Sankhyā : The Indian Journal of Statistics
2006, Volume 68, Part 1, pp. 1-23

A Necessary and Sufficient Condition for the Tail-Triviality of a Recursive Tree Process

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Abstract

Given a *recursive distributional equation* (RDE) and a solution μ of it, we consider the tree indexed invariant process called the *recursive tree process* (RTP) with marginal μ. We introduce a new type of bivariate uniqueness property which is different from the one defined by Aldous and Bandyopadhyay (2005), and we prove that this property is equivalent to tail-triviality for the RTP, thus obtaining a necessary and sufficient condition to determine tail-triviality for a RTP in general. As an application we consider Aldous' construction of the frozen percolation process on a infinite regular tree (Aldous, 2000) and show that the associated RTP has a trivial tail.

AMS (2000) *subject classification.* Primary 60K35, 60G10, 60G20.
Keywords and phrases. Bivariate uniqueness, distributional identities, endogeny, fixed point equations, frozen percolation process, recursive distributional equations, recursive tree process, tail-triviality.

1 Introduction, Background and Motivation

Fixed-point equations or distributional identities have appeared in the probability literature for quite a long time in a variety of settings. The recent survey of Aldous and Bandyopadhyay (2005) provides a general framework to study certain type of distributional equations.

Given a space S write $\mathcal{P}(S)$ for the set of all probabilities on S. A *recursive distributional equation* (RDE) (Aldous and Bandyopadhyay, 2005) is a fixed-point equation on $\mathcal{P}(S)$ defined as

$$X \stackrel{d}{=} g\left(\xi; \left(X_j : 1 \leq j \leq^* N\right)\right) \quad \text{on } S, \tag{1.1}$$

where it is assumed that $\left(X_j\right)_{j \geq 1}$ are i.i.d. S-valued random variables with same distribution as X, and are independent of the pair (ξ, N). Here N is a non-negative integer valued random variable, which may take the value ∞, and g is a given S-valued function. (In the above equation by "$\leq^* N$" we mean the left hand side is "$\leq N$" if $N < \infty$, and "$< N$" otherwise). In (1.1) the distribution of X is *unknown* while the distribution of the pair (ξ, N), and the function g are the *known* quantities. Perhaps a more conventional (analytic) way of writing the equation (1.1) would be

$$\mu = T(\mu) \ , \tag{1.2}$$

where $T : \mathcal{P} \to \mathcal{P}(S)$ is a function defined on $\mathcal{P} \subseteq \mathcal{P}(S)$ such that $T(\mu)$ is the distribution of the right-hand side of the equation (1.1), when $(X_j)_{j \geq 1}$ are i.i.d. $\mu \in \mathcal{P}$.

As outlined in Aldous and Bandyopadhyay (2005) in many applications RDEs play a very crucial role. Examples include study of Galton-Watson branching processes and related random trees, probabilistic analysis of algorithms with suitable recursive structure Rösler and Rüschendorf (2001), Rösler (1992), statistical physics models on trees, Aldous and Steele (2004), Aldous (2000), Gamarnik, Nowicki and Swirscsz (2004), Bandyopadhyay (2005), and statistical physics and algorithmic questions in the mean-field model of distance (Aldous, 1992, 2001, Aldous and Steele, 2004). In many of these applications, particularly in the last two types mentioned above, often one needs to construct a particular tree indexed *stationary* process related to a given RDE, which is called a *recursive tree process* (RTP) (Aldous and Bandyopadhyay, 2005).

1.1. Recursive tree process. More precisely, suppose the RDE (1.1) has a solution, say μ. Then as shown in Aldous and Bandyopadhyay (2005), using the consistency theorem of Kolmogorov (Billingsley, 1995), one can construct a process, say $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$, indexed by $\mathcal{V} := \{\emptyset\} \cup_{d \geq 1} \mathbb{N}^d$, such that

$$\begin{array}{ll} \text{(i)} & X_{\mathbf{i}} \sim \mu \ \ \forall \ \mathbf{i} \in \mathcal{V}, \\ \text{(ii)} & \text{For each } \ d \geq 0, (X_{\mathbf{i}})_{|\mathbf{i}|=d} \ \text{ are independent,} \\ \text{(iii)} & X_{\mathbf{i}} = g\left(\xi_{\mathbf{i}}; (X_{\mathbf{i}j} : 1 \leq j \leq^* N_{\mathbf{i}})\right) \ \ \forall \ \mathbf{i} \in \mathcal{V}, \\ \text{(iv)} & X_{\mathbf{i}} \ \text{ is independent of } \ \left\{ (\xi_{\mathbf{i}'}, N_{\mathbf{i}'}) \,\middle|\, |\mathbf{i}'| < |\mathbf{i}| \right\} \ \ \forall \ \mathbf{i} \in \mathcal{V}, \end{array} \tag{1.3}$$

where $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ are taken to be i.i.d. copies of the pair (ξ, N), and by $|\cdot|$ we mean the length of a finite word. The process $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ is called an invariant *recursive tree process* (RTP) with marginal μ. The i.i.d. random variables $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ are called the *innovation process.* In some sense an invariant RTP with marginal μ, is an almost sure representation of a solution μ of the RDE

(1.1). Here we note that there is a natural tree structure on $\mathcal{V}$. Taking $\mathcal{V}$ as the vertex set, we join two words $\mathbf{i}, \mathbf{i}' \in \mathcal{V}$ by an edge, if and only if, $\mathbf{i}' = \mathbf{i}j$ or $\mathbf{i} = \mathbf{i}'j$, for some $j \in \mathbb{N}$. We will denote this tree by $\mathbb{T}_\infty$. The empty-word $\emptyset$ will be taken as the root of the tree $\mathbb{T}_\infty$, and we will write $\emptyset j = j$ for $j \in \mathbb{N}$.

In the applications mentioned above the variables $(X_\mathbf{i})_{\mathbf{i}\in\mathcal{V}}$ of a RTP are often used as auxiliary variables to define or to construct some useful random structures. To be more precise in Aldous (2001) they were used to obtain "almost optimal matching", while in Aldous (2000) they were used to define the percolation clusters. In such applications typically the innovation process defines the "internal" variables while the RTP is constructed "externally" using the consistency theorem. It is then natural to ask whether the RTP is measurable only with respect to the i.i.d. innovation process $(\xi_\mathbf{i}, N_\mathbf{i})$.

DEFINITION 1.1. *An invariant RTP with marginal μ is called* endogenous*, if the root variable $X_\emptyset$ is measurable with respect to the σ-algebra*

$$\mathcal{G} := \sigma\left(\left\{(\xi_\mathbf{i}, N_\mathbf{i}) \,\middle|\, \mathbf{i} \in \mathcal{V}\right\}\right).$$

This notion of endogeny has been the main topic of discussion in Aldous and Bandyopadhyay (2005). The authors provide a necessary and sufficient condition for endogeny in the general setup (Aldous and Bandyopadhyay, 2005, Theorem 11). A non-trivial application of this result is given in Bandyopadhyay (2002), where it is proved that the invariant RTP associated with the *logistic* RDE, which appears in the study of the mean-field random assignment problem (Aldous, 2001) is endogenous. Another interesting example arise in the construction of the *frozen percolation* on an infinite 3-regular tree by Aldous (2000), where a particular RTP has been used to carry on the construction. This example is one of our main motivations, so we discuss this example in more detail in Section 1.5.

As discussed in Aldous and Bandyopadhyay (2005) in some sense, the concept of endogeny tries to capture the idea of having "no influence of the boundary at infinity" on the root. In this direction a closely related concept would be the tail-triviality of a RTP. To give a formal definition of the tail of a RTP, let $(X_\mathbf{i})_{\mathbf{i}\in\mathcal{V}}$ be an invariant RTP with marginal μ, where μ is a solution of the RDE (1.1). The tail σ-algebra of $(X_\mathbf{i})_{\mathbf{i}\in\mathcal{V}}$ is defined as

$$\mathcal{H} = \bigcap_{n\geq 0} \mathcal{H}_n, \tag{1.4}$$

where

$$\mathcal{H}_n := \sigma\left(\left\{X_\mathbf{i} \,\middle|\, |\mathbf{i}| \geq n\right\}\right). \tag{1.5}$$

Naturally, we will say an invariant RTP has trivial tail if the tail σ-algebra $\mathcal{H}$ is trivial. Because the innovation process $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is i.i.d. so it is natural to expect that if a RTP is endogenous, then it has a trivial tail.

