, then there exists no symmetric equilibrium.*

Proof. See Appendix A. □

Our analysis shows that the GSP is a simple, symmetric bidding game where the single-crossing condition applied by Athey (2001) and McAdams (2003) to games with a discrete action space is satisfied and, still, symmetric equilibria fail to exist when the action space is continuous. To gain some intuition for the result, suppose a bidder is bidding $\hat{\beta}$ in a pooling region $[\underline{w}, \overline{w}]$. If none of the other bidders have a type realization in $[\underline{w}, \overline{w}]$, then $\hat{\beta}$ yields the same payoff as either slightly overbidding $\hat{\beta} + \varepsilon$ or slightly underbidding $\hat{\beta} - \varepsilon$. The actual benefit from bidding $\hat{\beta}$ (as opposed to $\hat{\beta} + \varepsilon$) is that he may tie with some other bidders *and* be randomly chosen last among all bidders with which he is tied. In which case he would simply be competing with the types below v: getting a lower position but paying less. Note that, however, if he preferred to compete with the lower bidders, he could have guaranteed this competition by simply underbidding $\hat{\beta} - \varepsilon$.

In a discrete version of the GSP game (with discrete type spaces and discrete action spaces), such epsilon-deviations are not possible, and the results in Athey (2001) and McAdams (2003) imply that there exist monotone Bayes–Nash equilibria (but not necessarily efficient, since there is pooling). Denote by $v_1 \leqslant v_2 \leqslant \cdots \leqslant v_N$ the support of types and by $\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_K$ the support of actions. A monotone equilibrium associates to each type $v \in \{v_1, \ldots, v_N\}$ a bid $\beta(v) \in \{\beta_1, \ldots, \beta_K\}$ such that $\beta(v) \leqslant \beta(\hat{v})$ if and only if $v < \hat{v}$. Computing such equilibria for generic discrete supports of actions and types require, however, numerical methods which are out of the scope of this work.

6. Discussion

The model of the GSP studied in this paper makes the following assumptions: First, the game is static, and advertisers bid only once. Second, advertisers have no information about each others' values per click, except for the distribution from which they are drawn (i.e., we follow a Bayesian approach). Third, the advertisers' values per click are the same across positions and the click–through rates are homogeneous across advertisers (i.e., private information is unidimensional). Fourth, the advertisers' values are independently and identically distributed (i.e., the model is ex-ante symmetric).

In practice, the GSP is run as a continuous-time auction in which advertisers can continuously revise their bids. Arguably, a realistic model of this auction would entail a dynamic game where advertisers partially learn each others' values as time elapses. Moreover, click–through rates are likely to be advertiser-specific, and values per click may well depend on the position in which advertisers are displayed (see, for example, Borgers et al., 2008 and Blumrosen et al., 2008, among others).

Although its obvious limitations, we believe that the model of this paper offers a useful benchmark for studying the GSP auction. First, the i.i.d. private-values Bayesian setting is an important (perhaps canonical) model of auction theory, therefore providing a common ground for comparing the GSP with alternative multi-unit auction formats (such as the Vickrey–Clark–Groves mechanism and the generalized first-price auction). Second, the assumption of i.i.d. values is likely to hold for "broad" keywords (where advertisers are sufficiently differentiated) or for novel or infrequent keywords (where the advertisers estimates of the value of a click are sufficiently imprecise). Third, the static nature of our model is likely to be a good approximation of reality for keywords where dynamic learning about the other advertisers' private information is slow (e.g., when the keyword being auctioned is either infrequent or new). Fourth, in the model of this paper, advertisers can only condition their bids on their own private information. Arguably, advertisers in reality are likely to lack computational (or even cognitive) resources to employ more complex bidding strategies, such as conditioning one's bid on the previous allocations of slots (as a dynamic model of the GSP would require), or on the other advertisers' characteristics (as required in complete information models). In such cases, the Bayesian approach appears to be a natural modeling option.

Yet, the analysis developed above is worth extending in a number of interesting directions. In what follows, we will briefly describe various lines for future research.

Multi-dimensional private information

The private information of advertisers requires a multi-dimensional representation whenever (i) click–through rates are advertiser-specific (and unknown to the platform), or (ii) the values per click of advertisers are position-specific. A Bayes–Nash analysis of the GSP under either (i) or (ii) (or both) is an important (and yet challenging) step forward relative to the analysis of this paper.

