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Containment Threshold for Small Subgraphs

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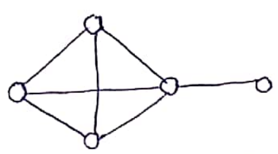
1. INTRODUCTION

<u>The Containment Problem</u> $\rightarrow$ Given a fixed graph G, the property that $\mathcal{G}(n,p)$ or $\mathcal{G}(n,M)$ contains G as a subgraph, is monotone. Thus, this property has a threshold. Our aim is to find the threshold and investigate whether its sharp.

Let $f(n,G)$ denote the # of distinct copies of G in K_n ~~$\mathcal{G}(n,p)$~~. If X_G denote the # of copies of G in $\mathcal{G}(n,p)$, we shall find that $\mathbb{E}X_G = f(n,G) \cdot p^{e_G}$. Again, we can show $f(n,G) \asymp n^{v_G}$ and thus $\mathbb{E}X_G \asymp n^{v_G} p^{e_G}$.

If $p \ll n^{-v_G/e_G}$, we find that $\mathbb{E}X_G \rightarrow 0$ and by 1st moment method, $P(X_G > 0) \rightarrow 0$. Now we ask whether $P(X_G > 0) \rightarrow 1$ when $p \gg n^{-v_G/e_G}$. Consider the following graph G:



We see that $K_4 \subseteq G$. Take $p = n^{-29/41}$, we see $p \gg n^{-5/7}$ i.e $p \gg n^{-v_G/e_G}$. But again, $p \ll n^{-2/3} = n^{-v_{K_4}/e_{K_4}}$. So, $P(X_{K_4} > 0) \rightarrow 0$ and so does $P(X_G > 0)$, as G contains K_4. So, why does $p = n^{-29/41}$ doesn't work?

2. ASYMPTOTIC EQUIVARIANCE [of $\mathcal{G}(n,p)$ & $\mathcal{G}(n,M)$, when $p = M/N$]

Once we get a threshold for containment of a fixed graph G in $\mathcal{G}(n,p)$, our aim is to find the same for $\mathcal{G}(n,M)$ while incorporating the fact that subgraph containment property is increasing.

Our goal is to establish conditions under ~~with~~ which convergence of $P(\mathcal{G}(n,p) \in Q)$ to 0 or 1 implies converges of $P(\mathcal{G}(n,M) \in Q)$ to 0 or 1. We may consider the special case when $p = M/N$. The machinery going to be used are elementary laws of probability along with a result from analysis. We shall find that convexity of Q is a useful feature for our purpose.

Lemma 2.1. :- Let $\mathcal{Q}$ be a convex property of subsets Γ, and let M, M_1, M_2 be three integer functions of N satisfying $0 \leq M_1 \leq M \leq M_2 \leq N$. Then

$$P(\mathcal{G}(n, M) \in \mathcal{Q}) \geq P(\mathcal{G}(n, M_1) \in \mathcal{Q}) + P(\mathcal{G}(n, M_2) \in \mathcal{Q}) - 1$$

Moreover, if $P(\mathcal{G}(n, M_i) \in \mathcal{Q}) \to 1$ as $N \to \infty$ for $i = 1, 2$; then $P(\mathcal{G}(n, M) \in \mathcal{Q}) \to 1$.

Claim 2.2. :- A property is convex if and only if it is the intersection of an increasing and a decreasing property.

Proof (Claim 2.2.) :-

"$\Leftarrow$ part" $\rightarrow$ Let $\mathcal{Q}_1$ & $\mathcal{Q}_2$ be an increasing and a decreasing property respectively.

Consider, $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$

Take, $A, B, C \in 2^{\Gamma}$ s.t. $A \subseteq B \subseteq C$ & $A, C \in \mathcal{Q}$.

Now, $A \in \mathcal{Q} \Rightarrow A \in \mathcal{Q}_1 \cap \mathcal{Q}_2 \Rightarrow A \in \mathcal{Q}_1$

$\Rightarrow B \in \mathcal{Q}_1$ .... *)

$[\because A \subseteq B$ & $\mathcal{Q}_1 \uparrow]$

Similarly, $C \in \mathcal{Q} \Rightarrow C \in \mathcal{Q}_1 \cap \mathcal{Q}_2 \Rightarrow C \in \mathcal{Q}_2$

$\Rightarrow B \in \mathcal{Q}_2$ .... **)

$[\because B \subseteq C$ & $\mathcal{Q}_2 \downarrow]$

From *) & **), we conclude that $B \in \mathcal{Q}_1 \cap \mathcal{Q}_2$ i.e $B \in \mathcal{Q}$.