PROPOSITION 1.1. *Suppose μ is a solution of the RDE (1.1) and $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ be an invariant RTP with marginal μ. Then the tail of $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is trivial if it is endogenous.*

Thus one way to conclude that a RTP is not endogenous will be to show that it has a non-trivial tail. The following easy example shows that the converse may not hold.

EXAMPLE 1.1. Take $S := \{0, 1\}$. Let $0 < q < 1$ and $\xi \sim \text{Bernoulli}\,(q)$. Consider the RDE

$$X \stackrel{d}{=} \xi + X_1 \quad (\,mod \quad 2\,)\,, \tag{1.6}$$

where X_1 has same distribution as X, and is independent of ξ.

If T is the associated operator defined by the right-hand side of the equation (1.6), then it is easy to see that T maps a Bernoulli (p) distribution to a Bernoulli (p') distribution where

$$p' = p\,(1-q) + q\,(1-p)\,.$$

Thus the unique solution of the RDE (1.6) is Bernoulli $\left(\frac{1}{2}\right)$.

In this example because there is no branching ($N \equiv 1$), so the invariant RTP with marginal Bernoulli $\left(\frac{1}{2}\right)$ can be indexed by the non-negative integers, we denote it by $(X_i)_{i\geq 0}$, where X_0 is the root variable and it satisfy

$$X_i = \xi_i + X_{i+1} \quad \text{a.s.} \;\; \forall \;\; i \geq 0,$$

where $(\xi_i)_{i\geq 0}$ are i.i.d. Bernoulli (q). It is then easy to see that we must have

$$X_{i+1} \text{ and } (\xi_0, \xi_1, \ldots, \xi_i) \quad \text{are independent, for all} \;\; i \geq 0.$$

Therefore, X_0 is independent of the innovation process $(\xi_i)_{i\geq 0}$, thus the RTP is not endogenous. The following proposition whose proof we defer till Section 2, states that the RTP $(X_i)_{i\geq 0}$ has trivial tail. This gives an example of an invariant RTP which is not endogenous but has trivial tail.

PROPOSITION 1.2. *The invariant RTP with marginal Bernoulli $\left(\frac{1}{2}\right)$ associated with the RDE (1.6) has trivial tail.*

So proving tail-triviality of a RTP is weaker than proving endogeny, but in some cases it might help to prove non-endogeny by showing that the tail is not trivial. Also in general, studying the tail of a stochastic process is mathematically interesting.

In this article we provide a necessary and sufficient condition to determine the tail-triviality for an invariant RTP. This condition is in the same spirit of the equivalence theorem of Aldous and Bandyopadhyay (2005, Theorem 11). But before we state our main result we first introduce a new type of *bivariate uniqueness* property, which is different than the one introduced in Aldous and Bandyopadhyay (2005), we will call it the *bivariate uniqueness property of the second kind.*

1.2. Bivariate uniqueness property of the second kind. Consider a general RDE given by (1.1) and let $T\colon \mathcal{P} \to \mathcal{P}(S)$ be the induced operator. We will consider a bivariate version of it. Write $\mathcal{P}^{(2)}$ for the space of probability measures on $S^2 = S \times S$, with marginals in $\mathcal{P}$. We can now define a map $T \otimes T : \mathcal{P}^{(2)} \to \mathcal{P}\left(S^2\right)$ as follows

DEFINITION 1.2. *For a probability $\mu^{(2)} \in \mathcal{P}^{(2)}$, $(T \otimes T)\left(\mu^{(2)}\right)$ is the joint distribution of*

$$\begin{pmatrix} g\left(\xi, X_j^{(1)}, 1 \leq j \leq^* N\right) \\ g\left(\eta, X_j^{(2)}, 1 \leq j \leq^* M\right) \end{pmatrix}$$

where we assume

1. $\left(X_j^{(1)}, X_j^{(2)}\right)_{j \geq 1}$ *are independent with joint distribution* $\mu^{(2)}$ *on* S^2*;*
2. (ξ, N) *and* (η, M) *are i.i.d;*
3. *the families of random variables in 1 and 2 are independent.*

We note that here we use *independent* copies of the innovation pair in the two coordinates. We also note that this is preciously where this bivariate operator differs from the bivariate operator defined in Aldous and Bandyopadhyay (2005), where the innovation pair was kept same at each coordinate.

From the definition it follows that

LEMMA 1.1. *(a) If μ is a fixed point for T, then the associated* product measure $\mu \otimes \mu$ *is a fixed point for $T \otimes T$.*

(b) If $\mu^{(2)}$ is a fixed point for $T \otimes T$, then each marginal distribution is a fixed point for T.

So if μ is a fixed point for T then $\mu \otimes \mu$ is a fixed point for $T \otimes T$ and there may or may not be other fixed points of $T \otimes T$ with marginal μ.

DEFINITION 1.3. *An invariant RTP with marginal μ has the* bivariate uniqueness property of the second kind *if $\mu \otimes \mu$ is the unique fixed point of $T \otimes T$ with marginal μ.*

1.3. Main result : an equivalence theorem. Our main theorem is the following general result linking the tail triviality of an invariant RTP with the bivariate uniqueness property of the second kind.

THEOREM 1.1. *Suppose S is a* Polish *space. Consider an invariant RTP with marginal distribution μ.*

(a) If the RTP has trivial tail then the bivariate uniqueness property of the second kind holds.

(b) Suppose the bivariate uniqueness property of the second kind holds. If also $T \otimes T$ is continuous with respect to weak convergence on the set of bivariate distributions with marginals μ, then the tail of the RTP is trivial.

(c) Further, the RTP has a trivial tail if and only if

$$(T \otimes T)^n \left(\mu^{\nearrow}\right) \stackrel{d}{\longrightarrow} \mu \otimes \mu,$$

where $\mu^{\nearrow}$ is the diagonal measure with marginal μ, that is, if $(X, Y) \sim \mu^{\nearrow}$, then $\mathbf{P}(X = Y) = 1$ and $X, Y \sim \mu$.

1.4. Heuristic behind the equivalence theorem. Suppose μ is a solution of the RDE (1.1) and let $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ be an invariant RTP with marginal μ. Let $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ be the i.i.d. innovation process, and $\mathcal{G}_n := \sigma\left(\left\{(\xi_{\mathbf{i}}, N_{\mathbf{i}}) \,\middle|\, |\mathbf{i}| \leq n\right\}\right)$ be the σ-algebra for the innovations in first n-generations of the tree $\mathbb{T}_\infty$. From the construction (1.3) of the RTP we note that for any $n \geq 0$ the root variable $X_\emptyset$ is measurable with respect to the σ-algebra $\sigma\left(\mathcal{G}_n \cup \mathcal{H}_{n+1}\right)$, where $\mathcal{H}_{n+1}$ is as defined in (1.5). So heuristically to check whether the tail of the RTP $\mathcal{H} = \cap_{n \geq 0} \mathcal{H}_n$ contains any non-trivial information, we may want to do the following :

> Start with "same input at infinity." Take two independent but identical copies of the innovation process and run through the

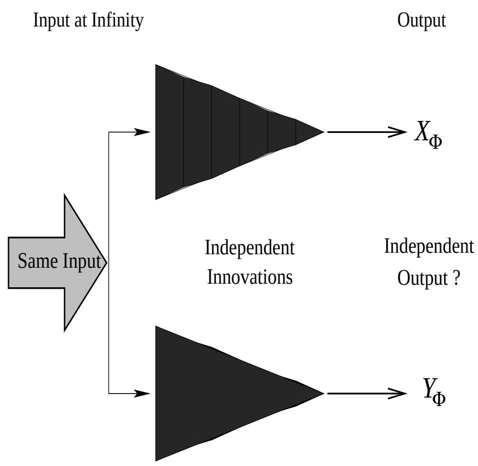

Figure 1: Intuitive picture for the bivariate uniqueness of the second kind

> recursions in (1.3). Finally obtaining two copies of the RTP, say $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ and $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$, with same marginal μ. Check if the root variables $X_\emptyset$ and $Y_\emptyset$, are independent or not.

