Stat-134, Section 02
Fall 2002

Solution of The Extra Credit Problem Set

1. Observe that from definition $X = R\cos\Theta$ and $Y = R\sin\Theta$.

   (a) **Step-1 :** The range of values of Θ is $[0, 2\pi)$.
   Step-2 : Fix $\alpha \in [0, 2\pi)$,

$$
\begin{aligned}
F_{\Theta}(\alpha) &= \mathbf{P}(\Theta \le \alpha) \\
&= \iint\limits_{\theta \le \alpha} \frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2+y^2\right)}\, dx\, dy \\
&= \int_0^{\alpha}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{r^2}{2}}\, r\, dr\, d\theta \\
&= \left(\int_0^{\alpha} \frac{d\theta}{2\pi}\right)\left(\int_0^{\infty} r e^{-\frac{r^2}{2}}\, dr\right) \\
&= \frac{\alpha}{2\pi}
\end{aligned}
$$

   So the CDF of Θ is given by

$$
F_{\Theta}(\alpha) = \begin{cases} 0 & \text{if } \alpha \le 0 \\ \frac{\alpha}{2\pi} & \text{if } 0 < \alpha \le 2\pi \\ 1 & \text{if } \alpha > 2\pi. \end{cases}
$$

   Clearly, $\Theta \sim \text{Unif}(0, 2\pi)$.

   (b) Observe that (Z, W) is nothing but a rotation of (X, Y) by an angle of $\theta = \frac{\pi}{4}$. Now recall the following fact proved in the class
   Fact : If X and Y are i.i.d Normal$(0, 1)$ random variables and (Z, W) is a rotation of (X, Y) then (Z, W) are also i.i.d Normal$(0, 1)$.
   So using this we conclude that (Z, W) are i.i.d. Normal$(0, 1)$ random variables.

2. (a) Notice that $F : \mathbb{R} \to (0, 1)$ is defined as $F(x) = \int_{-\infty}^{x} f(t)\, dt$, where $f(t) > 0$ for all $-\infty < t < \infty$. Thus F is a strictly increasing function, and hence $F^{-1} : (0, 1) \to \mathbb{R}$ exists.

   (b) Observe that the range of values of $Y = F^{-1}(U)$ is $\mathbb{R}$. Fix $-\infty < y < \infty$, then

$$
\begin{aligned}
F_Y(y) &= \mathbf{P}(Y \le y) \\
&= \mathbf{P}\left(F^{-1}(U) \le y\right) \\
&= \mathbf{P}(U \le F(y)) \\
&= F(y)
\end{aligned}
$$

   The last equality follows from the fact that $U \sim \text{Unif}(0, 1)$, and $F(y) \in (0, 1)$. So the CDF of $Y = F^{-1}(U)$, is F. In otherwords Y has the same distribution as of X.

(c) First note that from definition $W = F(X) \in (0,1)$. Fix $w \in (0,1)$.

$$
\begin{aligned}
F_W(w) &= \mathbf{P}\left(W \le w\right) \\
&= \mathbf{P}\left(F(X) \le w\right) \\
&= \mathbf{P}\left(X \le F^{-1}(w)\right) \\
&= F\left(F^{-1}(w)\right) \\
&= w
\end{aligned}
$$

So clearly $W \sim \text{Unif}(0,1)$.

3. Suppose $U_1, U_2, U_3, \ldots$ be the **i.i.d.** sequence of Unif$(0,1)$ random variables produced by the computer routine.

   (a) Define $W_1 := -\frac{1}{\lambda}\log U_1$, and use change of variable formula to conclude that $W_1 \sim \text{Exponential}(\lambda)$ (done in class).

   (b) For each $i \ge 1$, define $W_i := -\frac{1}{\lambda}\log U_i$. So using part (a), $W_1, W_2, W_3, \ldots$ are **i.i.d.** Exponential(λ). Let $T_r := W_1 + W_2 + \cdots + W_r$, where $r \ge 1$. Define N to be the first r such that $T_{r+1} \ge 1$; that is, $N = 0$ if $T_1 \ge 1$; $N = 1$ if $T_1 < 1$ but $T_2 \ge 1$, $N = 2$ if $T_2 < 1$ but $T_3 \ge 1$, and so on. Naturally, N is nothing but the number of arrivals in $[0,1)$ for the Poisson arrival process of rate λ defined through the Exponential variables $W_1, W_2, W_3, \ldots$, so clearly $N \sim \text{Poisson}(\lambda)$.

4. First note that $\mathbf{E}\left[Z\right] = \mathbf{E}\left[(X-Y)^2\right] = \mathbf{E}\left[X^2\right] + \mathbf{E}\left[Y^2\right] - 2\mathbf{E}\left[XY\right]$. Now using the *Law of Iterated Expectations* and the first given condition we get

$$
\begin{aligned}
\mathbf{E}\left[XY\right] &= \mathbf{E}\left[\mathbf{E}\left[XY \middle| Y\right]\right] \\
&= \mathbf{E}\left[Y\mathbf{E}\left[X \middle| Y\right]\right] \\
&= \mathbf{E}\left[Y^2\right]
\end{aligned}
$$

   Similarly using the second condition we will get $\mathbf{E}\left[XY\right] = \mathbf{E}\left[X^2\right]$. So $\mathbf{E}\left[Z\right] = 0$, hence using the given math fact we conclude that $\mathbf{P}\left(Z = 0\right) = 1 \Leftrightarrow \mathbf{P}\left(X = Y\right) = 1$.

