Solutions to Assignment 9
Stat 155: Game Theory

Question 1

Using translation invariance of optimal strategies, we solve the following modified payoff matrix made by adding 3 to each payoff. Recall that the optimal strategies of these modified game is same as that of the original game and the value is increased by 3.

$$\begin{pmatrix} 3 & 4 & 5 \\ 0 & 3 & 6 \\ 5 & 4 & 3 \end{pmatrix}$$

One possible choice of pivots are as follows: in the first step, 5 at $(3,1)^{th}$ position, in the second step, $\frac{8}{5}$ at the $(1,2)^{th}$ position.

Then the optimal strategies become, $(\frac{1}{2}, 0, \frac{1}{2})$ for Player I and $(0, 1, 0)$ for Player II. The value is $4 - 3 = 1$.

Question 2

Firstly, we notice that the following change of variable, $y_1 = 2z_1, y_2 = z_2, y_3 = 3z_3$, makes the problem easier to parse. The new optimization problem is,

$$\text{maximize } y_1 + y_2 + y_3$$

$$\text{subject to}$$

$$\begin{aligned}
\frac{1}{2}y_1 + \frac{1}{2}y_2 &\leq 1 \\
\frac{1}{2}y_2 + \frac{1}{2}y_3 &\leq 1 \\
\frac{1}{2}y_1 + \frac{1}{2}y_3 &\leq 1 \\
y_1, y_2, y_3 &\geq 0
\end{aligned}$$

We make another change of variable of the form $x_i = \frac{y_i}{y_1+y_2+y_3}$ and $v = y_1 + y_2 + y_3 \geq 0$. Obviously, $x_1 + x_2 + x_3 = 1$. Then the optimization problem becomes,

$$\text{maximize } \frac{1}{v}$$

$$\text{subject to}$$

$$\begin{aligned}
\frac{1}{2}x_1 + \frac{1}{2}x_2 &\leq v \\
\frac{1}{2}x_2 + \frac{1}{2}x_3 &\leq v \\
\frac{1}{2}x_1 + \frac{1}{2}x_3 &\leq v \\
x_1, x_2, x_3 &\geq 0 \\
x_1 + x_2 + x_3 = 1
\end{aligned}$$

As $v \geq 0$, maximizing $\frac{1}{v}$ is same as minimizing v, which means the optimization problem is same as,

$$\text{minimize } v$$

$$\text{subject to}$$

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 \leq v$$
$$\frac{1}{2}x_2 + \frac{1}{2}x_3 \leq v$$
$$\frac{1}{2}x_1 + \frac{1}{2}x_3 \leq v$$
$$x_1, x_2, x_3 \geq 0$$
$$x_1 + x_2 + x_3 = 1$$

which is exactly a Two-Person-Zero-Sum Game with the following payoff matrix where the optimization equations are written from Player IIś perpective.
a)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & o \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

b) It is easy to notice that the optimal strategies are uniform strategies $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the value is $\frac{1}{3}$.

c) This means the optimal values for $y_i = 1$, hence the optimal value of the original optimization problem is 3, and the optimal solutions are $z_1 = \frac{1}{2}, z_2 = 1, z_3 = \frac{1}{3}$.