As pointed out by Benisch et al. (2008), the gap between the dimension of the advertisers' types and the dimension of a mechanism's bidding language may lead to large efficiency losses. On this regard, the main result of our analysis (namely, that the Bayes–Nash GSP fails to possess an efficient equilibrium under a wide range of parameter values) is likely to be reinforced when types are multi-dimensional. On top of the bid shading phenomenon documented here (which is a feature of the "next-price" payment rule of the GSP), the simplicity of the GSP's bidding language is likely to render more difficult sustaining the efficient allocation as a Bayes–Nash equilibrium.

Correlated valuations and asymmetric bidders

The analysis of the paper assumed that advertisers' valuations are independent and identically distributed. This important assumption allows us to derive the bidding functions that implement the efficient allocation by invoking the Revenue Equivalence Principle. As is well know, this principle no longer holds when either the independence or the symmetry assumptions are relaxed. As such, extending the analysis of this paper to settings with either correlation or asymmetric distributions is an interesting problem that requires techniques that go beyond the scope of this paper (see, for example, Maskin and Riley, 2000 and Lucier and Paes Leme, 2011).

Budget constraints and externalities

In practice, advertisers face budget constraints when submitting their bids. It is an interesting direction of future research to study Bayes–Nash equilibria of the GSP with budget-constrained advertisers.

It is also an important open problem to study the GSP when advertisers impose allocative externalities on each other (e.g., when click–through rates depend on the identity of all the advertisers displayed in the sponsored list).[16]

Appendix A

Proof of Example 1. The analytic solution to the differential equation (10) with boundary condition $\beta(0) = 0$ is given by (for $c_2 \neq \frac{1}{2}, \frac{2}{3}$)

$$\beta(v) = \big(v + (1 - c_2) \cdot v\big)^{\frac{1}{1-c_2}} \cdot \frac{B(c_2, v)}{A(c_2, v)},$$

where

$$A(c_2, v) \equiv \frac{1}{6 \cdot (c_2 - \frac{1}{2}) \cdot (c_2 - \frac{2}{3}) \cdot \big(\frac{v+(1-c_2)\cdot v}{c_2}\big)^{\frac{1}{1-c_2}}}$$

and

$$B(c_2, v) = \big(c_2^{\frac{2c_2-1}{c_2-1}} - 8c_2^{\frac{c_2}{c_2-1}} \cdot v + 8c_2^{\frac{2c_2-1}{c_2-1}} \cdot v + 2c_2^{\frac{1}{c_2-1}} \cdot v - c_2^{\frac{4c_2-2}{c_2-1}} \cdot \big(v + (1 - c_2) \cdot v\big)^{\frac{2c_2-1}{1-c_2}}\big).$$

When $c_2 = \frac{1}{2}$, the solution is simply $\beta(v) = \log(1 + v)$. When $c_2 = \frac{2}{3}$, the solution is

$$\beta(v) = 3v + 4\log(2) + 2v\log(2) - 4\log(2 + v) - 2v\log(2 + v).$$

It is straight-forward (but long) to compute $\beta'(v)$ and see that $\beta'(v) \geqslant 0$ for all $v \in [0, 1]$ if and only if $c_2 \leqslant \frac{3}{4}$. □

Proof of Proposition 4. By the same argument in the proof of Proposition 1, the expected payment of a bidder with value per click $v \geqslant r$ in any quasi-efficient equilibrium of the GSP with reserve price r is given by

$$E\big[P^{PE}(v)\big] = \sum_{s=1}^{s=S} c_s \left\{ z_s^i(v)v - \int_r^v z_s^i(t)\,dt \right\} = \sum_{s=1}^{s=S} c_s \int_r^v \frac{dz_s^i(t)}{dt} t\,dt + z_s^i(r)r.$$

[16] See Constantin et al. (2010) and the references therein for an in-depth discussion of this topic.

From the rules of the GSP, we know that the expected payment for a bidder with value per click $v \geqslant r$ is given by

$$E\big[P^{GSP}(v)\big] = \sum_{s=1}^{s=S} c_s \cdot \binom{N-1}{s-1}\big(1-F(v)\big)^{s-1} \int_0^r r(N-s)F^{N-s-1}(t)f(t)\,dt$$
$$+ \sum_{s=1}^{s=S} c_s \cdot \binom{N-1}{s-1}\big(1-F(v)\big)^{s-1} \int_r^v \hat{\beta}(t,r)(N-s)F^{N-s-1}(t)f(t)\,dt.$$