Since, A, B, C were arbitrary elements of 2^{Γ} and $A \subseteq B \subseteq C$ with $A, C \in \mathcal{Q}$ implied $B \in \mathcal{Q}$; we conclude $\mathcal{Q}$ is convex.

" $\Rightarrow$ part" $\rightarrow$ Let $\mathcal{Q}$ be a convex property.

Define, $\mathcal{Q}_1 = \{A \in 2^{\Gamma} \mid \exists B \in \mathcal{Q} \text{ with } B \subseteq A\}$

$\mathcal{Q}_2 = \{A \in 2^{\Gamma} \mid \exists B \in \mathcal{Q} \text{ with } B \supseteq A\}$

We shall show that $\mathcal{Q}_1$ & $\mathcal{Q}_2$ are increasing and decreasing properties, respectively. Let $B \in \mathcal{Q}_1$ & $A \supseteq B$. Since $B \in \mathcal{Q}_1$, $\exists B' \subseteq B$ s.t. $B' \in \mathcal{Q}$; this implies $B' \subseteq B \subseteq A$. As $B' \in \mathcal{Q}$, $A \in \mathcal{Q}_1$. So $B \in \mathcal{Q}_1 \Rightarrow A \in \mathcal{Q}_1$ for $A \supseteq B$, hence $\mathcal{Q}_1$ is increasing. By similar argument, one can show that $\mathcal{Q}_2$ is decreasing.

It remains to show $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$.

a) Take $B \in \mathcal{Q}_1 \cap \mathcal{Q}_2$, hence $\exists A, C$ s.t. $A \subseteq B$ & $A \in \mathcal{Q}$; and $B \subseteq C$ & $C \in \mathcal{Q}$. By the convexity of $\mathcal{Q}$, $B \in \mathcal{Q}$.

b) Take $B \in \mathcal{Q}$; since $B \subseteq B$, $B \in \mathcal{Q}_1$ & $B \in \mathcal{Q}_2$. Therefore, $B \in \mathcal{Q}_1 \cap \mathcal{Q}_2$.

— Above statements together imply $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$. ▨

<u>Proof</u> (Lemma 2.1):— ~~Define~~, $A = \cancel{P(\mathcal{G}}$

Since $\mathcal{Q}$ is convex, by ~~ten~~ Claim 2.2., $\exists \mathcal{Q}_1$ increasing & $\mathcal{Q}_2$ decreasing properties s.t. $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$.

Define, $A = \mathbb{P}(\mathcal{G}(n,M) \in \mathcal{Q}_1)$, $B = \mathbb{P}(\mathcal{G}(n,M) \in \mathcal{Q}_2)$.

Hence, $A \cap B = \mathbb{P}(\mathcal{G}(n,M) \in \mathcal{Q}_1 \cap \mathcal{Q}_2) = (\mathcal{G}(n,M) \in \mathcal{Q})$

Since,
$$
\begin{aligned}
P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\
&\geq P(A) + P(B) - 1 \\
&\geq P(\mathcal{G}(n, M_1) \in \mathcal{Q}_1) + P(\mathcal{G}(n, M_2) \in \mathcal{Q}_2) - 1 \\
&\qquad [\because M_1 \leq M \leq N \ \& \ \mathcal{Q}_1 \uparrow, \mathcal{Q}_2 \downarrow] \\
&\geq P(\mathcal{G}(n, M_1) \in \mathcal{Q}) + P(\mathcal{G}(n, M_2) \in \mathcal{Q}) - 1 \\
&\qquad [\because \mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2]
\end{aligned}
$$

Therefore,

$$P(\mathcal{G}(n,M) \in \mathcal{Q}) \cancel{\leq} \geq P(\mathcal{G}(n,M_1) \in \mathcal{Q}) + P(\mathcal{G}(n,M_2) \in \mathcal{Q}) - 1.$$

If, as $N \to \infty$, for $i = 1, 2$, $P(\mathcal{G}(n, M_i) \in \mathcal{Q}) \to 1$, we invoke 'Squeeze Thm'. to conclude $P(\mathcal{G}(n,M) \in \mathcal{Q}) \to 1$ as ~~n~~ $N \to \infty$.

