Indian Statistical Institute, Kolkata

Master of Statistics (M.Stat.) IInd Year
Advanced Probability I

Semester I (2008-2009)
Surprise Test 1

Instructor: Antar Bandyopadhyay

Date: October 21, 2008 **Total Points: 15**
Time: 17:15 - 18:00 **Duration: 45 minutes**

Note:

- **Please write your name and roll number on top of your answer paper.**
- **There are 3 problems each carrying 5 points. Solve as many as you can. Show all your works and write explanations when needed.**
- **This is an open note examination. You are allowed to use your own hand written notes (such as class notes, your homework solutions, list of theorems, formulas etc). Please note that no printed materials or photo copies are allowed, in particular you are not allowed to use books, photocopied class notes etc.**

1. Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be two martingales with $\mathbf{E}\left[X_n^2\right] < \infty$ and $\mathbf{E}\left[Y_n^2\right] < \infty$ for all $n \geq 0$. Show that
$$\mathbf{E}\left[X_n Y_n\right] - \mathbf{E}\left[X_0 Y_0\right] = \sum_{m=1}^{n} \mathbf{E}\left[\left(X_m - X_{m-1}\right)\left(Y_m - Y_{m-1}\right)\right] .$$

2. Suppose $(X_n)_{n\geq 1}$ is an *exchangeable* sequence of zero mean random variables with $\mathbf{E}\left[X_1^2\right] < \infty$. Show that any two of them have non-negative correlation.

3. Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be two non-negative sequences of random variables which are $(\mathcal{F}_n)_{n\geq 0}$ adapted where $(\mathcal{F}_n)_{n\geq 0}$ is a filtration. Assume that
$$\mathbf{E}\left[X_{n+1} \,\middle|\, \mathcal{F}_n\right] \leq \left(1 + Y_n\right)\, X_n \,.$$
Then show that if $\sum_{n=1}^{\infty} Y_n < \infty$ a.s. then $\lim_{n\to\infty} X_n$ exists a.s. and is finite.

Good Luck