# Practice Final
# Stat 155: Game Theory

## True-False

(20-30 mins)

For each of these statements state whether they are true of false. With reason.

a) If a $m \times n$ matrix has 2 saddle points, they have the same value.

b) For any matrix $A_{m\times n} = ((a_{ij}))$, $\max_i \min_j a_{ij} = \min_j \max_i a_{ij}$

c) This payoff matrix can be reduced to $2 \times 2$ using domination,

$$\begin{pmatrix} 5 & 4 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 1 & 4 & 3 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

d) The value of a finite 2P0S symmetric game is 0.

e) $x \oplus y \leq x + y$, where $\oplus$ is the Nim-Sum.

## Question 1

(30-40 mins)

Consider the following combinatorial game, known as *Wythoffs game* - To start with, a queen is placed on some square of the $8 \times 8$ chessboard. The queen may only be moved vertically down, or horizontally to the left or diagonally down to the left. When the queen reaches the lower left corner, the game is over and the player to move last wins.

Denote the lower left corner square (terminal position) as $(0,0)$. Denote the square $x$ columns to the right of $(0,0)$ and $y$ rows above $(0,0)$ as $(x,y)$. Show that the Sprague-Grundy function, $g(x,y) = x + y$. Hint: use induction.

## Question 2

(30-40 mins)

a) This game called *Two-Dimensional Nim (modified)* - it is played on a $m \times n$ board with a finite number of counters placed on squares. A move consists in taking a counter and moving it any number of squares to the left on the same row, or moving it to any square whatever on any lower row. The game ends when all counters reach the lower-left corner of the board. A square is allowed to contain any number of counters. Describe the P & N positions of this game.

b) Consider a 2-person 0-sum game as follows - Think of a $3 \times 3$ board. Each player chooses a position on the board to place a counter. Then they play the resulting modified 2D Nim and whoever wins gets \$1 from the other. Player I gets the first move. Write down the payoff matrix, NEs and value of the game.

## Question 3

(60-70 mins)

A game is played on a graph as follows. Players alternate moves. A move consists of removing a vertex and all edges incident to that vertex, with the exception that a vertex without any incident edges may not be removed. That is, at least one edge must be removed. Last player to move wins.

a) Find the Sprague-Grundy value of $S_n$, the star with $n$ points. (The star with $n$ points is the graph with one central vertex and $n$ points connected to it.) *Note:* If you haven't understood the game correctly, the game on $S_n$ can be won in one move.

b) SG function of the complete graph with $n$ vertices.

c) SG function of $L_n$, the line with $n$ edges and $n+1$ vertices for $n = 1, 2, 3, 4, 5, 6$. The value for general $n$ is $n$ mod 3. (You don't need to prove this)

d) SG function of $C_n$, the cycle with $n$ vertices and $n$ edges. Assume the result about lines.

## Question 4

(30-45 mins)

a) Find the SG function, $g(.)$ of the subtraction game where the subtraction set at position $x$ is the set of all powers of 2 that divide $x$, not including $x$. For example, the subtraction set for 27 is $\{1\}$, for 24 is $\{1, 2, 4, 8\}$, for 8 is $\{1, 2, 4\}$.

b) Now consider the 2-person 0-sum game on the srategy set $\{124, 663, 37\}$ given by $a_{ij} = (-1)^{i+j} g(|i-j|)$. Find the NE(s) and value of this game.

c) Do you think the same game on the infinite strategy space $\{1, 2, 3, \dots\}$ has a value?

d) Find the value of the game on infinite strategy space when $a_{ij} = (-\frac{1}{2})^{i+j} g(|i-j|)$

## Question 5

(30-45 mins)

a) Suppose a million players play the following game - they each choose a number from $\{0, 1, 2, \dots, 100\}$. Whoever chooses the number closest to two-thirds of the mean of the numbers wins \$100 and everyone else loses \$1. What do you think the NE will be?

b) Now suppose, the players have never taken Stat 155 or an equivalent course and are not conversant with NE calculations. And suppose you have entered this game and have somehow through hook, crook and nuclear missiles, come to know that the distribution of responses is $Bin(100, 0.2)$. What is your best response? What is your expected payoff?

Aside: A $Bin(n, p)$ random variable takes values in $\{0, 1, 2, \dots, n\}$ with probability $P(Bin(n, p) = k) = \binom{n}{k} p^k (1-p)^{n-k}$

## Question 6

(60 -75 mins)

Universal, Warner Brothers and Columbia each have a movie lined up for the month of July, which has 4 Fridays. Each studio has to choose when to release there movie. Their combined PR agencies and focus groups have made the following observations about the moviegoing audience in general.

Each person is going to watch one of the movies at random that have released on the first Friday. Each person likes a movie with chance $\frac{1}{2}$. Everyone who likes a movie on a certain week comes back next week to watch a movie, they watch the same movie with chance $\frac{1}{5}$, and with chance $\frac{4}{5}$ a random new release if there are any, a random different old release otherwise. Of course, if there are no other options, they watch the same movie again. Everyone who didn't like the last movie they saw, comes back on subsequent weeks with chance $\frac{1}{2}$ only if there is an option of watching a different movie and proceeds to watch a random new release if there is one, a random old release different from the movie she didn't like otherwise.

The studios need to decide when to release their respective movies so as to maximize profits (make more people watch their movie).

Frame this as a 3-player general sum game by writing down the utilities to each studio. Find one NE.

Solve the same game if there were only 2 players using linear programming methods.