Solutions to Assignment 10
Stat 155: Game Theory

Question 1

a) When the mixed strategies of PI and PII are x and y respectively and the payoff is A,then the expected payoff to PI is formally written as

$$\sum_i \sum_j a_{ij} x_i y_j = \sum_i x_i (\sum_{j=1}^{i-1} y_j - \sum_{j=i+1}^{\infty} y_j) =: \sum_i c_i(y) x_i (\text{let})$$

It is enough to show that the series $\sum_i c_i(y)x_i$ converges *absolutely* for every possible choice of y. Now, for any fixed choice of y,

$$\begin{aligned} |c_i(y)| &= \left| \sum_{j=1}^{i-1} y_j - \sum_{j=i+1}^{\infty} y_j \right| \\ &\leq \left| \sum_{j=1}^{i-1} y_j \right| + \left| \sum_{j=i+1}^{\infty} y_j \right| \\ &= \sum_{j=1}^{i-1} y_j + \sum_{j=i+1}^{\infty} y_j \\ &= \sum_j y_j - y_i = 1 - y_i \leq 1 \end{aligned}$$

Thus the series, $\sum_{i=n}^{\infty} |c_i(y)x_i| = \sum_{i=n}^{\infty} |c_i(y)| x_i \leq \sum_{i=n}^{\infty} x_i$ converges to 0 as $n \to \infty$ which means that the tail sum of the series $\sum_i c_i(y)x_i$ converges to 0, which means the series converges for every choice of y. Thus $\sum_i c_i(y)x_i$ converges absolutely and hence the is always defined.

b) For the second example, lets choose the mixed strategies in the following way, $x_i = y_i = 2^{-i}$. Observe that $\sum_i x_i = \sum_j y_j = 1$ so they are actually mixed strategies. Then the expected payoff to PI is,

$$\begin{aligned}
P :=& \sum_i \left(\sum_{j<i} \frac{4^j}{2^i 2^j} - \sum_{j>i} \frac{4^i}{2^i 2^j} \right) \\
=& \sum_i \left(\sum_{j<i} \frac{2^j}{2^i} - \sum_{j>i} \frac{2^i}{2^j} \right) \\
=& \sum_i \left(\frac{2^i - 2}{2^i} - 1 \right) \\
=& \sum_i \frac{-1}{2^{i-1}} = -2
\end{aligned}$$

But again,

$$\begin{aligned}
P =& \sum_i \left(\sum_{j<i} \frac{2^j}{2^i} - \sum_{j>i} \frac{2^i}{2^j} \right) \\
=& \sum_j \left(-\sum_{i<j} \frac{2^i}{2^j} + \sum_{i>j} \frac{2^j}{2^i} \right) \\
=& -\sum_j \left(\sum_{i<j} \frac{2^i}{2^j} - \sum_{i>j} \frac{2^j}{2^i} \right) \\
=& -\sum_i \left(\sum_{j<i} \frac{2^j}{2^i} - \sum_{j>i} \frac{2^i}{2^j} \right) \quad \text{(by renaming the dummy variables)} \\
=& -P = 2
\end{aligned}$$

which means that the series 'converges' to both 2 and -2, i.e. it is not convergent.

Question 2

It is given that (i^*, j^*) is a saddle point of A, which means that

$$\min_j a_{i^* j} = a_{i^* j^*} = \max_i a_{i j^*}$$

That is,

$$i^* = \operatorname*{argmax}_i a_{ij^*} = \operatorname*{argmax}_i \min_j a_{ij}$$

which means that i^* is a safety strategy for PI. Similarly, j^* is a safety strategy for PII. Be mindful of the negative sign which counters the tranpose max/min.

Then, (i^*, j^*) is also a Nash equilibrium as the following holds,

$$\begin{aligned} a_{i^*j^*} &\geq a_{ij^*} \text{ (saddle point, hence column max, in)} A \\ b_{i^*j} &\leq b_{i^*j^*} \text{ (saddle point, hence row minimum, in)} - B \end{aligned}$$

This proves that the pair of safety strategies we found is also a NE.