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# Right-most position of a last progeny modified time inhomogeneous branching random walk

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In this work, we consider a modification of the time *inhomogeneous* branching random walk, where the driving increment distribution changes over time macroscopically. Following Bandyopadhyay and Ghosh (2021), we give certain independent and identically distributed (i.i.d.) displacements to all the particles at the last generation. We call this process *last progeny modified time inhomogeneous branching random walk (LPMTI-BRW)*. Under very minimal assumptions on the underlying point processes of the displacements, we show that the maximum displacement converges to a non-trivial limit after an appropriate centering which is either linear or linear with a logarithmic correction. Interestingly, the limiting distribution depends only on the first set of increments. We also derive Brunet–Derrida-type results of point process convergence of our LPMTI-BRW to a decorated Poisson point process. As in the case of the maximum, the limiting point process also depends only on the first set of increments. Our proofs are based on a method of coupling the maximum displacement with an appropriate linear statistics, which was introduced by Bandyopadhyay and Ghosh (2021).



## 1. Introduction

One dimensional *Branching random walk (BRW)* was introduced by Hammersley (1974) in the early '70s and since then it has received significant attention from various researches. Some references on this classical model with *homogeneous* displacements which are relevant to our work are Kingman (1975), Biggins (1976), Bramson (1978), Biggins and Kyprianou (1997), Bramson and Zeitouni (2007), Hu and Shi (2009), Bramson and Zeitouni (2009), Addario-Berry and Reed (2009), Aïdékon (2013), Aïdékon and Shi (2014), Madaule (2017). Under reasonable assumptions, it is well known from these literature that in the homogeneous case the maximum displacement grows linearly, with a logarithmic correction, and is tight around its median. The *inhomogeneous* case has received some attention in recent years (Bramson and Zeitouni, 2009; Fang, 2012; Fang and Zeitouni, 2012). Under certain uniform regularity assumptions, Bramson and Zeitouni (2009) and Fang (2012) showed that in the inhomogeneous case also the maximum displacement re-centered around its median is tight. Later Fang and Zeitouni (2012) showed that in the binary branching with independent Gaussian displacements the exact coefficients of the centering terms, both for the linear term and also for the logarithmic correction term differ based on the increasing/decreasing variance of the time inhomogeneous displacements. This particular example is interesting and relevant for our work and is described in more details in Section 4.

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Bandyopadhyay and Ghosh (2021) introduced a new modification of the classical homogeneous BRW, where they added a set of i.i.d. displacements with a specific form at the last generation. This new process was termed as the *last progeny modified BRW (LPM-BRW)* (Bandyopadhyay and Ghosh, 2021). In this work, we consider an inhomogeneous version with the same modification. Our results complement and work of Fang and Zeitouni (2012) in the context of this new model of last progeny modified version of BRW. We shall show that under mild conditions the maximum displacement after appropriate centering, which can either be linear or linear with a logarithmic correction, has a weak limit, where the limiting distribution depends only on the point process of the first set of displacements. This result is unusual and can have some non-trivial statistical applications (see Remark 2.1).

### 1.1. Model

We fix $k \in \mathbb{N}$. For each $i \in \{1, 2, \ldots, k\}$, we let $Z_i$ be a point process with $N_i := Z_i(\mathbb{R}) < \infty$ a.s. and $q_i$ be a sequence of integers satisfying $\sum_{i=1}^{k} q_i(n) = n$, and we write $t_m = \sum_{i=1}^{m} q_i(n)$. A *time inhomogeneous branching random walk (TI-BRW)* is a discrete-time stochastic process that can be described for each $n \geq 1$ as follows:

At the 0-th generation, we start with an initial particle at the origin. At time $t \in (t_{m-1}, t_m]$, each of the particles at generation $(t-1)$ gives birth to a random number of offspring distributed according to $N_m$. The offspring are then given random displacements independently and according to a copy of the point process $Z_m$.

For a particle $v$ in the $t$th generation, we write $|v| = t$ and $S(v)$ denotes its position, which is the sum of all the displacements the particle $v$ and its ancestors have received. We shall call the process $\{S(v) : |v| = t, 0 \leq t \leq n, n \geq 1\}$ a *time inhomogeneous branching random walk (TI-BRW)*. We denote $R_n := \max_{|v|=n} S(v)$ as the right-most position at the $n$th generation.