Since $E[P^{PE}(v)] = E[P^{GSP}(v)]$ for all $v \geqslant r$, we can differentiate both expressions to obtain that

$$\sum_{s=1}^{s=S} c_s \frac{dz_s^i(v)}{dv} v = \hat{\beta}(v,r) \sum_{s=1}^{s=S} \binom{N-1}{s-1}\big(1-F(v)\big)^{s-1} c_s (N-s)F^{N-s-1}(v)f(v)$$
$$- \int_r^v \hat{\beta}(t,r) \sum_{s=1}^{s=S} \binom{N-1}{s-1}(s-1)\big(1-F(v)\big)^{s-2} f(v) c_s (N-s)F^{N-s-1}(t)f(t)\,dt$$
$$- r\int_0^r \sum_{s=1}^{s=S} \binom{N-1}{s-1}(s-1)\big(1-F(v)\big)^{s-2} f(v) c_s (N-s)F^{N-s-1}(t)f(t)\,dt.$$

Following the same steps as in the proof of Proposition 1 we can rearrange the equality above to obtain the following Volterra equation of the second kind:

$$v - \hat{\beta}(v,r) = \hat{\phi}(v,r) + \int_r^v K_1(v,t)\big(t - \hat{\beta}(t,r)\big)\,dt,$$

where $\hat{\phi}(v,r)$ is given by (17) and $K_1(v,t)$ is given by (3). From Debnath and Mikusinski (1999) (p. 235, Theorem 5.5.1) we know that the Volterra equation above has a unique solution given by (16) provided that the functions $\hat{\phi}(v,\cdot) \in L^2([0,\bar{v}])$ and $K_1(v,t) \in L^2([0,\bar{v}]^2)$. As in the proof of Proposition 1, we can apply the Constraint Simplification Theorem (Milgrom, 2004, p. 105) to conclude that an efficient Bayes–Nash equilibrium exists if and only if (16) is strictly increasing in v.

The total revenue from a quasi-efficient equilibrium equals

$$N\int_r^{\bar{v}} E\big[P^{PE}(v)\big]f(v)\,dv = \sum_{s=1}^{S} \int_r^{\bar{v}} c_s \cdot z_s(t) \cdot \left(t - \frac{(1-F(t))}{f(t)}\right) f(t)\,dt,$$

after changing the order of integration. It is then immediate from the formula above that r^* has to satisfy $r^* - \frac{(1-F(r^*))}{f(r^*)}$. $\square$

Proof of Proposition 6. Fix $N+1$ bidders and click vector $c = (c_1, c_2, \dots, c_N, 0)$. Suppose all bidders are using the same (monotone) bid function $\beta(\cdot)$ where exactly the interval of types $(v_*, v^*]$ pool on a common $\hat{\beta}$. Further suppose that $\hat{\beta}$ is the lowest pooling bid so that $\beta(\cdot)$ is strictly increasing on $[0, v_*]$. In particular, if $\beta(\cdot)$ is an equilibrium strategy, $\beta(\cdot)$ must satisfy the integral equation found in the proof of Proposition 1 on the domain of $[0, v_*]$. Similarly, the highest non-pooling type v_* must be exactly indifferent between bidding $\beta(v_*)$ and pooling at $\hat{\beta}$; i.e. $u(\beta(v_*) \mid v_*) = u(\hat{\beta} \mid v_*)$. This implies

$$\sum_{a=0}^{a=N}\sum_{t=1}^{t=N-a} p_{at}\frac{c_{a+1}+\dots+c_{a+t}}{t+1}(v_* - \hat{\beta}) = \sum_{a=0}^{a=N}\sum_{t=1}^{t=N-a} p_{at}\frac{tc_{t+1}}{t+1}\big(v_* - E\big[\beta\big(v^{1:N-a-t}\big) \mid v^{1:N-a-t} < v_*\big]\big),$$

where $p_{at} = \binom{N}{a}\binom{N-a}{t}(1-F(v^*))^a(F(v^*)-F(v_*))^t F^{N-a-t}(v_*)$ is the probability that t bidders submit bids tying $\hat{\beta}$ and a bidders submit bids above $\hat{\beta}$.

If we think of $\beta(\cdot)$ on $[0, v_*]$ to be fixed and consider varying the upper bound of the pooling region, the equality above (along with the choice of v^*) uniquely pins down the pooling bid $\hat{\beta}$. Let $\hat{\beta}|_v$ be the pooling bid corresponding to any given choice of v. Two important notes: (i) If $v^* = v_*$ then $\hat{\beta}|_{v^*=v_*} = \beta(v_*)$ and (ii) if $\beta(\cdot)$ is incentive compatible for all types in $[0, v^*]$, then $\beta^\dagger(\cdot)$ is incentive compatible on $[0, v]$ where

$$\beta^\dagger = \begin{cases} \beta & \text{on } [0, v_*], \\ \hat{\beta}|_v & \text{on } [v_*, v]. \end{cases}$$

Both of notes (i) and (ii) follow from payoff equivalence and the continuity of the indirect utility function. The remainder of the proof shows that $\hat{\beta}|_v$ – as a function of v – is decreasing for $v \approx v_*$. Thus, in some neighborhood of v_* we can construct a $\beta^\dagger$ which is non-monotone and, therefore, not incentive compatible. This contradicts our supposition that $\beta(\cdot)$ was an equilibrium.