Figure 1 gives this intuitive picture. The part (c) of the Theorem 1.1 makes this process rigorous. Moreover we notice from definition the bivariate process $(X_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is a RTP associated with the operator $T \otimes T$. This leads to the notion of bivariate uniqueness property of the second kind. We would like to note that the proof of the Theorem 1.1 is nothing but to make this heuristic rigorous.

1.5. Application to frozen percolation. As mentioned earlier, one of our main motivating example arise in the context of frozen percolation process on an infinite regular tree. For sake of completeness we here provide a very brief background on frozen percolation process, readers are advised to look at Aldous (2000), Aldous and Bandyopadhyay (2005) for more details.

Frozen percolation process was first studied by Aldous (2000) where he constructed the process on a infinite 3-regular tree. Let $\mathcal{T}_3 = (\mathcal{V}, \mathcal{E})$ denote the infinite 3-regular tree. Let $(U_e)_{e\in\mathcal{E}}$ be i.i.d. Uniform$[0,1]$ edge weights.

Consider a collection of random subsets $\mathcal{A}_t \subseteq \mathcal{E}$ for $0 \leq t \leq 1$, whose evolution is described informally by :

> $\mathcal{A}_0$ is empty; for each $e \in \mathcal{E}$, at time $t = U_e$ set $\mathcal{A}_t = \mathcal{A}_{t-} \cup \{e\}$ if each end-vertex of e is in a finite cluster of $\mathcal{A}_{t-}$; otherwise set $\mathcal{A}_t = \mathcal{A}_{t-}$. $(*)$

(A *cluster* is formally a connected component of edges, but we also consider it as the induced set of vertices). Qualitatively, in the process $(\mathcal{A}_t)$ the clusters may grow to infinite size but, at the instant of becoming infinite they are "frozen", in the sense that no extra edge may be connected to an infinite cluster. The final set $\mathcal{A}_1$ will be a forest on $\mathcal{T}_3$ with both infinite and finite clusters, such that no two finite clusters are separated by a single edge. Aldous (2000) defines this process $(\mathcal{A}_t)$ as the *frozen percolation process.*

Although this process is intuitively quite natural, rigorously speaking it is not clear that it exists or that $(*)$ does specify a unique process. In fact Itai Benjamini and Oded Schramm have an argument that such a process does not exist on the $\mathbb{Z}^2$-lattice (see the remarks in Section 5.1 of Aldous, 2000). But for the infinite 3-regular tree, Aldous (2000) gives a rigorous construction of an automorphism invariant process satisfying $(*)$. This construction uses the following RDE

$$Y \stackrel{d}{=} \Phi\left(Y_1 \wedge Y_2; U\right) \quad \text{on } I := \left[\tfrac{1}{2}, 1\right] \cup \{\infty\}, \tag{1.7}$$

where (Y_1, Y_2) are i.i.d. with same distribution as Y, and are independent of $U \sim \text{Uniform}[0,1]$, and $\Phi : I \times [0,1] \to I$ is a function defined as

$$\Phi(x; u) = \begin{cases} x & \text{if } x > u \\ \infty & \text{otherwise} \end{cases}. \tag{1.8}$$

We will call (1.7) the *frozen percolation RDE.*

It turns out (Aldous, 2000) that the RDE (1.7) has many solutions. In particular, solutions having no atom in $\left[\frac{1}{2}, 1\right]$ are given by

$$\nu_a(dx) = \tfrac{dx}{2x^2} \;\; \tfrac{1}{2} < x < a; \;\; \nu_a\left(\{\infty\}\right) = \tfrac{1}{2a}, \tag{1.9}$$

where $a \in \left[\frac{1}{2}, 1\right]$, thus $\nu = \nu_1$ is the unique solution with support I.

Notice that for the RDE (1.7) $N \equiv 2$, so a RTP with marginal ν essentially lives on a rooted binary tree, we will denote the vertex set in this case by $\widetilde{\mathcal{V}}$. Let $(Y_{\mathbf{i}})_{\mathbf{i} \in \widetilde{\mathcal{V}}}$ be an invariant RTP with marginal ν. Aldous' construction of the frozen percolation process (Aldous, 2000) uses these *externally* defined random variables $(Y_{\mathbf{i}})_{\mathbf{i} \in \widetilde{\mathcal{V}}}$. We refer the readers to look at Aldous (2000) for the technical details of this construction. Here we only mention briefly what is the significance of the RTP $(Y_{\mathbf{i}})_{\mathbf{i} \in \widetilde{\mathcal{V}}}$. Let $e = (u, v)$ be an edge of the infinite regular binary tree $\mathcal{T}_3$ and let $\overrightarrow{e} = \overrightarrow{(u, v)}$ be a direction of it which is from the vertex u to vertex v. Naturally the directed edge $\overrightarrow{e}$ has two directed edges coming out of it, which can be considered as two children of it. Continuing in similar manner we notice that each directed edge $\overrightarrow{e}$ represent a rooted infinite binary tree, which is isomorphic to $\widetilde{\mathcal{V}}$, and the weights are defined appropriately using the i.i.d. Uniform edge weights (U_e). If the frozen percolation process exists, then the time for the edge e to join to infinite along the subtree defined by $\overrightarrow{e}$ is given by the variable $Y_{\emptyset}$. More preciously, such time should satisfy the distributional recursion (1.7). However to prove the existence of the process such times are then *externally* constructed using the RTP construction. Naturally it make sense to ask whether these variables can only be defined using the i.i.d. Uniform$[0, 1]$ edge weights (see Remark 5.7 in Aldous, 2000), which is same as asking whether the RTP is endogenous.

THEOREM 1.2. *Any invariant* recursive tree process *associated with the RDE (1.7) with marginal ν has trivial tail.*

This result does not resolve the question of endogeny, but it proves that the version of the frozen percolation process constructed by Aldous (2000) on an infinite 3-regular tree has trivial tail.

To give a bit of history, for several years we conjectured in seminar talks that the RTP with marginal ν is non-endogenous. Because the simulation results suggested one of the condition equivalent to endogeny from Aldous and Bandyopadhyay (2005, Theorem 11) fails for the solution ν of the RDE (1.7). In recent days for some time we thought we can prove the opposite, but it turned out that our argument had some flaw in it. Fresh simulations confirm our earlier belief that the RTP with marginal ν is non-endogenous. Till date to best of our knowledge a rigorous proof is yet to be found.

It is interesting to observe that if the RTP with marginal ν is non-endogenous then the frozen percolation process would have a kind of "spatial chaos" property, that the behavior near the root would be affected by the behavior at infinity. On the other hand in light of the Theorem 1.2, we note that possible influence of infinity at the root is not coming from the tail of the process. Such examples are rare, our Example 1.1 is one such. But so far we do not know a non-trivial example of this kind. Of course if non-endogeny for frozen percolation is proved, then that together with Theorem 1.2 will provide one such.

1.6. Outline of the rest of the paper. The rest of the article is divided as follows. In the following section we provide some basic connection between the root variable $X_\emptyset$ of an RTP with the tail σ-algebra $\mathcal{H}$, and also give proofs of Propositions 1.1 and 1.2. In Section 3 we give a proof of the equivalence theorem and Section 5 contains the proof of the Theorem 1.2. We conclude with Section 6 which contains some further discussion.

2 Connection Between Root and Tail of a RTP

Because of the recursive structure one would expect that the tail σ-algebra $\mathcal{H}$ is trivial, if and only if the root variable $X_\emptyset$ is independent of it. The following lemma preciously states that.

LEMMA 2.1. *$X_\emptyset$ is independent of $\mathcal{H}$, if and only if $\mathcal{H}$ is trivial.*

PROOF. If the tail $\mathcal{H}$ is trivial then naturally $X_\emptyset$ is independent of it. For proving the converse we will need the following standard measure theoretic fact whose proof is a straightforward application of Dynkin's π-λ Theorem (Billingsley, 1995), so we omit it here.