Following our earlier work Bandyopadhyay and Ghosh (2021), in this model we also introduce a non-negative real number $\theta > 0$, which should be thought of as a scaling parameter for the additional displacement we give to each particle at the $n$th generation. The additional displacements are of the form $\frac{1}{\theta}(-\log E_v)$, where $\{E_v\}_{|v|=n}$ are i.i.d. Exponential(1) and are independent of the process $\{S(v) : |v| \leq n\}$. We denote by $R_n^* \equiv R_n^*(\theta)$ the right-most position of this *last progeny modified time inhomogeneous branching random walk (LPMTI-BRW)*.

### 1.2. Assumptions

we first introduce the following important quantities. For each point process $Z_i = \sum_{j \geq 1} \delta_{\xi_j^{(i)}}$ with $1 \leq i \leq k$, we define

$$\nu_i(a) := \log \mathbb{E}\left[\int_{\mathbb{R}} e^{ax} Z_i(dx)\right] = \log \mathbb{E}\left[\sum_{i=1}^{N_i} e^{a\xi_j^{(i)}}\right], \tag{1}$$

for $a \in \mathbb{R}$, whenever the expectations exist. Needless to say that for each $i \in \{1, 2, \ldots, k\}$, $\nu_i$ is the logarithm of the moment-generating function of the point process $Z_i$.

Throughout this paper, for each $i \in \{1, 2, \ldots, k\}$, we assume the following:

**(A1)** $\nu_i(a)$ is finite for all $a \in (-\vartheta, \infty)$ for some $\vartheta > 0$.

**(A2)** The point process $Z_i$ is *non-trivial*, and the extinction probability of the underlying branching process is 0, i.e., $\mathbb{P}(N_i = 1) < 1$, $\mathbb{P}(Z_i(\{a\}) = N_i) < 1$ for any $a \in \mathbb{R}$ and $\mathbb{P}(N_i \geq 1) = 1$.

**(A3)** $N_i$ has finite $(1+p)$-th moment for some $p > 0$.

### 1.3. Outline

In Section 2, we state our main results, which are proved in Section 3. In Section 4 we consider an important example and compare our results with that of the existing literature.

## 2. Main results

We first introduce some constants related to the point processes $Z_i$'s. For $i \in \{1, 2, \ldots, k\}$, we define

$$\theta_{(i)} := \inf\left\{a > 0 : \frac{\nu_i(a)}{a} = \nu_i'(a)\right\}. \tag{2}$$

From our earlier work (Bandyopadhyay and Ghosh, 2021), we note that $\nu_i$'s are strictly convex under assumption **(A1)** and **(A2)**, thus, the above set is at most singleton. If it is a singleton, then $\theta_{(i)}$ is the unique point in $(0, \infty)$ such that a tangent from the origin to the graph of $\nu_i(a)$ touches the graph at $a = \theta_{(i)}$. And if it is empty, then by definition $\theta_{(i)}$ takes value $\infty$, and there does not exist any tangent from the origin to the graph of $\nu_i(a)$ on the right half-plane.

### 2.1. Asymptotic limits

Our first result is a centered asymptotic limit of the right-most position, which is similar to the results in below-the-boundary case for *last progeny modified BRW (LPM-BRW)* shown by Bandyopadhyay and Ghosh (2021).

**Theorem 2.1.** *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$, there exists a random variable $H^{\infty}_{\theta,(1)}$ depending only on $\theta$ and $Z_1$, such that,*

$$R_n^*(\theta) - \sum_{i=1}^{k} \frac{q_i(n)\nu_i(\theta)}{\theta} \xrightarrow{d} H^{\infty}_{\theta,(1)}. \tag{3}$$

**Theorem 2.2.** *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$ and $\theta_{(1)} < \min_{i\neq 1} \theta_{(i)} \le \infty$, then there exists a random variable $H^{\infty}_{\theta_{(1)},(1)}$ depending only on $Z_1$, such that,*

$$R_n^*\big(\theta_{(1)}\big) - \sum_{i=1}^{k} \frac{q_i(n)\nu_i\big(\theta_{(1)}\big)}{\theta_{(1)}} + \frac{1}{2\theta_{(1)}} \log\left(q_1(n)\right) \xrightarrow{d} H^{\infty}_{\theta_{(1)},(1)}. \tag{4}$$

**Remark 2.1.** It is very interesting to note that the centered asymptotic limit only depends on the point process of the first set of displacements. More interestingly, the result is valid as long as $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$. In particular, the rate of divergence of $q_1(n)$ can be very slow but we will still have the centered asymptotic limit depends only on the distribution of $Z_1$. Thus our model LPMTI-BRW may be used as a very efficient "*statistical sheave*" to filter out the distribution of the first set of displacements (may be thought as the "*signal*") from a number of others which may be considered as "*noise*" and of much larger in numbers compared to that of the "*signal*". We thus feel this result may have greater statistical significance.