∎

Theorem 2.3 :- Let $\{x_n\}_{n \in \mathbb{N}}$ be a seq. of real numbers and let $x \in \mathbb{R}$ be fixed. If for every subsequence $\{x_{n_k}\}_{n_k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$, there exists a further subsequence that converges to x; then as $n \to \infty$, $x_n \to x$.

— this result is called the 'subsequence principle'.

Proposition 2.4 :- Let $\mathcal{Q}$ be a convex property of subsets of Γ & let $0 \leq M \leq N$. If $P(\mathcal{G}(n, M/N) \in \mathcal{Q}) \to 1$ as $n \to \infty$, then $P(\mathcal{G}(n,M) \in \mathcal{Q}) \to 1$.

Proof:- (Prop. 2.4) :- Define, $x_n = P(\mathcal{G}(n,M) \in \mathcal{Q})$. We take a subseq. $\{x_{n_k}\}_{n_k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$. Then we may have 2 possibilites for the expression $\frac{M(N-M)}{N}$ as for the subseq. $\{x_{n_k}\}_{n_k \in \mathbb{N}}$ —

i) $$\limsup_{k \to \infty} \frac{M(n_k)(N(n_k) - M(n_k))}{N(n_k)} = \infty$$

ii) $$\limsup_{k \to \infty} \frac{M(n_k)(N(n_k) - M(n_k))}{N(n_k)} = a, \quad a \geq 0.$$

The first case happens when there is a subseq. n_{k_ℓ} s.t. $\frac{M(n_{k_\ell})(N(n_{k_\ell}) - M(n_{k_\ell}))}{N(n_{k_\ell})} \to \infty$. Since $M \leq N$, the second case occurs either when $\exists\, n_{k_\ell}$ s.t. $M(n_{k_\ell}) = O(1)$ or $N(n_{k_\ell}) - M(n_{k_\ell}) = \cancel{\Theta(1)}\ O(1)$.

Therefore everything boils down to consider three cases —

a) ~~$N \notin M - 1$~~ ~~$N \notin$~~ $\dfrac{M(N-M)}{N} \to \infty$,

b) $M = O(1)$ or c) $N - M = O(1)$.

Case – a) :- We have, by the law of total probability

$$P(\mathcal{G}(n, M/N) \in \mathcal{Q}) = \sum_{K=0}^{N} P(\mathcal{G}(n, M/N) \in \mathcal{Q} \,|\, |\mathcal{G}(n, M/N)| = K) \cdot P(|\mathcal{G}(n, M/N)| = K)$$

$$= \sum_{K=0}^{N} P(\mathcal{G}(n, K) \in \mathcal{Q}) \cdot P(|\mathcal{G}(n, M/N)| = K)$$

$[\because \mathcal{G}(n, p) \,|\, |\mathcal{G}(n, p)| = K \sim \mathcal{G}(n, K)]$

Let, M_1 & M_2 maximize $P(\mathcal{G}(n, K) \in \mathcal{Q})$ for $K \leq M$ & $K \geq M$, respectively.

Hence,

$$P(\mathcal{G}(n, M/N) \in \mathcal{Q}) \leq P(\mathcal{G}(n, M_1) \in \mathcal{Q}) \cdot P(|\mathcal{G}(n, M/N)| \leq M) + P(|\mathcal{G}(n, M/N)| > M) \quad \dots *)$$

Now, $|\mathcal{G}(n, M/N)| \overset{d}{=} X_1 + X_2 + \dots + X_N$, where $X_i \overset{iid}{\sim} Ber(M/N)$.

Hence, by CLT, since $\mathbb{E}(|\mathcal{G}(n, M/N)|) = M$, we have $P(|\mathcal{G}(n, M/N)| \leq M) \to 1/2$ & $P(|\mathcal{G}(n, M/N)| > M) \to 1/2$ as well.

