UNIVERSITY OF CALIFORNIA, BERKELEY

DEPARTMENT OF STATISTICS

STAT 134: Concepts of Probability

Spring 2014

Instructor: Antar Bandyopadhyay

Practice Final Examination (I)

Date Given: April 25, 2014 **Duration: 180 minutes** **Total Points: 100**

Note: There are ten problems with a total of 100 points. Show all your works.

1. State whether the following statements are **true** or **false**. Write brief explanations supporting your answers. [5×2]

   (a) If $A \subseteq B$ then $\mathbf{P}\left(A \middle| B\right) \geq \mathbf{P}(A)$.

   (b) Suppose X and Y are **i.i.d.** Normal$(0,1)$ then $\mathbf{P}(|X - Y| > 2) > \frac{1}{2}$.

   (c) If X and Y are independent random variables with finite expectations then $\mathbf{E}\left[\frac{X}{Y}\right] = \frac{\mathbf{E}[X]}{\mathbf{E}[Y]}$.

   (d) If X is a random variable then $\mathbf{E}\left[X^2\right] \geq (\mathbf{E}[X])^2$.

   (e) If X and Y are independent continuous random variables with density f_X and f_Y then $\mathbf{P}(X = Y) = 0$.

2. Find the density of the random variable Y, if $X = \log Y \sim$ Normal$(0,1)$. [10]

3. A Geiger counter is recording background radiation as a Poisson arrival process of 3 hits per minute.

   (a) Find the chance that the 3^{rd} particle arrives after 3 minutes. [5]

   (b) Find the probability that the 6^{th} particle arrives within 2 minutes of the 3^{rd} particle. [5]

4. Suppose $X \sim$ Normal$(0,1)$. Given $[X = x]$ the conditional distribution of Y is Normal$(x, 1)$.

   (a) Find the marginal distribution of Y. [5]

   (b) Calculate $\mathbf{E}\left[X \middle| Y = y\right]$. [5]

5. A box contains 15 red, 15 green and 20 white balls. You are doing sampling **with replacement**. Every time you draw a green ball you receive \$2, and every time you draw a red ball you have to pay back \$1. If you get a white ball then you gain or lose nothing. Suppose you start with \$0 and you are allowed to borrow as much as you need throughout the sampling. Find the chance that after 100 draws you will at least have a profit of \$45. [10]

6. Suppose the joint density of (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{1}{5}\left(3x^2 + 4xy + 6y^2 + 2x\right) & \text{if } \ 0 < x < 1, \ 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal density of X. [5]

(b) Calculate $\mathbf{P}(X \le Y)$. [5]

7. A jar has three coins, two of which are unbiased, and one is biased. Suppose that the probability of getting a head from the bias coin is $\left(\frac{1}{2} + \theta\right)$, where $0 < \theta < \frac{1}{2}$. I pick a coin at random from the jar and then toss it independently until I get a head. Let T be the number of tosses.

(a) Find the distribution of T. [6]

(b) Given that $[T = 10]$ find the conditional probability that I picked the biased coin. [4]

8. Suppose that $X \sim$ Exponential(1). Let $Y = [X]$, where $[x]$ is the greatest integer less of equal to x, for example $[1.4] = 1$, while $[0.99] = 0$.

(a) Find the possible values of Y. [2]

(b) For each value k of Y, find $\mathbf{P}(Y = k)$. [6]

(c) Recognize the distribution of Y. [2]

9. Harry and Hermione decided to meet at 12:00 noon in the library to study potion. From the past experience Hermione knows that Harry on an average is always 5 minutes late, while if she plans to arrive by 12:00 noon then she on an average reaches 5 minutes earlier. Assume that they arrive independently and their arrival times are Normal random variables with appropriate means and each with standard deviation 3 minutes.

(a) Calculate the chance that Harry actually arrives before Hermione. [5]

(b) Calculate the probability that Hermione has to wait more than 10 minutes for Harry. [5]

10. Suppose that X is a non-negative random variable such that $\mathbf{P}(X > t) = e^{-\lambda t}$, for $t > 0$; where $\lambda > 0$ is a given number.

(a) For each $n \ge 0$, find $\mathbf{E}[X^n]$. [7]

(b) For $z < \lambda$, compute $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!} \mathbf{E}[X^n]$. [3]