As we will see in the proof of the above theorem (see Section 3), we have a slightly stronger result. As in Theorem 2.5 of Bandyopadhyay and Ghosh (2021), we let

$$\hat{H}^{\infty}_{\theta,(1)} = \frac{1}{\theta} \log D^{\infty}_{\theta,(1)},$$

where $D^{\infty}_{\theta,(1)}$ is the unique solution of the following *linear recursive distributional equation* with mean 1.

$$\Delta \stackrel{\mathrm{d}}{=\!=} \sum_{|v|=1} e^{\theta S(v) - \nu_1(\theta)} \Delta_v, \tag{5}$$

where $\Delta_v$ are i.i.d. and has the same distribution as that of $\Delta$. As in Theorem 2.3 of Bandyopadhyay and Ghosh (2021), we also let

$$\hat{H}^{\infty}_{\theta_{(1)},(1)} = \frac{1}{\theta_{(1)}} \left[\log D^{\infty}_{\theta_{(1)},(1)} + \frac{1}{2}\log\left(\frac{2}{\pi\sigma_1^2}\right)\right],$$

where

$$D^{\infty}_{\theta_{(1)},(1)} \stackrel{\text{a.s.}}{=\!=} \lim_{n\to\infty} - \sum_{|v|=q_1(n)} \big(\theta_{(1)}S_v - q_1(n)\nu\big(\theta_{(1)}\big)\big)\, e^{\theta_{(1)}S_v - q_1(n)\nu(\theta_{(1)})}, \tag{6}$$

$$\sigma_1^2 := \mathbb{E}\left[\sum_{|v|=1} \big(\theta_{(1)}S_v - \nu\big(\theta_{(1)}\big)\big)^2 e^{\theta_{(1)}S_v - \nu(\theta_{(1)})}\right]. \tag{7}$$

Then we have

**Theorem 2.3.** *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$,*

$$R_n^*(\theta) - \sum_{i=1}^{k} \frac{q_i(n)\nu_i(\theta)}{\theta} - \hat{H}^{\infty}_{\theta,(1)} \xrightarrow{d} -\log E, \tag{8}$$

*where $E \sim$ Exponential$(1)$.*

**Theorem 2.4.** *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$ and $\theta_{(1)} < \min_{i\neq 1} \theta_{(i)} \le \infty$, then*

$$R_n^*\big(\theta_{(1)}\big) - \sum_{i=1}^{k} \frac{q_i(n)\nu_i\big(\theta_{(1)}\big)}{\theta_{(1)}} + \frac{1}{2\theta_{(1)}} \log\left(q_1(n)\right) - \hat{H}^{\infty}_{\theta_{(1)},(1)} \xrightarrow{d} -\log E, \tag{9}$$

*where $E \sim$ Exponential$(1)$.*

**Remark 2.2.** Note that $H^{\infty}_{\theta,(1)}$ in Theorem 2.1 has the same distribution as $\hat{H}^{\infty}_{\theta,(1)} - \log E$, where $E \sim \text{Exponential}(1)$ and is independent of $\hat{H}^{\infty}_{\theta,(1)}$. Similarly, $H^{\infty}_{\theta_{(1)},(1)}$ in Theorem 2.2 has the same distribution as $\hat{H}^{\infty}_{\theta_{(1)},(1)} - \log E$, where $E \sim \text{Exponential}(1)$ and is independent of $\hat{H}^{\infty}_{\theta_{(1)},(1)}$.

As a corollary of the above results, we obtain that if the centering term converges after dividing by $n$, then $R^*_n/n$ has a limit in probability. In particular, we have the following result:

**Theorem 2.5.** *If for all $1 \le i \le k$, $q_i(n) \longrightarrow \infty$ satisfying $\lim_{n\to\infty} q_i(n)/n = \alpha_i \ge 0$, then for any $\theta < \min_i \theta_{(i)} \le \infty$ and also for $\theta = \theta_{(1)} < \min_{i\ne 1} \theta_{(i)} \le \infty$,*

$$\frac{R^*_n(\theta)}{n} \xrightarrow{p} \sum_{i=1}^{k} \frac{\alpha_i \nu_i(\theta)}{\theta}. \tag{10}$$