From $*$),

$$1 = \lim_{n \to \infty} P(\mathcal{G}(n, M/N) \in \mathcal{Q}) \leq \frac{1}{2} \cdot \liminf_{n \to \infty} P(\mathcal{G}(n, M_1) \in \mathcal{Q}) + \frac{1}{2}$$

$$\Rightarrow P(\mathcal{G}(n, M_1) \in \mathcal{Q}) \to 1.$$

Similarly, we may arrive at;

$$P(\mathcal{G}(n, M/N) \in \mathcal{Q}) \leq P(|\mathcal{G}(n, M/N)| \leq M) + P(\mathcal{G}(n, M_2) \in \mathcal{Q}) \cdot P(|\mathcal{G}(n, M/N)| > M)$$

And, using the fact that $P(|\mathcal{G}(n, M/N)| > M) \to 1/2$, we get

$$1 = \lim_{n\to\infty} P(\mathcal{G}(n, M/N) \in Q)$$

$$\leq \frac{1}{2} + \liminf_{n\to\infty} P(\mathcal{G}(n, M_2) \in Q) \cdot \frac{1}{2}$$

$$\Rightarrow P(\mathcal{G}(n, M_2) \in Q) \to 1.$$

By Lemma 2.1., since $M_1 \leq M \leq M_2$, Q is convex & $P(\mathcal{G}(n, M_i) \in Q) \to 1$ for $i = 1, 2$ we conclude $P(\mathcal{G}(n, M) \in Q) \to 1$.

Case-b) :- By the law of total probability,

$$P(\mathcal{G}(n, M/N) \notin Q)$$

$$= \sum_{K=0}^{N} P(\mathcal{G}(n, M/N) \notin Q \mid |\mathcal{G}(n, M/N)| = k) \cdot P(|\mathcal{G}(n, M/N)| = k)$$

$$= \sum_{K=0}^{N} P(\mathcal{G}(n, k) \notin Q) \cdot P(|\mathcal{G}(n, M/N)| = k)$$

$$[\because \mathcal{G}(n, p) \mid |\mathcal{G}(n, p)| = k \sim \mathcal{G}(n, k)]$$

$$\geq P(\mathcal{G}(n, M) \notin Q) \cdot P(|\mathcal{G}(n, M/N)| = M)$$

$$= P(\mathcal{G}(n, M) \notin Q) \cdot \binom{N}{M}\left(\frac{M}{N}\right)^M\left(\frac{N-M}{N}\right)^{N-M} \quad \ldots *)$$

We call $B(N, M) = {}^N C_M \dfrac{M^M \cdot (N-M)^{N-M}}{N^N}$

We have, for $0 < k \leq n$, $\quad {}^n C_k \geq \left(\frac{n}{k}\right)^k$

$$\therefore B(N, M) = {}^N C_M \cdot \left(\frac{M}{N}\right)^M \cdot \left(1 - \frac{M}{N}\right)^N \cdot \left(1 - \frac{M}{N}\right)^{-M}$$

$$\geq \left(1 - \frac{M}{N}\right)^N \cdot \left(1 - \frac{M}{N}\right)^{-M}$$

Since, $M = O(1)$, say $M \leq C$, $\forall n$, for some C.

So, $B(N, M) \geq \left(1 - \frac{C}{N}\right)^N \cdot \left(1 - \frac{C}{N}\right)^{-M} \quad \ldots **)$

as $n \to \infty$, $\left(1-\frac{C}{N}\right)^N \to e^{-C}$.

and $C \geqslant M \geqslant 0$ implies

$$-C\log\left(1-\frac{C}{N}\right) \geqslant -M\log\left(1-\frac{C}{N}\right) \geqslant 0$$

as $n \to \infty$, $-C\log\left(1-\frac{C}{N}\right) \to 0$; so by 'Squeeze Thm.',

$$-M\log\left(1-\frac{C}{N}\right) \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \left(1-\frac{C}{N}\right)^{-M} \to 1 \text{ as } n \to \infty.$$

$$\therefore \lim_{n\to\infty} B(N,M) \geqslant e^{-C} \qquad (\text{from } **)$$

Therefore, from $*$),

$$0 = \lim_{n\to\infty} P(\mathcal{G}(n, M/N) \notin Q)$$

$$\geqslant \lim_{n\to\infty} P(\mathcal{G}(n,M) \notin Q) \cdot \lim_{n\to\infty} B(N,M)$$

$$\geqslant e^{-C} \lim_{n\to\infty} P(\mathcal{G}(n,M) \notin Q) \geqslant 0$$

Hence, $\lim_{n\to\infty} P(\mathcal{G}(n,M) \notin Q) = 0$, so $P(\mathcal{G}(n,M) \in Q) \to 1$.