Totally differentiate the indifference equation with respect to the upper bound v, and evaluate at $v = v_*$ to yield:

$$\sum_{a=0}^{a=N}\sum_{t=1}^{t=N-a}\left[\frac{d}{dv}p_{at}\right]\frac{c_{a+1}+\cdots+c_{a+t}}{t+1}\big(v_*-\beta(v_*)\big)-\sum_{a=0}^{a=N}\sum_{t=1}^{t=N-a}p_{at}\frac{c_{a+1}+\cdots+c_{a+t}}{t+1}\left[\frac{d}{dv}\hat{\beta}|_v\right]$$
$$=\sum_{a=0}^{a=N}\sum_{t=1}^{t=N-a}\left[\frac{d}{dv}p_{at}\right]\frac{tc_{t+1}}{t+1}\big(v_*-E\big[\beta\big(v^{1:N-a-t}\big)\mid v^{1:N-a-t}<v_*\big]\big).$$

We want to show that $[\frac{d}{dv}\hat{\beta}]<0$ or

$$\sum_{a=0}^{a=N}\sum_{t=1}^{t=N-a}\left[\frac{d}{dv}p_{at}\right]\frac{c_{a+1}+\cdots+c_{a+t}}{t+1}\big(v_*-\beta(v_*)\big)$$
$$<\sum_{a=0}^{a=N}\sum_{t=1}^{t=N-a}\left[\frac{d}{dv}p_{at}\right]\frac{tc_{t+1}}{t+1}\big(v_*-E\big[\beta\big(v^{1:N-a-t}\big)\mid v^{1:N-a-t}<v_*\big]\big).$$

The remaining derivative has a nice structure:

$$\left[\frac{d}{dv}p_{at}\right]_{v=v_*}=N\binom{N-1}{a}\big(1-F(v_*)\big)^aF^{N-a-1}(v_*).$$

The desired inequality now becomes:

$$\sum_{a=0}^{a=N}\sum_{t=1}^{t=N-a}\binom{N-1}{a}\big(1-F(v_*)\big)^aF^{N-a-1}(v_*)\frac{c_{a+1}+\cdots+c_{a+t}}{t+1}\big(v_*-\beta(v_*)\big)$$
$$<\sum_{a=0}^{a=N}\sum_{t=1}^{t=N-a}\binom{N-1}{a}\big(1-F(v_*)\big)^aF^{N-a-1}(v_*)\frac{tc_{t+1}}{t+1}\big(v_*-E\big[\beta\big(v^{1:N-a-t}\big)\mid v^{1:N-a-t}<v_*\big]\big).$$

From the integral equation derivation of $\beta(\cdot)$ on $[0, v_*]$, we have

$$\big(v_*-\beta(v_*)\big)\sum_{k=0}^{k=N-1}\binom{N-1}{k}\big(1-F(v_*)\big)^kF^{N-1-k}(v_*)c_{k+1}$$
$$=\sum_{k=1}^{N}\binom{N-1}{k-1}\big(1-F(v_*)\big)^{k-1}F^{N-k}(v_*)c_{k+1}\big(v_*-E\big[\beta\big(v^{1:N-k}\big)\mid v^{1:N-k}<v_*\big]\big),$$

where we have intentionally changed the indexing variable to k to avoid confusion below. Note that $(v_* - \beta(v_*))$ appears in both the desired inequality as well as the integral equation. With this in hand, the following string of manipulations verifies the desired inequality. Throughout, we will make repeated use of the following lemma from McAfee (2002):