*2.2. Brunet-Derrida type results*

Here we present results of the type Brunet and Derrida (2011) for our LPMTI-BRW.
For any $\theta < \min_i \theta_{(i)} \le \infty$, we define

$$Z_n(\theta) = \sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - \sum_{i=1}^{k} q_i(n)\nu_i(\theta) - \theta \hat{H}^{\infty}_{\theta,(1)}\right\}}, \tag{11}$$

and for $\theta_{(1)} < \min_{i\ne 1} \theta_{(i)} \le \infty$, we define

$$Z_n\left(\theta_{(1)}\right) = \sum_{|v|=n} \delta_{\left\{\theta_{(1)} S_v - \log E_v - \sum_{i=1}^{k} q_i(n)\nu_i(\theta_{(1)}) + \frac{1}{2}\log(q_1(n)) - \theta_{(1)} \hat{H}^{\infty}_{\theta_{(1)},(1)}\right\}}, \tag{12}$$

where $\hat{H}^{\infty}_{\theta,(1)}$ and $\hat{H}^{\infty}_{\theta_{(1)},(1)}$ are as in Theorems 2.3 and 2.4. Our first result is the weak convergence of the point processes $(Z_n(\theta))_{n\ge 0}$, which is similar to the results for LPM-BRW as shown by Bandyopadhyay and Ghosh (2021).

**Theorem 2.6.** *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$ and also for $\theta = \theta_{(1)} < \min_{i\ne 1} \theta_{(i)} \le \infty$,*

$$Z_n(\theta) \xrightarrow{d} \mathcal{Y},$$

*where $\mathcal{Y}$ is a decorated Poisson point process. In particular, $\mathcal{Y} = \sum_{j\ge 1} \delta_{-\log \zeta_j}$, where $\mathcal{N} = \sum_{j\ge 1} \delta_{\zeta_j}$ is a homogeneous Poisson point process on $\mathbb{R}_+$ with intensity 1.*

The following is a slightly weaker version of the above theorem.

**Theorem 2.7.** *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$,*

$$\sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - \sum_{i=1}^{k} q_i(n)\nu_i(\theta)\right\}} \xrightarrow{d} \sum_{j\ge 1} \delta_{-\log \zeta_j + \theta \hat{H}^{\infty}_{\theta,(1)}}, \tag{13}$$

*and for $\theta_{(1)} < \min_{i\ne 1} \theta_{(i)} \le \infty$,*

$$\sum_{|v|=n} \delta_{\left\{\theta_{(1)} S_v - \log E_v - \sum_{i=1}^{k} q_i(n)\nu_i(\theta_{(1)}) + \frac{1}{2}\log(q_1(n))\right\}} \xrightarrow{d} \sum_{j\ge 1} \delta_{-\log \zeta_j + \theta_{(1)} \hat{H}^{\infty}_{\theta_{(1)},(1)}}, \tag{14}$$

*where $\mathcal{N} = \sum_{j\ge 1} \delta_{\zeta_j}$ is a homogeneous Poisson point process on $\mathbb{R}_+$ with intensity 1, which is independent of the process $\{S(v) : |v| \le n\}$.*

Let $\mathcal{Y}_{\max}$ be the right-most position of the point process $\mathcal{Y}$, and $\overline{\mathcal{Y}}$ be the point process $\mathcal{Y}$ viewed from its right-most position, i.e.,

$$\overline{\mathcal{Y}} = \sum_{j\ge 1} \delta_{-\log \zeta_j - \mathcal{Y}_{\max}}.$$

Then as a corollary of the above theorem, we get the following result, which confirms the validity of the *Brunet–Derrida Conjecture* for LPMTI-BRW for any $\theta < \min_i \theta_{(i)} \le \infty$.

**Theorem 2.8.** *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$ and also for $\theta = \theta_{(1)} < \min_{i\ne 1} \theta_{(i)} \le \infty$,*

$$\sum_{|v|=n} \delta_{\left\{\theta S(v) - \log E_v - \theta R^*_n(\theta)\right\}} \xrightarrow{d} \overline{\mathcal{Y}}.$$

## 3. Proofs of the main results

### 3.1. Proof of Theorems 2.1–2.4

To prove these theorems, we need the following technical result. We define the linear statistics

$$W_n(\theta) \equiv W_n(\theta)(q_1(n), \ldots, q_k(n), Z_1, \ldots, Z_k) := \sum_{|v|=n} e^{\theta S(v)}. \tag{15}$$

Then we have

**Lemma 3.1.** *Suppose $q_i(n) \longrightarrow \infty$ for all $1 \le i \le k$, then for any $\theta < \min_i \theta_{(i)} \le \infty$ and also for $\theta = \theta_{(1)} < \min_{i \neq 1} \theta_{(i)} \le \infty$,*

$$\frac{W_n(\theta)(q_1(n), \ldots, q_k(n), Z_1, \ldots, Z_k) \cdot e^{-\sum_{i=1}^k q_i(n)\nu_i(\theta)}}{W_{q_1(n)}(\theta)(q_1(n), Z_1) \cdot e^{-q_1(n)\nu_1(\theta)}} \xrightarrow{p} 1$$

**Proof.** Without loss of generality we can assume that $\nu_i(\theta) = 0$ for all $i \in \{1, 2, \ldots, k\}$. This can be made to satisfy by centering each point process $Z_i$ by $\nu_i(\theta)$.