Case – c) :- We observe, $B(N,M) = B(N, N-M)$

Hence, if $N-M = O(1)$, $\lim_{n\to\infty} B(N, N-M) \geqslant e^{-C}$. So

we similarly shall arrive at (as in case b)),

$$\lim_{n\to\infty} P(\mathcal{G}(n,M) \notin Q) = 0 \Rightarrow P(\mathcal{G}(n,M) \in Q) \to 1.$$

– Hence, for all subsequences of $\{x_n\}_{n\in\mathbb{N}} = \{P(\mathcal{G}(n,M) \in Q)\}_{n\in\mathbb{N}}$,

we get a further subsequence, for which $\{x_{n_{k_l}}\}_{n_{k_l}\in\mathbb{N}} \to 1$

as $l \to \infty$. We therefore use Theorem 2.3 to ~~conclude~~

complete the proof. ∎

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<u>Corollary 2.5.</u> :- Let Q be an increasing property of subsets of Γ, and let $M = M(n) \to \infty$.

i) If $P(\mathcal{G}(n, M/N) \in Q) \to 1$ then $P(\mathcal{G}(n, M) \in Q) \to 1$.

ii) If $P(\mathcal{G}(n, M/N) \in Q) \to 0$ then $P(\mathcal{G}(n, M) \in Q) \to 0$.

<u>Proof</u>:- A property Q is increasing or decreasing implies Q is convex. We can take $A, B, C \in 2^{\Gamma}$ with $A \subsetneq B \subsetneq C$ & $A, C \in Q$. If Q is increasing, $A \in Q \Rightarrow B \in Q$; if Q is decreasing, $C \in Q \Rightarrow B \in Q$.

Again, a property Q is increasing if and only if its complement in 2^{Γ} is decreasing.

i) → Since an increasing property is convex, ~~Theorem~~ Prop. 2.4. proves i).

ii) → Since Q^c is decreasing and hence convex, Prop. 2.4. proves ii) while using the fact that

$$P(A \in Q) \to 0 \iff P(A \notin Q) \to 1. \qquad \blacksquare$$

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3. THE THRESHOLD

For a fixed graph G, the problem of $\mathcal{G}(n, M)$ containing at least one copy of G was first studied by Erdös and Rényi in 1960. Since G is fixed, and $\mathcal{G}(n, M)$ grows as $n \to \infty$, copies of G in $\mathcal{G}(n, M)$ are called small subgraphs.

They found the threshold for the case when G is balanced. In 1981, Bollobás solved the problem in full generality. We shall present a simpler version of the proof, given by Rucinski and Vince (1985). It is a classical example where method of moments have been applied.

Once we find the threshold for $\mathcal{G}(n, p)$, we would use the tools in Section 2 to find the threshold for $\mathcal{G}(n, M)$.

Let G be a fixed graph; we denote the # of copies of G in $\mathcal{G}(n,p)$ by X_G.

<u>Lemma 3.1.</u> :- $\mathbb{E}X_G = \binom{n}{v_G} \cdot \frac{v_G!}{aut(G)} \cdot p^{e_G}$,

where for a fixed graph G, v_G & e_G respectively denote the # of vertices and edges of G. And $aut(G)$ is the size of the automorphism group of G.

<u>Proof</u> :- For an n, the complete graph K_n contains ${}^nC_{v_H} \cdot a_H$ many distinct copies of a graph H, where a_H is the number of copies of H in K_{v_H}.