Lemma 2 *(Covariance of non-decreasing functions non-negative). (See McAfee (2002).) Let Y, Z be random variables formed by non-decreasing functions y, z of a real-valued random variable X. Let F be the distribution of X. Suppose that the variances of Y and Z are finite. Then $\operatorname{cov}(Y, Z) \geqslant 0$.*

$$\frac{\sum_{a=0}^{a=N-1}\sum_{t=1}^{t=N-a}\binom{N-1}{a}(1-F(v_*))^aF^{N-a-1}(v_*)\frac{tc_{t+1}}{t+1}(v_*-E[\beta(v^{1:N-a-t})|v^{1:N-a-t}<v_*])}{\sum_{a=0}^{a=N-1}\sum_{t=1}^{t=N-a}\binom{N-1}{a}(1-F(v_*))^aF^{N-a-1}(v_*)\frac{c_{a+1}+\cdots+c_{a+t}}{t+1}}$$
$$>\frac{\sum_{a=0}^{a=N-1}\sum_{t=1}^{t=N-a}\binom{N-1}{a}(1-F(v_*))^aF^{N-a-1}(v_*)\frac{tc_{t+1}}{t+1}(v_*-E[\beta(v^{1:N-a-t})|v^{1:N-a-t}<v_*])}{\sum_{a=0}^{a=N-1}\sum_{t=1}^{t=N-a}\binom{N-1}{a}(1-F(v_*))^aF^{N-a-1}(v_*)\frac{tc_{a+1}}{t+1}}$$
$$>\frac{\sum_{a=0}^{a=N-1}\sum_{t=1}^{t=N-a}\binom{N-1}{a}(1-F(v_*))^aF^{N-a-1}(v_*)c_{t+1}(v_*-E[\beta(v^{1:N-a-t})|v^{1:N-a-t}<v_*])}{\sum_{a=0}^{a=N-1}\sum_{t=1}^{t=N-a}\binom{N-1}{a}(1-F(v_*))^aF^{N-a-1}(v_*)c_{a+1}}$$
$$=\frac{\sum_{a=0}^{a=N-1}\binom{N-1}{a}(1-F(v_*))^aF^{N-a-1}(v_*)\sum_{t=1}^{t=N-a}c_{t+1}(v_*-E[\beta(v^{1:N-a-t})|v^{1:N-a-t}<v_*])}{\sum_{a=0}^{a=N-1}\binom{N-1}{a}(N-a-1)(1-F(v_*))^aF^{N-a-1}(v_*)c_{a+1}}$$

$$= \frac{\sum_{a=0}^{a=N-1} \binom{N-1}{a}(1-F(v_*))^a F^{N-a-1}(v_*) \sum_{t=1}^{t=N-a} c_{t+1}(v_* - E[\beta(v^{1:N-a-t})|v^{1:N-a-t} < v_*])}{(N-1)\sum_{a=0}^{a=N-1} \binom{N-2}{a}(1-F(v_*))^a F^{N-a-1}(v_*)c_{a+1}}$$

$$= \frac{\sum_{a=0}^{a=N-1} \frac{1}{N-1}\binom{N-1}{a}(1-F(v_*))^a F^{N-a-1}(v_*) \sum_{t=1}^{t=N-a} c_{t+1}(v_* - E[\beta(v^{1:N-a-t})|v^{1:N-a-t} < v_*])}{\sum_{a=0}^{a=N-1} \binom{N-2}{a}(1-F(v_*))^a F^{N-a-1}(v_*)c_{a+1}}$$

$$= \frac{\sum_{a=0}^{a=N-1} \binom{N-2}{a}(1-F(v_*))^a F^{N-a-1}(v_*) \sum_{t=1}^{t=N-a} \frac{c_{t+1}}{(N-a-1)}(v_* - E[\beta(v^{1:N-a-t})|v^{1:N-a-t} < v_*])}{\sum_{a=0}^{a=N-1} \binom{N-2}{a}(1-F(v_*))^a F^{N-a-1}(v_*)c_{a+1}}$$

$$> \frac{\sum_{k=1}^{N} \binom{N-1}{k-1}(1-F(v_*))^{k-1} F^{N-k}(v_*)c_{k+1}(v_* - E[\beta(v^{1:N-k})|v^{1:N-k} < v_*])}{\sum_{k=0}^{k=N-1} \binom{N-1}{k}(1-F(v_*))^k F^{N-1-k}(v_*)c_{k+1}}.$$

The relation between the first and last lines establishes the desired inequality. Recall that the goal of these tedious manipulations was simply to show that $[\frac{d}{dv}\hat{\beta}|_v] < 0$ in neighborhood $v = v_*$. Given that $\hat{\beta}|_v$ is locally decreasing, our original $\beta(\cdot)$ is not incentive compatible. We conclude that there cannot be pooling in any symmetric equilibrium. This, together with the previous monotonicity result, concludes the proof of the proposition: If no efficient equilibrium exists, then no symmetric equilibrium exists. □

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