We prove the lemma by induction. Note that for $k = 1$, the lemma holds trivially. We assume the lemma holds for $k = m - 1$ for some $m \in \mathbb{N}$.

Now, take $k = m$. For each $v$ such that $|v| = q_1(n)$, we define

$$\overline{W}_{n,v}(\theta) = \sum_{|u|=n, v<u} e^{\theta(S(u)-S(v))}. \tag{16}$$

Here, $v < u$ means $u$ is a descendant of $v$. Notice that $\left\{\overline{W}_{n,v}(\theta)\right\}_{|v|=q_1 n}$ are i.i.d. and have the same distribution as

$$W_{n-q_1(n)}(\theta)(q_2(n), \ldots, q_m(n), Z_2, \ldots, Z_m),$$

which by our induction hypothesis and Proposition 4.2 (ii) of Bandyopadhyay and Ghosh (2021) converges in probability to $D^{\infty}_{\theta,(2)}$. Since both of them has mean 1, we also have

$$W_{n-q_1(n)}(\theta)(q_2(n), \ldots, q_m(n), Z_2, \ldots, Z_m) \xrightarrow{L_1} D^{\infty}_{\theta,(2)}. \tag{17}$$

Now, observe that

$$\frac{W_n(\theta)(q_1(n), \ldots, q_m(n), Z_1, \ldots, Z_m)}{W_{q_1(n)}(\theta)(q_1(n), Z_1)} - 1 = \sum_{|v|=q_1(n)} \frac{e^{\theta S(v)}}{\sum_{|u|=q_1(n)} e^{\theta S(u)}} \left(\overline{W}_{n,v}(\theta) - 1\right). \tag{18}$$

Now, from (5.5) and (5.6) of Bandyopadhyay and Ghosh (2021), we know that

$$M_n(\theta) := \max_{|v|=q_1(n)} \frac{e^{\theta S(v)}}{\sum_{|u|=q_1(n)} e^{\theta S(u)}} \xrightarrow{p} 0.$$

Let $\mathcal{F}_n$ be the $\sigma$-field generated by $\{S(v) : |v| \le q_1(n)\}$. Then using Lemma 2.1 of Biggins and Kyprianou (1997), which is a particular case of Lemma 2.2 in Kurtz (1972), we get that for every $0 < \varepsilon < 1/2$,

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{W_n(\theta)(q_1(n), \ldots, q_m(n), Z_1, \ldots, Z_m)}{W_{q_1(n)}(\theta)(q_1(n), Z_1)} - 1\right| > \varepsilon \,\middle|\, \mathcal{F}_n\right) \\
&\le \frac{2}{\varepsilon^2}\left(\int_0^{\frac{1}{M_n(\theta)}} M_n(\theta)t \cdot \mathbb{P}\left(\left|W_{n-q_1(n)}(\theta)(q_2(n), \ldots, q_m(n), Z_2, \ldots, Z_m) - 1\right| > t\right) dt\right. \\
&\qquad \left. + \int_{\frac{1}{M_n(\theta)}}^{\infty} \mathbb{P}\left(\left|W_{n-q_1(n)}(\theta)(q_2(n), \ldots, q_m(n), Z_2, \ldots, Z_m) - 1\right| > t\right) dt\right) \\
&\le \frac{2}{\varepsilon^2}\left(\int_0^{\infty} \mathbb{P}\left(\left|W_{n-q_1(n)}(\theta)(q_2(n), \ldots, q_m(n), Z_2, \ldots, Z_m) - D^{\infty}_{\theta,(2)}\right| > t/2\right) dt\right. \\
&\qquad \left. + \int_0^{\frac{1}{M_n(\theta)}} M_n(\theta)t \cdot \mathbb{P}\left(\left|D^{\infty}_{\theta,(2)} - 1\right| > t/2\right) dt + \int_{\frac{1}{M_n(\theta)}}^{\infty} \mathbb{P}\left(\left|D^{\infty}_{\theta,(2)} - 1\right| > t/2\right) dt\right)
\end{aligned} \tag{19}$$

By using the dominated convergence theorem, the second and the third term on the right-hand side of (19) converges to 0 as $n \to \infty$, and by (17), the first term also tends to 0 as $n \to \infty$. Then by taking expectation and using the dominated convergence theorem again, we get