For $H = G$, we ~~denot~~ order the $\binom{n}{v_G} \cdot a_G$ many distinct copies of G by G_t; $t = 1, \ldots, {}^nC_{v_G} \cdot a_G$. Then,

$$X_G = \sum_{t=1}^{{}^nC_{v_G} \cdot a_G} \mathbb{1}(\text{all } e_G \text{ edges of } G_t \text{ appear in } \mathcal{G}(n,p))$$

The indicator random variables in the summands are iid and follow $Ber(p^{e_G})$. Therefore,

$$\mathbb{E}X_G = {}^nC_{v_G} \cdot a_G \cdot p^{e_G}.$$

<u>Claim 3.2.</u> :- $a_G \times aut(G) = v_G!$

<u>Proof</u> (Claim 3.2.) :- Each permutation σ of $[v_G] = \{1, \ldots, v_G\}$ defines a copy of G:

A copy of G corresponds to a set of e_G edges of K_{v_G}. The copy G_σ for the perm. σ has edges $\{(x_{\sigma(i)}, y_{\sigma(i)}) : 1 \leq i \leq e_G\}$, where $\{(x_j, y_j) : 1 \leq j \leq e_G\}$ is a fixed copy of G.

But $G = G_\sigma$ if $\exists \tau$ s.t. for each i, $\exists j$ s.t. $(x_j, y_j) = (x_{\tau\sigma(i)}, y_{\tau\sigma(i)})$, for $i, j \leq e_G$; i.e. if τ is an automorphism. Hence in every v_G-vertex subgraph, the # of copies of G in it repeats $aut(G)$ times. Therefore, $a_G = v_G! / aut(G)$. ▨

From Claim 3.2., the proof of Lemma 3.1. follows. ▨

Theorem 3.3. :- For a graph G with v_G vertices & e_G edges, if $p = o(n^{-v_G/e_G})$, then $P(\mathcal{G}(n,p)$ contains a copy of $G) \to 0$ as $n \to \infty$.

Proof: Define, $X_G \to$ # of copies of G in $\mathcal{G}(n,p)$. Suppose that, $p(n) = \delta(n) \cdot n^{-v_G/e_G}$; since $p = o(n^{-v_G/e_G})$, as $n \to \infty$, $\delta \to 0$.

Again, for v_G fixed (since G is fixed),

$$\binom{n}{v_G} \cdot \frac{v_G!}{aut(G)} \leq C \cdot n^{v_G}$$; for some $C > 0$; ~~for some~~ $\forall n > n_\varepsilon$, for some n_ε.

Therefore,

$$\mathbb{E} X_G \leq C \cdot n^{v_G} \cdot \delta^{e_G} \cdot n^{-e_G \cdot v_G / e_G}, \text{ for } n > n_\varepsilon$$

i.e. $\quad 0 \leq \mathbb{E} X_G \leq C \cdot \delta^{e_G}$, for $n > n_\varepsilon$. As $n \to \infty$, we use 'Squeeze Thm.' to conclude $\mathbb{E} X_G \to 0$.

Hence, using Markov's inequality,

$$0 \leq P_{\mathcal{G}(n,p)}(X_G > 0) = P_{\mathcal{G}(n,p)}(X_G \geq 1)$$
$$\leq \mathbb{E} X_G \to 0 \text{ as } n \to \infty.$$

Applying 'Squeeze Thm.' for another time completes the proof. ▨

Definition 3.4. :- Consider a graph H with v_H vertices & e_H edges. Suppose $v_H > 0$.

The quantity $d(H) = e_H / v_H$ is said to be the 'density' of H.

Consider, $m(H)$ ~~= max~~ defined by, $m(H) = \max_{\substack{H' \subseteq H \\ v_{H'} > 0}} d(H')$ –

$m(H)$ is called the 'maximum density' of H.

H is called 'balanced' if $d(H) = m(H)$, i.e if for every $H' \subseteq H$, $d(H') \leq d(H)$. If $d(H)$ satisfies $d(H) > d(H')$ for $H' \subsetneq H$, we call H to be 'strictly balanced'.

– –

From Theorem 3.3. we have seen, $P_{\mathcal{G}(n,p)}(X_G > 0) \to 0$ for $p \ll n^{-v_G/e_G}$ i.e for $p \ll n^{-1/d(G)}$. One can thus expect $P_{\mathcal{G}(n,p)}(X_G > 0) \to 1$ for $p \gg n^{-1/d(G)}$ i.e if $p / n^{-1/d(G)} \to \infty$.

Say $p = \omega(n) \cdot n^{-1/d(G)}$ where $\omega(n) \to \infty$ as $n \to \infty$.

