Solutions to Assignment 8
Stat 155: Game Theory

Question 1

Firstly, observe that Column 3 dominates Column 2 for Player II. The new payoff matrix is

$$\begin{pmatrix} 8 & 0 & 5 \\ 0 & 4 & 1 \end{pmatrix}$$

Then, assigning probabilities p and $1-p$ to rows 1 and 2 for Player I, we have that the expected payoffs for each choice of strategy of Player II are $8p, 4(1-p), 4p+1$. It is easy to see that the lower envelope of these expected payoffs is the piecewise linear curve,

$$\begin{array}{ll} 8p & \text{if } p \leq \dfrac{1}{4} \\ 4p+1 & \text{if } \dfrac{1}{4} \leq p \leq \dfrac{1}{3} \\ 4-4p & \text{if } \dfrac{1}{3} \leq p \leq \dfrac{3}{8} \end{array}$$

and the changepoints are $(\frac{1}{4}, 2)$ and $(\frac{3}{8}, \frac{5}{2})$. Thus, the optimal strategy for Player I is to play $(\frac{3}{8}, \frac{5}{8})$ and the optimal strategy for Player II is to play $(0, 0, \frac{1}{2}, \frac{1}{2})$ and the value of the game is $\frac{5}{2}$.

In the transpose question, the payoff matrix is,

$$\begin{pmatrix} 8 & 0 \\ 3 & 4 \\ 0 & 4 \\ 5 & 1 \end{pmatrix}$$

Here, Row 3 is dominated by Row 2 and then Row 4 is dominated by $\frac{1}{2}\times$ Row1 $+\frac{1}{2}\times$ Row2.

Then, it is easy to solve a 2×2 matrix to ascertain that the optimal strategies for Players I and II are $(\frac{1}{9},\frac{8}{9})$, $(\frac{4}{9},\frac{5}{9})$ and the value is $\frac{32}{9}$.

Question 2

It can be observed that Column3 is dominated by $\frac{3}{8}\times$ Column1 $+\frac{5}{8}\times$ Column2. Which gives the reduced payoff matrix,

$$\begin{pmatrix} 0 & 8 \\ 8 & 4 \\ 12 & -4 \end{pmatrix}$$

Assigning probabilities q and $1-q$ to the actions of Player II, we get the set of expected payoffs $8(1-q), 4+4q, 16q-4$. The upper envelope is the piecewise linear curve,

$$\begin{aligned} &8p && \text{if } p \leq \frac{1}{4} \\ &4p+1 && \text{if } \frac{1}{4} \leq p \leq \frac{1}{3} \\ &4-4p && \text{if } \frac{1}{3} \leq p \leq \frac{3}{8} \end{aligned}$$

and the changepoints are $(\frac{1}{3},\frac{16}{3})$ and $(\frac{2}{3},\frac{20}{3})$. Thus, the optimal strategy for Player I is to play $(\frac{1}{3},0,\frac{2}{3})$ and the optimal strategy for Player II is to play $(\frac{1}{3},\frac{2}{3},0)$ and the value of the game is $\frac{16}{3}$.