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\frac{W_n(\theta)(q_1(n),\ldots,q_m(n),Z_1,\ldots,Z_m)}{W_{q_1(n)}(\theta)(q_1(n),Z_1)}-1\right|>\varepsilon\right)=0 \;\Rightarrow\; \frac{W_n(\theta)(q_1(n),\ldots,q_m(n),Z_1,\ldots,Z_m)}{W_{q_1(n)}(\theta)(q_1(n),Z_1)}\xrightarrow{p}1. \tag{20}$$

So if the lemma holds for $k=m-1$, it also holds for $k=m$. Therefore, by using induction we complete the proof. □

Now, note that

$$\theta R_n^*(\theta)=\max_{|v|=n}(\theta S(v)-\log E_v)=-\log\left(\min_{|v|=n}\frac{E_v}{e^{\theta S(v)}}\right)\stackrel{d}{=}-\log\left(\frac{E}{\sum_{|v|=n}e^{\theta S(v)}}\right)=\log W_n(\theta)-\log E, \tag{21}$$

where $E\sim$ Exponential (1) and is independent of the process $\{S(v):|v|\le n\}$. Similarly,

$$\theta R_n^*(\theta)-\log W_n(\theta)=\max_{|v|=n}(\theta S(v)-\log E_v-\log W_n(\theta))=-\log\left(\min_{|v|=n}E_v\left(\frac{e^{\theta S(v)}}{\sum_{|u|=n}e^{\theta S(u)}}\right)^{-1}\right)\stackrel{d}{=}-\log E. \tag{22}$$

Now, by Proposition 4.2 (ii) of Bandyopadhyay and Ghosh (2021), for any $\theta<\theta_{(1)}\le\infty$,

$$W_{q_1(n)}(\theta)(q_1(n),Z_1)\cdot e^{-q_1(n)\nu_1(\theta)}\to D^\infty_{\theta,(1)} \text{ a.s.,} \tag{23}$$

and by Theorem 1.1 of Aïdékon and Shi (2014),

$$\sqrt{q_1(n)}\cdot W_{q_1(n)}(\theta)(q_1(n),Z_1)\cdot e^{-q_1(n)\nu_1(\theta)}\xrightarrow{p}\left(\frac{2}{\pi\sigma_1^2}\right)^{1/2}\cdot D^\infty_{\theta_{(1)},(1)}. \tag{24}$$

Now, combining Lemma 3.1 together with Eqs. (21), (22), (23) and (24) proves the theorems.

### 3.2. Proof of Theorems 2.6 and 2.7

Let $\mathcal{G}_n$ be the $\sigma$-algebra generated by the TI-BRW defined up to generation $n$. We know that conditioned on $\mathcal{G}_n$, $\left\{E_vW_n(\theta)e^{-\theta S(v)}\right\}_{|v|=n}$ are independent, and

$$E_vW_n(\theta)e^{-\theta S(v)}\big|\,\mathcal{G}_n\sim\text{ Exponential }\left(\frac{e^{\theta S(v)}}{W_n(\theta)}\right).$$

Now, for any $\theta<\min_{i\neq1}\theta_{(i)}$, we choose $a$ such that $\theta<a<\min_{i\neq1}\theta_{(i)}$ and note that

$$\begin{aligned}&\mathbb{E}\left[\frac{\max_{|v|=n}e^{\theta S(v)-\sum_{i=1}^k q_i(n)\nu_i(\theta)}}{W_{q_1(n)}(\theta)(q_1(n),Z_1)e^{-q_1(n)\nu_1(\theta)}}\right]\\ &\le\mathbb{E}\left[\sum_{|u|=q_1(n)}\frac{e^{\theta S(u)}}{W_{q_1(n)}(\theta)(q_1(n),Z_1)}\left(\sum_{|v|=n,u<v}e^{a(S(v)-S(u))}\right)^{\theta/a}e^{-\sum_{i=2}^k q_i(n)\nu_i(\theta)}\right]\\ &=\mathbb{E}\left[\left(W_{n-q_1(n)}(a)(q_2(n),\ldots,q_k(n),Z_2,\ldots,Z_k(n))\right)^{\theta/a}\right]\cdot e^{-\sum_{i=2}^k q_i(n)\nu_i(\theta)}\end{aligned} \tag{25}$$

As discussed in the proof of Lemma 3.1,

$$W_{n-q_1(n)}(a)(q_2(n),\ldots,q_k(n),Z_2,\ldots,Z_k(n))\cdot e^{-\sum_{i=2}^k q_i(n)\nu_i(\theta)}\xrightarrow{L_1}D^\infty_{\theta,(2)}. \tag{26}$$