LEMMA 2.2. *Suppose $(\Omega, \mathcal{I}, \mathbf{P})$ be a probability space and let $\mathcal{F}^*, \mathcal{G}^*$ and $\mathcal{H}^*$ be three sub-σ-algebras such that $\mathcal{F}^*$ is independent of $\mathcal{H}^*$; $\mathcal{G}^*$ is independent of $\mathcal{H}^*$; and $\mathcal{F}^*$ and $\mathcal{G}^*$ are independent given $\mathcal{H}^*$. Then $\sigma\left(\mathcal{F}^* \cup \mathcal{G}^*\right)$ is independent of $\mathcal{H}^*$.*

To complete the proof of the Lemma 2.1 we denote $\mathcal{F}_n^0 := \sigma\left(X_{\mathbf{i}}, |\mathbf{i}| = n\right)$ and $\mathcal{F}_n := \sigma\left(X_{\mathbf{i}}, |\mathbf{i}| \leq n\right)$. From assumption $X_{\mathbf{i}}$ is independent of $\mathcal{H}$ for all $\mathbf{i} \in \mathcal{V}$. Fix $n \geq 1$ and let $\mathbf{i} \neq \mathbf{i}'$ be two vertices at generation n. From the definition of RTP $X_{\mathbf{i}}$ and $X_{\mathbf{i}'}$ are independent, moreover they are independent given $\mathcal{H}_{n+k}$ for any $k \geq 1$. Letting $k \to \infty$ we conclude that $X_{\mathbf{i}}$ and $X_{\mathbf{i}'}$ are independent given $\mathcal{H}$. Thus by Lemma 2.2 we get that $(X_{\mathbf{i}}, X_{\mathbf{i}'})$ is independent of $\mathcal{H}$, and hence by induction $\mathcal{F}_n^0$ is independent of $\mathcal{H}$.

Now let $\mathcal{G}_n = \sigma\left(\left\{(\xi_{\mathbf{i}}, N_{\mathbf{i}}) \,\middle|\, |\mathbf{i}| \leq n\right\}\right)$, then $\mathcal{G}_n$ is independent of $\mathcal{H}$ from definition. Further $\mathcal{G}_n$ is independent of $\mathcal{F}_{n+1}^0$ given $\mathcal{H}_{n+k}$ for any $k \geq 1$. Once again letting $k \to \infty$ we conclude that $\mathcal{G}_n$ and $\mathcal{F}_{n+1}^0$ are independent given $\mathcal{H}$. So again using Lemma 2.2 it follows that $\sigma\left(\mathcal{G}_n \cup \mathcal{F}^0_{n+1}\right)$ is independent of $\mathcal{H}$. But $\mathcal{F}_n \subseteq \sigma\left(\mathcal{G}_n \cup \mathcal{F}^0_{n+1}\right)$ so $\mathcal{F}_n$ is independent of $\mathcal{H}$. But $\mathcal{F}_n \uparrow \mathcal{H}_0$ and hence $\mathcal{H}$ is independent of $\mathcal{H}_0 \supseteq \mathcal{H}$. This proves that $\mathcal{H}$ is trivial. $\square$

2.1. Proof of Proposition 1.1. Let $\mathcal{G}_n := \sigma\left(\left(\xi_{\mathbf{i}}, N_{\mathbf{i}}\right), |\mathbf{i}| \leq n\right)$. From definition we have $\mathcal{H}_n \downarrow \mathcal{H}$ and $\mathcal{G}_n \uparrow \mathcal{G}$. Also for each $n \geq 0$, $\mathcal{G}_n$ is independent of $\mathcal{H}_{n+1}$. So clearly $\mathcal{G}$ is independent of $\mathcal{H}$. Hence if the RTP is endogenous then $X_\emptyset$ is $\mathcal{G}$-measurable, so it is independent of $\mathcal{H}$. The rest follows from the Lemma 2.1.

2.2. Proof of Proposition 1.2. There are several ways one can prove Proposition 1.2, perhaps the simplest is to apply the equivalence theorem (Theorem 1.1). This will also illustrate an easy application of the equivalence theorem. A non-trivial application is given in Sections 4 and 5.

PROOF. We will show that the bivariate uniqueness of the second kind holds for the unique solution Bernoulli $\left(\frac{1}{2}\right)$ of the RDE (1.6). So by part (b) of the equivalence theorem (Theorem 1.1) the tail-triviality will follow (note that in this case the continuity condition trivially holds).

Let (X, Y) be S^2-valued random pair with some distribution such that the marginals are both Bernoulli$(1/2)$. Let

$$\theta = \mathbf{P}\left(X = 1, Y = 1\right) = \mathbf{P}\left(X = 0, Y = 0\right).$$

Suppose further that the distribution of (X, Y) satisfies the following bivariate RDE

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} X_1 + \xi \\ Y_1 + \eta \end{pmatrix} \pmod{2},$$

where (X_1, Y_1) is a copy of (X, Y) and independent of (ξ, η) which are i.i.d. Bernoulli(q). So we get the following equation for θ

$$\theta = q^2\theta + (1-q)^2\theta + 2q(1-q)(1/2 - \theta). \tag{2.1}$$

The only solution of (2.1) is $\theta = 1/4$, thus X and Y must be independent, proving the bivariate uniqueness of the second kind. $\square$

3 Proof of the Equivalence Theorem

(a) Let λ be a fixed point of $T \otimes T$ with marginals μ. Consider two independent and identical copies innovation processes given by $\left(\left(\xi_{\mathbf{i}}, N_{\mathbf{i}}\right), \mathbf{i} \in \mathcal{V}\right)$

and $((\eta_{\mathbf{i}}, M_{\mathbf{i}}), \mathbf{i} \in \mathcal{V})$. Using Kolmogorov's consistency theorem (Billingsley, 1995), we can then construct a bivariate RTP $\left(\left(X_{\mathbf{i}}^{(1)}, X_{\mathbf{i}}^{(2)}\right), \mathbf{i} \in \mathcal{V}\right)$ with $\lambda = \text{dist}(X_\emptyset^{(1)}, X_\emptyset^{(2)})$. We note that this construction is no different than what one does to obtain an univariate RTP as in (1.3), and the bivariate RTP has the similar properties as well. Notice that $\left(X_{\mathbf{i}}^{(1)}\right)_{\mathbf{i} \in \mathcal{V}}$ and $\left(X_{\mathbf{i}}^{(2)}\right)_{\mathbf{i} \in \mathcal{V}}$ are two (*univariate*) RTPs with marginal μ. So from assumption both has trivial tails.

We define the following σ-algebras

$$
\begin{aligned}
\mathcal{H}_n^{(1)} &:= \sigma\left(\left\{X_{\mathbf{i}}^{(1)} \,\middle|\, |\mathbf{i}| \geq n\right\}\right); & (3.1)\\
\mathcal{H}_n^{(2)} &:= \sigma\left(\left\{X_{\mathbf{i}}^{(2)} \,\middle|\, |\mathbf{i}| \geq n\right\}\right); & (3.2)\\
\mathcal{H}_n^{(*)} &:= \sigma\left(\left\{\left(X_{\mathbf{i}}^{(1)}, X_{\mathbf{i}}^{(2)}\right) \,\middle|\, |\mathbf{i}| \geq n\right\}\right), & (3.3)
\end{aligned}
$$

and we also define

$$
\begin{aligned}
\text{Tail of } \left(X_{\mathbf{i}}^{(1)}\right)_{\mathbf{i} \in \mathcal{V}} := \mathcal{H}^{(1)} &= \bigcap_{n \geq 0} \mathcal{H}_n^{(1)}; & (3.4)\\
\text{Tail of } \left(X_{\mathbf{i}}^{(2)}\right)_{\mathbf{i} \in \mathcal{V}} := \mathcal{H}^{(2)} &= \bigcap_{n \geq 0} \mathcal{H}_n^{(2)}; & (3.5)\\
\text{Tail of } \left(X_{\mathbf{i}}^{(1)}, X_{\mathbf{i}}^{(2)}\right)_{\mathbf{i} \in \mathcal{V}} := \mathcal{H}^{(*)} &= \bigcap_{n \geq 0} \mathcal{H}_n^{(*)}. & (3.6)
\end{aligned}
$$

