Midterm examination

Mathematical programming with applications to economics

September 10, 2018; Time: **10 AM to 2 PM**.

Be precise and clear in your arguments.

1. Let $G = (B \cup L, E)$ be an undirected bipartite graph such that for each edge $\{i, j\} \in E$, we have $i \in B$ and $j \in L$. Use the max-flow min-cut theorem to show that the size of the maximum matching equals the size of the minimum vertex cover of G (Konig's theorem). **(10 marks)**

   Answer. (Since most of you got this correct, I only give an outline).

   We construct a flow graph as follows. The flow graph G' has two new vertices $\{s, t\}$. The direction of every edge $\{i, j\}$ in G with $i \in B$ and $j \in L$ is (i, j). There is an edge (s, i) for every $i \in B$ and an edge (j, t) for every $j \in L$. The capacities of the new edges are 1 and the old edges are arbitrarily large integer (greater than $|B| + |L|$).

   By Ford-Fulkerson, the maximum flow of this graph is integral. Hence, each edge either carries zero or 1 unit of flow. This is a matching. Since every matching can be converted to a flow of G', we get that the maximum flow of G' is the maximum matching.

   Similarly, we can argue that the minimum capacity cut is the minimum vertex cover.

2. Consider an undirected graph G with n vertices and e edges. Suppose G is connected. Show the following.

   (a) $e \geq n - 1$. **(2 marks)**

   Answer. Since G is connected, there exists a subgraph G' of G which is a spanning tree of G. Such a spanning tree must have $(n - 1)$ edges. Hence, $e \geq n - 1$.

   (b) If G has exactly one cycle, then $e = n$. **(4 marks)**

   Answer. Suppose G is connected and has one cycle. Pick any edge $\{i, j\}$ in the cycle and delete it. Then, the resulting graph, call it G', has no cycle. Note that G' can be partitioned into $G^1, \ldots, G^q$ component subgraphs each of which is connected and acyclic. Then, we know that the number of edges in each G^k is $n_k - 1$, where n_k is the number of vertices in G^k. As a result, the total number of edges in G' is $\sum_{k=1}^{q} n_k - q = n - q$. Hence, $n - q = e - 1$, where e is the number of edges in G. As a result, $n = e + (q - 1) \geq e$. By the earlier question, $n \leq e + 1$.

Hence, $n \in \{e, e+1\}$. If $n = e+1$, then G is a tree (and, hence, acyclic). Since G has a cycle, it must be that $n = e$.

(c) If $e = n$, then G has exactly one cycle. **(4 marks)**

Answer. Let $d(i)$ denote the degree of vertex i. Since $n = e$, we get $2n = 2e = \sum_{i \in N} d(i)$. Since the graph is connected, every vertex has non-zero degree. We consider two cases.

Case 1. Suppose each vertex has degree at least two. Then, we get, $2n = \sum_{i \in N} d(i) \geq 2n$. Hence, the only way this is possible is if each vertex has degree exactly 1. Then, the graph has exactly one cycle (involving all the vertices).

Case 2. The other possibility is G has *some* vertex i of degree 1. Then, we can use induction. The base case when $n = 3$ is obvious. Suppose the claim is true when $n = K - 1$ and consider $n = K$. Then, since the graph has one vertex i with degree 1. Deleting that vertex and the unique edge to which it is the end point gives us a connected graph with $K - 1$ vertices and $K - 1$ edges. Hence, this must have exactly one cycle. Adding the deleted edge and vertex does not create a new cycle. So, G also has exactly one cycle.

3. Let $G = (N, E)$ be an undirected bipartite graph, i.e., there exists a partition $B \cup H$ of N such that for every $\{i, j\} \in E$, we have $i \in B$ and $j \in H$. We say a bipartite graph G is k-regular, where k is a positive integer, if degree of every vertex is k.

(a) Show that if G is a k-regular bipartite graph, then $|B| = |H|$. **(2 marks)**

Answer. Since G is k-regular, the sum of degree of vertices on B side is $k|B|$, which is equal to the sum of degree of vertices on H side. Hence, $k|B| = k|H|$. This implies that $|B| = |H|$.

(b) Show that if G is a k-regular bipartite graph, then the size of maximum matching in G is $|B|$. **(8 marks)**

Answer. Let C be a minimum vertex cover. Clearly, B is a vertex cover. Hence, $|C| \leq |B|$. Since each vertex has degree k, the number of edges C covers is less than or equal to $k|C|$. Since the number of edges in the graph is $k|B|$ and C is a vertex cover, this implies $k|B| \leq k|C|$ or $|B| \leq |C|$. Hence, $|C| = |B|$. So, B is a minimum vertex cover. By Konig's theorem, the size of maximum matching in G is $|B|$.