Since $\frac{\nu_i(\theta)}{\theta}>\frac{\nu_i(a)}{a}$ for all $i\ge2$, combining Eqs. (25) and (26), we get

$$\mathbb{E}\left[\frac{\max_{|v|=n}e^{\theta S(v)-\sum_{i=1}^k q_i(n)\nu_i(\theta)}}{W_{q_1(n)}(\theta)(q_1(n),Z_1)e^{-q_1(n)\nu_1(\theta)}}\right]\to0\;\Rightarrow\;\frac{\max_{|v|=n}e^{\theta S(v)-\sum_{i=1}^k q_i(n)\nu_i(\theta)}}{W_{q_1(n)}(\theta)(q_1(n),Z_1)e^{-q_1(n)\nu_1(\theta)}}\xrightarrow{p}0,$$

which, together with Lemma 3.1, suggests that for any $\theta<\min_i\theta_{(i)}\le\infty$ and also for $\theta=\theta_{(1)}<\min_{i\neq1}\theta_{(i)}\le\infty$,

$$\max_{|v|=n}\frac{e^{\theta S(v)}}{W_n(\theta)}\xrightarrow{P}0. \tag{27}$$

Also, note that

$$\sum_{|v|=n}\frac{e^{\theta S(v)}}{W_n(\theta)}=1,$$

Therefore by Lemma 5.2 of Bandyopadhyay and Ghosh (2021), for any positive integer $r$, Borel sets $B_1, B_2, \dots, B_r$ and non-negative integers $t_1, t_2, \dots, t_r$, we have

$$\mathbb{P}\left(\sum_{|v|=n}\delta_{E_v W_n(\theta)e^{-\theta S(v)}}(B_1)=t_1,\dots,\sum_{|v|=n}\delta_{E_v W_n(\theta)e^{-\theta S(v)}}(B_r)=t_r\,\middle|\,\mathcal{G}_n\right)\xrightarrow{P}\mathbb{P}\left(\mathcal{N}(B_1)=t_1,\dots,\mathcal{N}(B_r)=t_r\right).$$

Then, using the dominated convergence theorem, we get

$$\mathbb{P}\left(\sum_{|v|=n}\delta_{E_v W_n(\theta)e^{-\theta S(v)}}(B_1)=t_1,\dots,\sum_{|v|=n}\delta_{E_v W_n(\theta)e^{-\theta S(v)}}(B_r)=t_r\right)\to\mathbb{P}\left(\mathcal{N}(B_1)=t_1,\dots,\mathcal{N}(B_r)=t_r\right).$$

or equivalently (see Theorem 11.1.VII of Daley and Vere-Jones (2008)),

$$\sum_{|v|=n}\delta_{E_v W_n(\theta)e^{-\theta S(v)}}\xrightarrow{d}\mathcal{N}.$$

Since $-\log(.)$ is continuous and therefore Borel measurable, the above equation suggests that

$$\mathcal{U}_n:=\sum_{|v|=n}\delta_{\theta S_v-\log E_v-\log W_n(\theta)}\xrightarrow{d}\mathcal{Y}. \tag{28}$$

To simplify the notations, for all $\theta<\min_i\theta_{(i)}\le\infty$, we denote

$$A_n(\theta)=\sum_{i=1}^{k}q_i(n)\nu_i(\theta)+\log D^{\infty}_{\theta,(1)},$$

and for $\theta=\theta_{(1)}<\min_{i\neq 1}\theta_{(i)}\le\infty$, we denote

$$A_n(\theta_{(1)})=\sum_{i=1}^{k}q_i(n)\nu_i(\theta_{(1)})-\frac{1}{2}\log(q_1(n))+\log D^{\infty}_{\theta_{(1)},(1)}+\frac{1}{2}\log\left(\frac{2}{\pi\sigma_1^2}\right).$$

Recall that by Eqs. (23), (24) and Lemma 3.1, for $\theta<\min_i\theta_{(i)}\le\infty$ and also for $\theta=\theta_{(1)}<\min_{i\neq 1}\theta_{(i)}\le\infty$,

$$A_n(\theta)-\log W_n(\theta)\xrightarrow{P}0.$$

Now, take any positive integer $r$, non-negative integers $\{t_i\}_{i=1}^r$, and extended real numbers $\{a_i\}_{i=1}^r$ and $\{b_i\}_{i=1}^r$ with $a_i<b_i$ for all $i$. We choose $\delta\in\left(0,\min_{i=1}^r(b_i-a_i)/2\right)$. Then, we have