Let f and g be two bounded measurable functions. Fix $n \geq 0$,

$$
\begin{aligned}
& \mathbf{E}\left[f\left(X_\emptyset^{(1)}\right) g\left(X_\emptyset^{(2)}\right) \,\middle|\, \mathcal{H}_n^{(*)}\right] \\
= \; & \mathbf{E}\left[f\left(X_\emptyset^{(1)}\right) \,\middle|\, \mathcal{H}_n^{(*)}\right] \times \mathbf{E}\left[g\left(X_\emptyset^{(2)}\right) \,\middle|\, \mathcal{H}_n^{(*)}\right] \\
= \; & \mathbf{E}\left[f\left(X_\emptyset^{(1)}\right) \,\middle|\, \mathcal{H}_n^{(1)}\right] \times \mathbf{E}\left[g\left(X_\emptyset^{(2)}\right) \,\middle|\, \mathcal{H}_n^{(2)}\right], \qquad (3.7)
\end{aligned}
$$

where the first equality follows from the recursive construction and because the two innovation processes are independent. Taking limit as $n \to \infty$ and using the martingale convergence theorem we get

$$
\mathbf{E}\left[f\left(X_\emptyset^{(1)}\right) g\left(X_\emptyset^{(2)}\right) \,\middle|\, \mathcal{H}^{(*)}\right] = \mathbf{E}\left[f\left(X_\emptyset^{(1)}\right) \,\middle|\, \mathcal{H}^{(1)}\right] \times \mathbf{E}\left[g\left(X_\emptyset^{(2)}\right) \,\middle|\, \mathcal{H}^{(2)}\right]. \qquad (3.8)
$$

Because both $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ are trivial, so taking a further expectation we conclude that

$$
\mathbf{E}\left[f\left(X_\emptyset^{(1)}\right) g\left(X_\emptyset^{(2)}\right)\right] = \mathbf{E}\left[f\left(X_\emptyset^{(1)}\right)\right] \times \mathbf{E}\left[g\left(X_\emptyset^{(2)}\right)\right]. \qquad (3.9)
$$

So $X_\emptyset^{(1)}$ and $X_\emptyset^{(2)}$ are independent, that is, $\lambda = \mu \otimes \mu$, which implies that the bivariate uniqueness property of the second kind holds.

(b) Let $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ be the invariant RTP with marginal μ. $\mathcal{H}_n$ and $\mathcal{H}$ be as defined in (1.5) and (1.4) respectively. Observe that $\mathcal{H}_n \downarrow \mathcal{H}$. Now fix $\Lambda : S \to \mathbb{R}$ a bounded continuous function. So by reverse martingale convergence

$$\mathbf{E}\left[\Lambda(X_\emptyset)\middle|\mathcal{H}_n\right] \xrightarrow[\mathcal{L}_2]{\text{a.s.}} \mathbf{E}\left[\Lambda(X_\emptyset)\middle|\mathcal{H}\right]. \tag{3.10}$$

Let $(\eta_{\mathbf{i}}, M_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ be independent innovations which are independent of $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ and $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$. For $n \geq 1$, define $Y_{\mathbf{i}}^n := X_{\mathbf{i}}$ if $|\mathbf{i}| = n$, and then recursively define $Y_{\mathbf{i}}^n$ for $|\mathbf{i}| < n$ using RTP construction (1.3), but replacing $\xi_{\mathbf{i}}$ by $\eta_{\mathbf{i}}$ and $N_{\mathbf{i}}$ by $M_{\mathbf{i}}$ to get an invariant RTP $(Y_{\mathbf{i}}^n)$ of depth n. Observe that $X_\emptyset \stackrel{d}{=} Y_\emptyset^n$. Further given $\mathcal{H}_n$, the variables $X_\emptyset$ and $Y_\emptyset^n$ are conditionally independent and identically distributed. Now let

$$\bar{\sigma}_n^2(\Lambda) := \left\| \mathbf{E}\left[\Lambda(X_\emptyset)\middle|\mathcal{H}_n\right] - \mathbf{E}\left[\Lambda(X_\emptyset)\right] \right\|_2^2. \tag{3.11}$$

We calculate

$$\begin{aligned}
\bar{\sigma}_n^2(\Lambda) &= \mathbf{E}\left[\left(\mathbf{E}\left[\Lambda(X_\emptyset)\middle|\mathcal{H}_n\right] - \mathbf{E}\left[\Lambda(X_\emptyset)\right]\right)^2\right] \\
&= \mathbf{Var}\left(\mathbf{E}\left[\Lambda(X_\emptyset)\middle|\mathcal{H}_n\right]\right) \\
&= \mathbf{Var}\left(\Lambda(X_\emptyset)\right) - \mathbf{E}\left[\mathbf{Var}\left(\Lambda(X_\emptyset)\middle|\mathcal{H}_n\right)\right] \\
&= \mathbf{Var}\left(\Lambda(X_\emptyset)\right) - \tfrac{1}{2}\mathbf{E}\left[\left(\Lambda(X_\emptyset) - \Lambda(Y_\emptyset^n)\right)^2\right]. \qquad (3.12)
\end{aligned}$$

The last equality uses the conditional form of the fact that for any random variable U, one has $\mathbf{Var}(U) = \frac{1}{2}\mathbf{E}\left[(U_1 - U_2)^2\right]$, where U_1, U_2 are i.i.d. copies of U.

Now suppose we show that

$$(X_\emptyset, Y_\emptyset^n) \xrightarrow{d} (X^\star, Y^\star) \tag{3.13}$$

for some limit $(X^\star, Y^\star)$. From the construction,

$$\begin{bmatrix} X_\emptyset \\ Y_\emptyset^{n+1} \end{bmatrix} \stackrel{d}{=} (T \otimes T)\left(\begin{bmatrix} X_\emptyset \\ Y_\emptyset^n \end{bmatrix}\right),$$

and then the weak continuity assumption on $T \otimes T$ implies

$$\begin{bmatrix} X^\star \\ Y^\star \end{bmatrix} \stackrel{d}{=} (T \otimes T)\left(\begin{bmatrix} X^\star \\ Y^\star \end{bmatrix}\right).$$

Also by construction we have $X_\emptyset \stackrel{d}{=} Y_\emptyset^n \stackrel{d}{=} \mu$ for all $n \geq 1$, and hence $X^\star \stackrel{d}{=} Y^\star \stackrel{d}{=} \mu$. Now since we assume that the bivariate uniqueness property of the second kind holds, so $X^\star$ and $Y^\star$ must be independent. Since Λ is a bounded continuous function, (3.13) implies

$$\mathbf{E}\left[\left(\Lambda(X_\emptyset) - \Lambda(Y_\emptyset^n)\right)^2\right] \to \mathbf{E}\left[\left(\Lambda(X^\star) - \Lambda(Y^\star)\right)^2\right] = 2\mathbf{Var}\left(\Lambda(X_\emptyset)\right) \quad (3.14)$$

and so using (3.12) we see that $\bar{\sigma}_n^2(\Lambda) \longrightarrow 0$. Hence from (3.11) and (3.10) we conclude that $\Lambda(X_\emptyset)$ is independent of $\mathcal{H}$. This is true for every bounded continuous Λ, proving that $X_\emptyset$ is independent of $\mathcal{H}$, so from Lemma 2.1it follows that $\mathcal{H}$ is trivial.

Now all remains is to show that limit (3.13) exists. Fix $f : S \to \mathbb{R}$ and $h : S \to \mathbb{R}$, two bounded continuous functions. Again by reverse martingale convergence

$$\mathbf{E}\left[f(X_\emptyset)\middle|\mathcal{H}_n\right] \xrightarrow[\mathcal{L}_1]{\text{a.s.}} \mathbf{E}\left[f(X_\emptyset)\middle|\mathcal{H}\right],$$

and similarly for h. So

$$\begin{aligned} \mathbf{E}\left[f(X_\emptyset)h(Y_\emptyset^n)\right] &= \mathbf{E}\left[\mathbf{E}\left[f(X_\emptyset)h(Y_\emptyset^n)\middle|\mathcal{H}_n\right]\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[f(X_\emptyset)\middle|\mathcal{H}_n\right]\mathbf{E}\left[h(X_\emptyset)\middle|\mathcal{H}_n\right]\right], \end{aligned}$$

the last equality because of conditional on $\mathcal{H}_n$ $X_\emptyset$ and $Y_\emptyset^n$ are independent and identically distributed. Letting $n \to \infty$ we get

$$\mathbf{E}\left[f(X_\emptyset)h(Y_\emptyset^n)\right] \longrightarrow \mathbf{E}\left[\mathbf{E}\left[f(X_\emptyset)\middle|\mathcal{G}\right]\mathbf{E}\left[h(X_\emptyset)\middle|\mathcal{G}\right]\right]. \quad (3.15)$$

Moreover note that $X_\emptyset \stackrel{d}{=} Y_\emptyset^n \stackrel{d}{=} \mu$ and so the sequence of bivariate distributions $(X_\emptyset, Y_\emptyset^n)$ is tight. Tightness, together with convergence (3.15) for all bounded continuous f and h, implies weak convergence of $(X_\emptyset, Y_\emptyset^n)$.