5. Let A_i be the event that i^{th} ball goes into i^{th} box, for $1 \le i \le n$. We know from the Midterm Exam Problem # 3 that $\mathbf{P}\left(A_i\right) = (1 - 1/n)^n$ for any $1 \le i \le n$, and $\mathbf{P}\left(A_i \cap A_j\right) = (1 - 2/n)^n$ for $1 \le i \ne j \le n$. Let I_i be the indicator function of the event A_i. Naturally from definition $X = I_1 + I_2 + \cdots + I_n$. Thus $\mathbf{E}\left[X\right] = n \times (1 - 1/n)^n$. Notice that

$$
\begin{aligned}
X^2 &= \sum_{i=1}^{n} I_i^2 + \sum\sum_{1 \le i \ne j \le n} I_i I_j \\
&= \sum_{i=1}^{n} I_i + \sum\sum_{1 \le i \ne j \le n} I_{A_i \cap A_j}
\end{aligned}
$$

   So by taking expectation we get

$$
\begin{aligned}
\mathbf{E}\left[X^2\right] &= \sum_{i=1}^{n} \mathbf{P}\left(A_i\right) + \sum\sum_{1 \le i \ne j \le n} \mathbf{P}\left(A_i \cap A_j\right) \\
&= n \times \left(1 - \frac{1}{n}\right)^n + n(n-1) \times \left(1 - \frac{2}{n}\right)^n
\end{aligned}
$$

   Finally,

$$
\begin{aligned}
\mathbf{Var}\left(X\right) &= \mathbf{E}\left[X^2\right] - \left(\mathbf{E}\left[X\right]\right)^2 \\
&= n\left(1 - \frac{1}{n}\right)^n + n(n-1)\left(1 - \frac{2}{n}\right)^n - n^2\left(1 - \frac{1}{n}\right)^{2n}
\end{aligned}
$$

6. (a) From the formula for the joint density we get that both X and Y takes values in $(0,1)$, and have the same distribution. Fix $0 < x < 1$, then

$$\begin{aligned} f_X(x) &= \int_0^1 (2x + 2y - 4xy)\, dy \\ &= \left[2xy + y^2 - 2xy^2\right]_0^1 \\ &= 2x + 1 - 2x = 1. \end{aligned}$$

Thus X and Y are Unif$(0,1)$.

(b)

$$f_{Y|X}\left(y \middle| x = \frac{1}{4}\right) = \frac{f\left(\frac{1}{4}, y\right)}{f_X\left(\frac{1}{4}\right)} = \begin{cases} \frac{2}{4}y + 2y - \frac{4}{4}y = \frac{1}{2} + y & \text{if } 0 < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

(c) $\mathbf{E}\left[Y \middle| X = \frac{1}{4}\right] = \int_0^1 y\left(\frac{1}{2} + y\right)\, dy = \frac{7}{12}$

7. (a) Observe that Y only takes positive values. Fix $y > 0$,

$$\begin{aligned} f_Y(y) &= \int_0^y \lambda^3 x e^{-\lambda y}\, dx \\ &= \frac{\lambda^3}{2} y^2 e^{-\lambda y} \end{aligned}$$

Thus $Y \sim$ Gammma$(3, \lambda)$. Hence $\mathbf{E}[Y] = \frac{3}{\lambda}$.

(b) The conditional density of X given $Y = 1$ is

$$f_{X|Y}\left(x \middle| y = 1\right) = \frac{f(x,1)}{f_Y(1)} = \begin{cases} \frac{\lambda^3 x e^{-\lambda y}}{(\lambda^3/3) y^2 e^{-\lambda y}} = 2x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\mathbf{E}\left[X \middle| Y = 1\right] = \int_0^1 x(2x)\, dx = \frac{2}{3}$.

8. (a) Observe that Y only takes positive values. Fix $y > 0$,

$$\begin{aligned} f_Y(y) &= \int_0^y \frac{e^{-y/2}\, dx}{2\pi\sqrt{x(y-x)}} \\ &= \frac{1}{2\pi} e^{-y/2} \int_0^y \frac{dx}{\sqrt{x(y-x)}} \\ &= \frac{c}{2\pi} e^{-y/2}, \end{aligned}$$

where $c = \int_0^1 ds/\sqrt{s(1-s)}$, and the last equality can be obtained by substituting $x = ys$ in the previous integral. Since f_Y is a density so it must integrate out to 1, and so without any difficult integration evaluation we can show that $c = \pi$. Hence $Y \sim$ Exponential$(1/2)$.

(b) The conditional density of X given $Y = 1$ is

$$f_{X|Y}\left(x \middle| y = 1\right) = \frac{f(x,1)}{f_Y(1)} = \begin{cases} \frac{e^{-y/2}/(\pi\sqrt{x(1-x)})}{(1/2)e^{-y/2}} = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus Given $Y = 1$, X is a random variable which takes values in $(0,1)$. Notice (by inspection, or by using change of variable formula) that given $Y = 1$, $1 - X$ has the same distribution as that of X. Thus clearly with any difficult calculations we can conclude that $\mathbf{E}\left[X \middle| Y = 1\right] = \frac{1}{2}$.