4. Consider a directed graph $G = (N, E)$ with two unique vertices s and t such that there is a directed path from s to t. Let φ be a function $\varphi : N \to \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers) such that for every edge $(i, j) \in E$, $\varphi(j) - \varphi(i) \leq 1$.

   (a) Show that such a φ function always exists such that $\varphi(i) \neq \varphi(j)$ for some $i, j \in N$. **(2 marks)**

   Solution: φ can be chosen to be any constant function. For a non-constant function, just choose any one i and set $\varphi(i) = 1$ and set $\varphi(j) = 0$ for all $j \neq i$. It satisfies the potential inequalities.

   (b) Let P be a path from s to t such that the number of edges in P, denoted by $e(P)$, is less than or equal to the number of edges in any path from s to t. Show $\varphi(t) - \varphi(s) \leq e(P)$. **(3 marks)**

   Solution: Let $P' = (s, i^1, \ldots, i^k, t)$ be any arbitrary path from s to t. By definition, the number of edges in P', $e(P') \leq e(P)$. But $e(P') = [\varphi(i^1) - \varphi(s)] + [\varphi(i^2) - \varphi(i^1)] + \ldots + [\varphi(t) - \varphi(i^k)] = \varphi(t) - \varphi(s)$. Hence, $\varphi(t) - \varphi(s) \leq e(P)$.

5. There are n states (of the "economy"): set of states is $\{1, \ldots, n\}$. Let P be a $n \times n$ matrix, called the **transition probability matrix**, where every row of P is a probability distribution over the set of states. In particular, if P_{ij} denotes the entry in the i-th row and j-th column, then it satisfies:

$$P_{ij} \geq 0 \qquad \forall\, i, j \in \{1, \ldots, n\}$$
$$\sum_{j=1}^{n} P_{ij} = 1 \qquad \forall\, i \in \{1, \ldots, n\}.$$

   The interpretation is that P_{ij} is the probability with which state changes from i to j.

   A probability distribution over set of states is π such that $\pi_i \geq 0$ for all $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^{n} \pi_i = 1$. We say π is **stationary** for P if $P^T \cdot \pi = \pi$, where P^T is the transpose of P. In other words,

$$\sum_{i=1}^{n} P_{ij}\pi_i = \pi_j \qquad \forall\, j \in \{1, \ldots, n\}.$$

   Show that for every transition matrix P, there is a probability distribution π such that π is stationary for P. **(10 marks)**

   Answer. The primal problem is the following (with $\{\pi_j\}_{j\in\{1,\ldots,n\}}$ being the variables).

$$
\begin{aligned}
\sum_{i=1}^{n} P_{ij}\pi_i - \pi_j = 0 \qquad & \forall\ j \in \{1,\ldots,n\} \\
\sum_{j=1}^{n} \pi_j = 1 \qquad & \\
\pi_j \geq 0 \qquad & \forall\ j \in \{1,\ldots,n\}.
\end{aligned}
$$

Rewriting this, we get

$$
\begin{aligned}
(P_{jj}-1)\pi_j + \sum_{i\neq j}^{n} P_{ij}\pi_i = 0 \qquad & \forall\ j \in \{1,\ldots,n\} \\
\sum_{i=1}^{n} \pi_i = 1 \qquad & \\
\pi_i \geq 0 \qquad & \forall\ i \in \{1,\ldots,n\}.
\end{aligned}
$$

Hence, the Farkas alternative is as follows:

$$
\begin{aligned}
\delta < 0 \qquad & \\
(P_{ii}-1)y_i + \sum_{j\neq i} P_{ij}y_j + \delta \geq 0 \qquad & \forall\ i \in \{1,\ldots,n\}.
\end{aligned}
$$

This is equivalent to:

$$
(P_{ii}-1)y_i + \sum_{j\neq i} P_{ij}y_j > 0 \qquad \forall\ i \in \{1,\ldots,n\}.
$$

Rearranging terms, we can write this as:

$$
\sum_{j=1}^{n} P_{ij}y_j > y_i \qquad \forall\ i \in \{1,\ldots,n\}.
$$

Assume for contradiction Farkas alternative has a solution, i.e., the previous inequality has a solution $\{\bar{y}_i\}_i$. Let $\bar{y}_k = \max_j \bar{y}_j$. Since $P_{ij} \geq 0$ for all i, j, by considering the above inequality for $i = k$ we get

$$
\bar{y}_k < \sum_{j=1}^{n} P_{kj}\bar{y}_j \leq \sum_{j=1}^{n} P_{kj}\bar{y}_k = \bar{y}_k \sum_{j=1}^{n} P_{kj} = \bar{y}_k.
$$

This is a contradiction. Hence, Farkas alternative has no solution. By Farkas Lemma, the primal problem has a solution.