Statistics - 134 (Lecture - 2), Fall 2002

Solution of Practice Midterm

1. Since it is sampling **with replacement**, so each draw is independent success-failure experiment, with probability of success $p =$ proportion of green balls in the box $= 30/100 = 0.3$.

(a) $X :=$ number of green balls in the first 40 draws, hence $X \sim \text{Binomial}(40, 0.3)$.
$Y :=$ number of green balls in the last 60 draws, hence $Y \sim \text{Binomial}(60, 0.3)$.
$T :=$ total number of green balls in 100 draws, hence $T \sim \text{Binomial}(100, 0.3)$.

(b) **Yes**. The first 40 draws are independent of the last 60 draws, since we are doing sampling **with replacement**. Hence X and Y are independent.

(c) Fix $0 \le t \le 100$. Notice that given $[T = t]$ the values of X are $\{0, 1, 2, \ldots, t\}$. Fix $0 \le k \le t$, then

$$\begin{aligned}
\mathbf{P}\left(X = k \,\middle|\, T = t\right) &= \frac{\mathbf{P}\left(X = k,\, T = t\right)}{\mathbf{P}\left(T = t\right)} \\
&= \frac{\mathbf{P}\left(X = k,\, Y = t - k\right)}{\mathbf{P}\left(T = t\right)} \\
&= \frac{\mathbf{P}\left(X = k\right)\,\mathbf{P}\left(Y = t - k\right)}{\mathbf{P}\left(T = t\right)} \\
&= \frac{\binom{40}{k}\ 0.3^{k}\ 0.7^{40-k}\quad \binom{60}{t-k}\ 0.3^{t-k}\ 0.7^{60-(t-k)}}{\binom{100}{t}\ 0.3^{t}\ 0.7^{100-t}} \\
&= \frac{\binom{40}{k}\ \binom{60}{t-k}}{\binom{100}{t}}.
\end{aligned}$$

Note that given $[T = t]$ the distribution of X is $\text{Hypergeometric}(n = t; N = 100, G = 40)$.

(d) **No**. Consider $\mathbf{P}\left(X = 0, T = 0\right) = \mathbf{P}\left(T = 0\right)$ and certainly $\mathbf{P}\left(X = 0\right) \neq 1$.

2. Let D be the number of cards dealt before getting the first **King**. Since we are doing sampling **without replacement** so the values of D are $\{1, 2, \ldots, 49\}$. Hence if $n \ge 50$ then $p(n) = 0$. Fix $1 \le n \le 49$ then,

$$\begin{aligned}
p(n) &= \mathbf{P}\Big(\text{ The first } (n-1) \text{ cards has } \textbf{no King}, \text{ but the } n^{\text{th}} \text{ card is a } \textbf{King}\Big) \\
&= \left(\frac{48}{52} \times \frac{47}{51} \times \frac{46}{50} \times \cdots \times \frac{48 - n + 2}{52 - n + 2}\right) \times \frac{4}{52 - n + 1} \\
&= \frac{\binom{48}{n-1}}{\binom{52}{n-1}} \times \frac{4}{52 - n + 1}.
\end{aligned}$$

3. Observe that balls are being placed at random, thus the outcomes are equally likely. Further, the placement of the i^{th} ball is independent of the placement of the other balls.

(a)

$$\begin{aligned}\mathbf{P}\,(\text{ none of the boxes are empty }) &= \frac{\text{Number of placements with } \textbf{no} \text{ empty boxes}}{\text{Total number of placements}} \\ &= \frac{n!}{n^n}.\end{aligned}$$

(b) Fix $1 \leq i \leq n$, from definition $\mathbf{P}\,(A_i) = \dfrac{n^{n-1}}{n^n} = \dfrac{1}{n}$.

(c) Notice that from definition, $X = \mathbf{I}_{A_1} + \mathbf{I}_{A_2} + \cdots + \mathbf{I}_{A_n}$, where $\mathbf{I}_{A_i}$ is the indicator of the event A_i. But the events $A_1, A_2, \ldots, A_n$ are independent with $\mathbf{P}\,(A_i) = \frac{1}{n}$, for all $1 \leq i \leq n$. Thus $X \sim \text{Binomial}(n, \frac{1}{n})$.

(d) $\mathbf{E}[X] = n \times \frac{1}{n} = 1$, and $\mathbf{Var}(X) = n \times \frac{1}{n} \times \left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n}$.

(e) Note that the mean of X is 1 and the sd of X is $\sqrt{1 - \frac{1}{n}}$, so using **Poisson** approximation to Binomial $\left(n, \frac{1}{n}\right)$ with $n = 10,000$, we get that

$$\mathbf{P}\,(X < 2) \approx 2e^{-1} \approx 0.735759,$$

$$\mathbf{P}\,(X = 2) \approx \frac{1}{2}e^{-1}, \approx 0.183940,$$

$$\mathbf{P}\,(X > 2) \approx 1 - \frac{5}{2}e^{-1} \approx 0.080301.$$