(c) First assume that $(T \otimes T)^n \left(\mu^\nearrow\right) \xrightarrow{d} \mu \otimes \mu$, then with the same construction as done in part (b) we get that

$$\left(X_\emptyset, Y_\emptyset^n\right) \xrightarrow{d} \left(X^\star, Y^\star\right),$$

where $X^\star$ and $Y^\star$ are independent copies of $X_\emptyset$. Further recall that Λ is bounded continuous, thus using (3.12), (3.11) and (3.10) we conclude that $\Lambda(X_\emptyset)$ is independent of $\mathcal{H}$. Since it is true for any bounded continuous function Λ, thus $X_\emptyset$ is independent of $\mathcal{H}$. Thus again by Lemma 2.1 the RTP has trivial tail.

Conversely, suppose that the invariant RTP with marginal μ has trivial tail. Let Λ_1 and Λ_2 be two bounded continuous functions. Note that the variables $\left(X_\emptyset, Y_\emptyset^n\right)$, as defined in part (b) has joint distribution $(T \otimes T)^n \left(\mu^{\nearrow}\right)$. Further, given $\mathcal{H}_n$, they are conditionally independent and have same conditional law as of $X_\emptyset$ given $\mathcal{H}_n$. So

$$\begin{aligned}
\mathbf{E}\left[\Lambda_1(X_\emptyset)\,\Lambda_2(Y_\emptyset^n)\right] &= \mathbf{E}\left[\mathbf{E}\left[\Lambda_1(X_\emptyset)\middle|\mathcal{H}_n\right]\,\mathbf{E}\left[\Lambda_2(X_\emptyset)\middle|\mathcal{H}_n\right]\right] \\
&\to \mathbf{E}\left[\mathbf{E}\left[\Lambda_1(X_\emptyset)\middle|\mathcal{H}\right]\,\mathbf{E}\left[\Lambda_2(X_\emptyset)\middle|\mathcal{H}\right]\right] \\
&= \mathbf{E}\left[\Lambda_1(X_\emptyset)\right]\mathbf{E}\left[\Lambda_2(X_\emptyset)\right].
\end{aligned}$$

The convergence is by reverse martingale convergence, and the last equality is by tail triviality and Lemma 2.1. So from definition we get

$$(T \otimes T)^n \left(\mu^{\nearrow}\right) \stackrel{d}{=} \left(X_\emptyset, Y_\emptyset^n\right) \xrightarrow{d} \mu \otimes \mu.$$

4 Bivariate Uniqueness Property of the Second Kind for the Frozen Percolation RDE

In this section we prove the bivariate uniqueness property of the second kind for the frozen percolation RDE (1.7).

THEOREM 4.1. *Consider the following bivariate RDE,*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Phi\left(X_1 \wedge X_2; U\right) \\ \Phi\left(Y_1 \wedge Y_2; V\right) \end{pmatrix}, \tag{4.1}$$

where $(X_j, Y_j)_{j=1,2}$ are i.i.d. with same joint law as (X, Y) and have same marginal distribution ν given by

$$\nu\left(dx\right) = \tfrac{dx}{2x^2}, \quad \tfrac{1}{2} < x < 1; \quad \nu\left(\{\infty\}\right) = \tfrac{1}{2}, \tag{4.2}$$

and are independent of (U, V) which are i.i.d. with Uniform$[0, 1]$ distribution; and Φ is given by (1.8). Then the unique solution of this bivariate RDE (4.1) is the product measure $\nu \otimes \nu$.

PROOF. Since ν is a solution of the RDE (1.7), so by Lemma 1.1 (a), the product measure $\nu \otimes \nu$ is a solution of the bivariate RDE (4.1). We will show it is the unique solution. Suppose (X, Y) is a solution of (4.1), and let $F(x,y) := \mathbf{P}(X \le x, Y \le y)$, for $x, y \in [0,1]$ be the joint distribution function. Notice that if $(x,y) \in [0,1]^2 \setminus D$ where $D := \left[\frac{1}{2}, 1\right]^2$ then $F(x,y) = 0$. Now from equation (4.1) if $x, y \in \left[\frac{1}{2}, 1\right]$ then

$$
\begin{aligned}
& F(x,y) \\
=& \mathbf{P}\left(\Phi(X_1 \wedge X_2; U) \le x,\ \Phi(Y_1 \wedge Y_2; V) \le y\right) \\
=& \mathbf{P}\left(U < X_1 \wedge X_2 \le x,\ V < Y_1 \wedge Y_2 \le y\right) \\
=& \mathbf{E}\left[\left(1_{(X_1 \wedge X_2 > U)} - 1_{(X_1 \wedge X_2 > x)}\right)\left(1_{(Y_1 \wedge Y_2 > V)} - 1_{(Y_1 \wedge Y_2 > y)}\right) 1_{(U<x)}\, 1_{(V<y)}\right] \\
=& \int_0^x \int_0^y \left(G^2(x,y) - G^2(x,v) - G^2(u,y) + G^2(u,v)\right)\, dv\, du \qquad (4.3)
\end{aligned}
$$

where $G(x,y) := \mathbf{P}(X > x, Y > y)$, which can be written as

$$
\begin{aligned}
G(x,y) &= F(x,y) - \mathbf{P}(X \le x) - \mathbf{P}(Y \le y) + 1 \\
&= F(x,y) + \tfrac{1}{2x} + \tfrac{1}{2y} - 1\,. \qquad (4.4)
\end{aligned}
$$

Further notice that $G(x,y) = 1$ if $x, y \le \frac{1}{2}$; $G(x,y) = \frac{1}{2x}$ if $x \in \left[\frac{1}{2}, 1\right]$ and $y \le \frac{1}{2}$; and finally $G(x,y) = \frac{1}{2y}$ if $y \in \left[\frac{1}{2}, 1\right]$ and $x \le \frac{1}{2}$. So (4.3) can be written as

$$
\begin{aligned}
F(x,y) \;=\;& xy\, G^2(x,y) - \tfrac{1}{4x} - \tfrac{1}{4y} + \tfrac{3}{4} \\
& - x \int_{\frac{1}{2}}^{y} G^2(x,v)\, dv - y \int_{\frac{1}{2}}^{x} G^2(u,y)\, du + \int_{\frac{1}{2}}^{x} \int_{\frac{1}{2}}^{y} G^2(u,v)\, dv\, du\,, \qquad (4.5)
\end{aligned}
$$

when $x, y \in \left[\frac{1}{2}, 1\right]$. We know that $G_0(x,y) := \frac{1}{4xy}$ on $\left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right]$ is a solution of the equation (4.5) which represent the $\nu \otimes \nu$ solution of the bivariate equation (4.1). Let F_0 be the distribution function for this solution. Note that for this solution $F_0(1,1) = G_0(1,1) = \frac{1}{4}$ is the mass at the point (∞, ∞).

Let $H(x,y) = 1 - G(x,y)/G_0(x,y)$, where $0 \le x, y \le 1$. Notice that $H \equiv 0$ on $[0,1]^2 \setminus D$. Moreover for $(x,y) \in D$,

$$
\begin{aligned}
G(x,y) &= \mathbf{P}(X > x, Y > y) \\
&\le \min\left(\mathbf{P}(X > x), \mathbf{P}(Y > y)\right) \\
&= \frac{1}{2(x \vee y)} \\
&\le \frac{1}{2xy} = 2\, G_0(x,y)\,,
\end{aligned}
$$

where the last inequality follows because $\frac{1}{2} \leq x, y \leq 1$. Thus $-1 \leq H(x,y) \leq 1$ for all $(x,y) \in D$. To prove the bivariate uniqueness all we need to show is $H \equiv 0$ on D.