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{U}_n\left((a_1-\delta,b_1+\delta)\right)\le t_1,\dots,\mathcal{U}_n\left((a_r-\delta,b_r+\delta)\right)\le t_r\right)-\mathbb{P}\left(|A_n(\theta)-\log W_n(\theta)|>\delta\right)\\
&\le\mathbb{P}\left(Z_n(\theta)\left((a_1,b_1)\right)\le t_1,\dots,Z_n(\theta)\left((a_r,b_r)\right)\le t_r\right)\\
&\le\mathbb{P}\left(\mathcal{U}_n\left((a_1+\delta,b_1-\delta)\right)\le t_1,\dots,\mathcal{U}_n\left((a_r+\delta,b_r-\delta)\right)\le t_r\right)+\mathbb{P}\left(|A_n(\theta)-\log W_n(\theta)|>\delta\right).
\end{aligned}$$

Now, by Eq. (28), we have $\mathcal{U}_n\xrightarrow{d}\mathcal{Y}$. By Lemma 5.3 of Bandyopadhyay and Ghosh (2021), we also have that $\mathcal{Y}$ is a decorated Poisson point process, and hence it is continuous. Therefore, allowing $n\to\infty$ and then letting $\delta\to 0$, we obtain

$$\lim_{n\to\infty}\mathbb{P}\left(Z_n(\theta)\left((a_1,b_1)\right)\le t_1,\dots,Z_n(\theta)\left((a_r,b_r)\right)\le t_r\right)=\mathbb{P}\left(\mathcal{Y}\left((a_1,b_1)\right)\le t_1,\dots,\mathcal{Y}\left((a_r,b_r)\right)\le t_r\right),$$

or equivalently, $Z_n(\theta)\xrightarrow{d}\mathcal{Y}$. This, together with Lemma 5.3 of Bandyopadhyay and Ghosh (2021), completes the proof of Theorem 2.6.

Theorem 2.7 is a slightly weaker version and it follows from the argument similar to that mentioned above.

## 4. A specific example

In this section we consider a *time inhomogeneous Gaussian displacement binary BRW*, which is a specific example of inhomogeneous BRW introduced by Fang and Zeitouni (2012). Here we shall consider the last progeny modified version of the same example. To be precise, let $Z_1=\delta_{\xi_{11}}+\delta_{\xi_{12}}$, $Z_2=\delta_{\xi_{21}}+\delta_{\xi_{22}}$, $\xi_{11},\xi_{12}$ are i.i.d. $\mathrm{N}(0,\sigma_1^2)$, $\xi_{21},\xi_{22}$ are i.i.d. $\mathrm{N}(0,\sigma_2^2)$ and $q_1(n)=q_2(n)=n/2$. In this case we have

$$\nu_1(t)=\log 2+\frac{\sigma_1^2t^2}{2}\quad\text{and}\quad\nu_2(t)=\log 2+\frac{\sigma_2^2t^2}{2},$$

and

$$\theta_1=\frac{\sqrt{2\log 2}}{\sigma_1}\quad\text{and}\quad\theta_2=\frac{\sqrt{2\log 2}}{\sigma_2}.$$

Therefore by Theorem 2.2, we obtain that

**Theorem 4.1.** *Assume $\sigma_1 > \sigma_2$, then the following sequence of random variables*

$$R_n^*\left(\frac{\sqrt{2\log 2}}{\sigma_1}\right) - n\left(\sigma_1\sqrt{\frac{\log 2}{2}} + \frac{\sqrt{2\log 2}}{4\sigma_1}\left(\sigma_1^2+\sigma_2^2\right)\right) + \log n\left(\frac{\sigma_1}{2\sqrt{2\log 2}}\right)$$

*converges in distribution to a non-trivial distribution which depends only on $\sigma_1$.*

As comparison we note that in Fang and Zeitouni (2012), it is shown that for this example when $\sigma_1 > \sigma_2$, the following sequence of random variables

$$R_n - n\left((\sigma_1+\sigma_2)\sqrt{\frac{\log 2}{2}}\right) + \log n\left(\frac{3\,(\sigma_1+\sigma_2)}{2\sqrt{2\log 2}}\right)$$

is tight. Thus for our model we have been able to establish more than Fang and Zeitouni (2012) as we obtain a weak limit for the right-most position of the LMPTI-BRW after an appropriate centering. However, we only have this for the case when $\sigma_1 > \sigma_2$. In Fang and Zeitouni (2012) the other case when $\sigma_1 < \sigma_2$ has also been worked out and tightness of the right-most position has been proved with an appropriate centering.

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