Recall that G_0 satisfy (4.5), that is,

$$\begin{aligned} F_0(x,y) =& xy\, G_0^2(x,y) - \tfrac{1}{4x} - \tfrac{1}{4y} + \tfrac{3}{4} \\ & - x \int_{\frac{1}{2}}^{y} G_0^2(x,v)\, dv - y \int_{\frac{1}{2}}^{x} G_0^2(u,y)\, du + \int_{\frac{1}{2}}^{x} \int_{\frac{1}{2}}^{y} G_0^2(u,v)\, du\, dv \,. \end{aligned}$$

Further by (4.4) and definition of H we have

$$F_0(x,y) - F(x,y) = G_0(x,y) - G(x,y) = G_0(x,y) H(x,y).$$

So using (4.5) we get

$$\begin{aligned} & G_0(x,y) H(x,y) \\ = \;& F_0(x,y) - F(x,y) \\ = \;& xy \left(G_0^2(x,y) - G^2(x,y) \right) + \int_{\frac{1}{2}}^{x} \int_{\frac{1}{2}}^{y} \left(G_0^2(u,v) - G^2(u,v) \right)\, du\, dv \\ & -x \int_{\frac{1}{2}}^{y} \left(G_0^2(x,v) - G^2(x,v) \right)\, dv - y \int_{\frac{1}{2}}^{x} \left(G_0^2(u,y) - G^2(u,y) \right)\, du \,. \end{aligned} \tag{4.6}$$

Observe that

$$G_0^2 - G^2 = G_0^2 - G_0^2 (1 - H)^2 = G_0^2 \left(2H - H^2 \right) , \tag{4.7}$$

and also using $G_0(x,y) = \frac{1}{4xy}$ on D, we get

$$\begin{aligned} & G_0(x,y) H(x,y) - xy\, G_0^2(x,y) \left(2H(x,y) - H^2(x,y) \right) \\ = \;& G_0(x,y) H(x,y) \left(1 - xy\, G_0(x,y) \left(2 - H(x,y) \right) \right) \\ = \;& \tfrac{1}{4} G_0(x,y) H(x,y) \left(2 + H(x,y) \right) \\ = \;& H(x,y) \, \frac{\left(2 + H(x,y) \right)}{16xy} \,. \end{aligned} \tag{4.8}$$

Thus using (4.6), (4.7) and (4.8) we conclude that for $(x,y) \in D$,

$$\begin{aligned} H(x,y) =& \frac{16xy}{2+H(x,y)} \left[\int_{\frac{1}{2}}^{x} \int_{\frac{1}{2}}^{y} G_0^2(u,v) H(u,v) \left(2 - H(u,v)\right) \, dv \, du \right. \\ & - x \int_{\frac{1}{2}}^{y} G_0^2(x,v) H(x,v) \left(2 - H(x,v)\right) \, dv \\ & \left. - y \int_{\frac{1}{2}}^{x} G_0^2(u,y) H(u,y) \left(2 - H(u,y)\right) \, du \right] \\ =& \frac{xy}{2+H(x,y)} \left[\int_{\frac{1}{2}}^{x} \int_{\frac{1}{2}}^{y} \tfrac{1}{u^2 v^2} H(u,v) \left(2 - H(u,v)\right) \, dv \, du \right. \\ & - \tfrac{1}{x} \int_{\frac{1}{2}}^{y} \tfrac{1}{v^2} H(x,v) \left(2 - H(x,v)\right) \, dv \\ & \left. - \tfrac{1}{y} \int_{\frac{1}{2}}^{x} \tfrac{1}{u^2} H(u,y) \left(2 - H(u,y)\right) \, du \right] . \end{aligned} \tag{4.9}$$

Fix $0 < \varepsilon < \frac{1}{3}$ then there exists a partition $\frac{1}{2} = a_0 < a_1 < a_2 < \ldots < a_{k-1} < a_k = 1$ of $\left[\frac{1}{2}, 1\right]$ with equal lengths, such that

$$\int_{a_i}^{a_{i+1}} \int_{a_j}^{a_{j+1}} \frac{dv \, du}{u^2 v^2} + 2 \int_{a_i}^{a_{i+1}} \frac{du}{u^2} + 2 \int_{a_j}^{a_{j+1}} \frac{dv}{v^2} < \varepsilon \qquad \forall \; 0 \leq i, j \leq k-1 \, , \tag{4.10}$$

where $(x,y) \in D$. This we can do because the function $s \mapsto \frac{1}{s^2}$ is a continuous decreasing function on $\left[\frac{1}{2}, 1\right]$.

Put $B_{i,j} := [a_i, a_{i+1}] \times [a_j, a_{j+1}]$ and let $\parallel H \parallel_{i,j} := \sup_{x,y \in B_{ij}} |H(x,y)|$, for $0 \leq i,j \leq k-1$. Start with $i = j = 0$ and let $(x,y) \in B_{i,j}$, observe that from equation (4.9) we have

$$\begin{aligned} & |H(x,y)| \\ =& \left| \frac{xy}{2+H(x,y)} \right| \times \left| \int_{\frac{1}{2}}^{x} \int_{\frac{1}{2}}^{y} \tfrac{1}{u^2 v^2} \, H(u,v) \left(2 - H(u,v)\right) \, dv \, du \right. \\ & \left. - \tfrac{1}{x} \int_{\frac{1}{2}}^{y} \tfrac{1}{v^2} \, H(x,v) \left(2 - H(x,v)\right) \, dv - \tfrac{1}{y} \int_{\frac{1}{2}}^{x} \tfrac{1}{u^2} \, H(u,y) \left(2 - H(u,y)\right) \, du \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{xy}{2+H(x,y)} \right| \times \Bigg| \int_{a_i}^{x} \int_{a_j}^{y} \tfrac{1}{u^2 v^2}\, H(u,v)\,(2-H(u,v))\; dv\, du \\
&\qquad - \tfrac{1}{x} \int_{a_j}^{y} \tfrac{1}{v^2}\, H(x,v)\,(2-H(x,v))\; dv - \tfrac{1}{y} \int_{a_i}^{x} \tfrac{1}{u^2}\, H(u,y)\,(2-H(u,y))\; du \Bigg| \\
&\leq \frac{3xy \parallel H \parallel_{i,j}}{2+H(x,y)} \times \left[\int_{a_i}^{x} \int_{a_j}^{y} \tfrac{dv\, du}{u^2 v^2} + 2 \int_{a_j}^{y} \tfrac{dv}{v^2} + 2 \int_{a_i}^{x} \tfrac{du}{u^2} \right] \\
&\leq 3 \parallel H \parallel_{i,j} \times \left[\int_{a_i}^{a_{i+1}} \int_{a_j}^{a_{j+1}} \tfrac{dv\, du}{u^2 v^2} + 2 \int_{a_i}^{a_{i+1}} \tfrac{du}{u^2} + 2 \int_{a_j}^{a_{j+1}} \tfrac{dv}{v^2} \right] \qquad (4.11)
\end{aligned}
$$

where the last but one inequality follows because $(x,y) \in B_{i,j} \subseteq D = \left[\frac{1}{2}, 1\right]^2$, and so $x, y \geq \frac{1}{2}$, and also because $1 \leq 2-H \leq 3$ on D, and the last inequality follows because $2 + H \geq 1$ on D. So from (4.11) we get

$$\parallel H \parallel_{i,j} \leq 3\,\varepsilon \parallel H \parallel_{i,j} .$$

But we have chosen $\varepsilon < \frac{1}{3}$, so we must have

$$H(x,y) = 0 \;\text{ for all }\; (x,y) \in B_{i,j} .$$

Now we do induction on two indices i and j in the following way. For every fixed $0 \leq l \leq k-1$ we start with $i = j = l$ and then continue with $i \in \{l, l+1, \ldots, k-1\}$, and $j \in \{l, l+1, \ldots, k-1\}$, repeating the above argument in each step. This finally yields

$$H(x,y) = 0 \;\text{ for all }\; (x,y) \in D ,$$

which completes the proof. □

5 Proof of Theorem 1.2

Now to prove the Theorem 1.2 we will use the part (b) of our equivalence theorem (Theorem 1.1). The bivariate uniqueness of the second kind has been proved in Theorem 4.1, so it only remains to check the technical condition of Theorem 1.1(b).

For that suppose $\nu_n^{(2)} \xrightarrow{d} \nu^{(2)}$ where $\left\{\nu_n^{(2)}\right\}_{n \geq 1}$ and $\nu^{(2)}$ are bivariate distributions on I^2 with marginals ν. Let F_n be the distribution function for

$\nu_n^{(2)}$ and F be that for $\nu^{(2)}$. We define G_n and G in similar manner as done in equation (4.4). Following argument similar of derivation of the equation (4.3) we get that for $x, y \in \left[\frac{1}{2}, 1\right]$,

$$T \otimes T\left(F_n\right)(x, y) = \int_0^x \int_0^y \left(G_n^2(x, y) - G_n^2(x, v) - G_n^2(u, y) + G_n^2(u, v)\right)\, dv\, du\,.$$

The rest follows using the dominated convergence theorem.

6 Remarks and Complement

6.1. Tail-triviality and long range independence. Gamarnik et al. (2004) introduced the concept of *long range independence* for some particular RDEs, similar concept was also used in later works (Bandyopadhyay, 2005, Bandyopadhyay et al., 2006). Borrowing their idea we define the long range independence property for an invariant RTP as follows.

DEFINITION 6.1. *Suppose $\left(X_{\mathbf{i}}\right)_{\mathbf{i} \in \mathcal{V}}$ be an invariant RTP with marginal μ, then we will say that the* long range independence property *holds if*

$$\lim_{d \to \infty} \sup_{\substack{x_{\mathbf{i}} \in S \\ |\mathbf{i}| = d}} \rho\left(dist\left(X_\emptyset \,\middle|\, X_{\mathbf{i}} = x_{\mathbf{i}},\, |\mathbf{i}| = d\right)\,,\, \mu\right) = 0\,, \tag{6.1}$$

where ρ is a metric for the weak convergence topology on $\mathcal{P}(S)$.

PROPOSITION 6.1. *Suppose $\left(X_{\mathbf{i}}\right)_{\mathbf{i} \in \mathcal{V}}$ is an invariant RTP with marginal μ which has long range independence property as defined above, then it must have trivial tail.*

PROOF. Let $\mathcal{H} = \bigcap_{n \geq 0} \mathcal{H}_n$ be the tail of the RTP $\left(X_{\mathbf{i}}\right)_{\mathbf{i} \in \mathcal{V}}$ where $\mathcal{H}_n$ is as defined in (1.5). Let $\Lambda : S \to \mathbb{R}$ be a bounded continuous function and consider the conditional expectation $\mathbf{E}\left[\Lambda\left(X_\emptyset\right) \,\middle|\, \mathcal{H}_n\right]$, by martingale convergence theorem

$$\mathbf{E}\left[\Lambda\left(X_\emptyset\right) \,\middle|\, \mathcal{H}_n\right] \longrightarrow \mathbf{E}\left[\Lambda\left(X_\emptyset\right) \,\middle|\, \mathcal{H}\right] \quad \text{a.s.}$$

On the other hand from the long range independence property it follows

$$\mathbf{E}\left[\Lambda\left(X_\emptyset\right) \,\middle|\, \mathcal{H}_n\right] \longrightarrow \mathbf{E}\left[\Lambda\left(X_\emptyset\right)\right] \quad \text{a.s.}\,,$$

since $X_\emptyset \sim \mu$. Thus we get

$$\mathbf{E}\left[\Lambda\left(X_\emptyset\right) \,\middle|\, \mathcal{H}\right] = \mathbf{E}\left[\Lambda\left(X_\emptyset\right)\right] \quad \text{a.s.}\,,$$

which is true for every bounded continuous function Λ, hence we must have $X_\emptyset$ independent of $\mathcal{H}$. So by Lemma 2.1 we conclude that the tail of the RTP is trivial. $\square$

Now the converse is not necessarily true. To see this, we first note that in order for a RTP to have the long range independence, the underlying RDE need to satisfy certain properties. For example,

LEMMA 6.1. *Suppose an invariant RTP with marginal μ has long range independence property. If T is the associated operator for the RDE with domain $\mathcal{P}$ then for any $\mu' \in \mathcal{P}$ we must have*

$$T^n\left(\mu'\right) \stackrel{d}{\longrightarrow} \mu \quad as\ n \to \infty\,. \tag{6.2}$$

The proof of this lemma easily follows from equation (6.1), the details are left for the readers. But from this lemma we see that if an invariant RTP with marginal μ has long range independence property then the underlying RDE necessarily has unique solution μ. Now Aldous and Bandyopadhyay (2005) gives several examples of RDEs which may have multiple solutions but some of which can be endogenous. To give a specific example, we consider the Quicksort RDE, which is given by

$$X \stackrel{d}{=} UX_1 + (1-U)\,X_2 + 2U\log U + 2\,(1-U)\log\,(1-U) + 1 \quad \text{on } \mathbb{R}, \tag{6.3}$$

It is known that this RDE has a two parameter family of solutions (Fill et al., 2000), and only those with finite first moment are endogenous (see Aldous and Bandyopadhyay, 2005, Theorem 21). So an invariant RTP with a marginal which is a solution of (6.3) and has finite first moment, will be endogenous and hence from Proposition 1.1 has trivial tail. But by Lemma 6.1 we conclude that this invariant RTP can not have long range independence property because, the RDE (6.3) has many solutions.

Finally, even though it is not quite related to tail-triviality, but we still note that the above example also shows that endogeny does not imply long range independence property. Interesting enough the converse is not true either. It is in fact easy to show that the unique invariant RTP of the Example 1.1 discussed in Section 1 has long range independence property, but it is not endogenous. In light of Lemma 6.1 one may conjecture that if a RDE has unique solution with full domain of attraction, and the solution is endogenous, then it must have the long range independence property, but this to best of our knowledge remains as an open problem.

6.2. Frozen percolation on r-regular trees. Using exactly similar arguments as done in the case of infinite regular binary tree one can construct an automorphism invariant version of frozen percolation process on a infinite r-regular tree $\mathcal{T}_r$ in which each vertex has degree $r \geq 3$ (see Aldous, 2000) for details). In this setting the RDE is given by

$$Y^r \stackrel{d}{=} \Phi^r \left(Y_1^r \wedge Y_2^r \wedge \cdots \wedge Y_{r-1}^r ; U \right) \text{ on } I^r := \left[\tfrac{1}{r-1}, 1 \right] \cup \{\infty\}, \tag{6.4}$$

where $\left(Y_j^r \right)_{1 \leq j \leq r-1}$ are i.i.d. with same law as Y^r and are independent of $U \sim \text{Uniform}[0,1]$; and $\Phi^r : I^r \times [0,1] \to I^r$ is the function defined by equation (1.8). It is easy to check that the unique solution of this RDE with full support and having no atom in $\left[\frac{1}{r-1}, 1 \right]$ is given by

$$\nu^r (dy) = \frac{dy}{(r-2)(r-1)^{\frac{1}{r-2}} \, y^{\frac{r-1}{r-2}}}, \quad \tfrac{1}{r-1} < y < 1, \quad \nu^r(\{\infty\}) = \frac{1}{(r-1)^{\frac{1}{r-2}}}. \tag{6.5}$$

Naturally the case $r = 3$ gives back the RDE (1.7) and its fundamental solution ν. Interesting enough our argument to prove the bivariate uniqueness of the second kind for the frozen percolation RDE (1.7) extend essentially unchanged in this setting (only the constants need to be changed). So the invariant RTP associated with the RDE (6.4) with marginal ν^r also has trivial tail. Once again the question of non-endogeny remains open.

Acknowledgement. An incomplete version of the main result of this article appears in author's doctoral dissertation, written under the guidance of Professor David J. Aldous, whom the author would like to thank for many discussions, and also for his continuous encouragement. The author would also like to thank Professor Don Aronson for his insightful remarks on solving non-linear integral equations, and to Krishanu Maulik for carefully reading an earlier version of this paper. Thanks are also due to an anonymous referee for a thorough review of the paper.

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Paper received November 2005; revised January 2006.