Then,

$$\mathbb{E} X_G = \binom{n}{v_G} \cdot \frac{v_G!}{aut(G)} \cdot p^{e_G}$$

$$\geq n^{v_G} \cdot C \cdot p^{e_G} \quad \forall n \geq n_\varepsilon, \text{ for some } C \;\&\; n_\varepsilon$$

$$= n^{v_G} \cdot C \cdot \omega^{e_G} \cdot n^{-v_G}$$

$$= \omega^{e_G} \cdot C \to \infty \text{ as } n \to \infty. \quad \ldots (*)$$

Does it suffice for $\mathcal{G}(n,p)$ to contain a copy of G?

We shall find out that it is not. To see this, consider the graph in fig. 3.1.

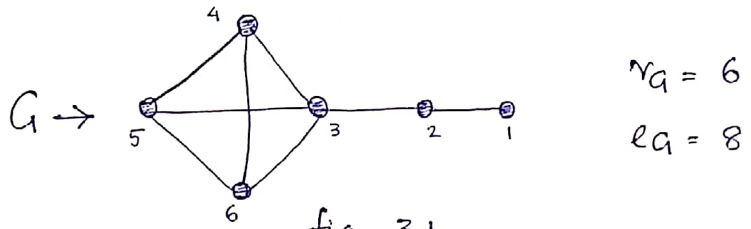

fig. 3.1.

We see that $d(G) = \frac{8}{6}$. G also contains K_4 as a subgraph; $d(K_4) = \frac{6}{4}$. We take a fraction in between $d(G)$ & $d(K_4)$, say $7/5$. Consider, $p = n^{-5/7}$.

We see, $p = n^{-5/7} = n^{-6/8} \cdot n^{(6/8 - 5/7)}$

$$= n^{-1/d(G)} \cdot n^{1/28}$$

Thus, we observe, $p / n^{-1/d(G)} = n^{1/28} \to \infty$ as $n \to \infty$.

Hence, $\mathbb{E}X_G \to \infty$ as $n \to \infty$. (from (*)).

Again, $p = n^{-5/7} = n^{-4/6} \cdot n^{(4/6 - 5/7)}$

$$= n^{-1/d(K_4)} \cdot n^{-1/21}$$

So, $p / n^{-1/d(K_4)} = n^{-1/21} \to 0$ as $n \to \infty$. From Theorem 3.3, we conclude $\mathbb{E}X_{K_4} \to 0$ as $n \to \infty$. We see that, for $p = n^{-1/d(G)}$, $P(X_G > 0) \leq P(X_{K_4} > 0) \to 0$; thus $\mathcal{G}(n,p)$ won't contain a copy of K_4 that is a copy of G, a.a.s. This happens because G contains a subgraph (K_4) denser than itself.

<u>Theorem 3.5.</u> :- Let G be a fixed graph with $e_G > 0$. Then,

$$\lim_{n \to \infty} P(\mathcal{G}(n,p) \supset G) = \begin{cases} 0 & , \text{ if } p \ll n^{-1/m(G)} \\ 1 & , \text{ if } p \gg n^{-1/m(G)} \end{cases}$$

Proof:-

'0'-statement → Let $H \subseteq G$ be s.t. $m(G) = d(H)$.

Then,

$$P_{\mathcal{G}(n,p)}(X_G > 0) \leq P_{\mathcal{G}(n,p)}(X_H > 0) \to 0$$ as $n \to \infty$, from Theorem 3.3.; where $X_{H'}$ denotes # copies of H' in $\mathcal{G}(n,p)$.

Thus, when $p = o(n^{-1/m(G)})$, a.a.s. there is not copy of H in $\mathcal{G}(n,p)$, ~~i.e.~~ and hence, no copy of G.

'1'-statement → The second moment method will be required to prove this. We define,

$$\Phi_G = \Phi_G(n,p) = \min_{H \subseteq G,\, e_H > 0} \mathbb{E}_{\mathcal{G}(n,p)}(X_H).$$

Since, $\mathbb{E} X_H \asymp p^{e_H} \cdot n^{v_H}$, for any H fixed; we get

$$\Phi_G \asymp \min_{H \subseteq G,\, e_H > 0} n^{v_H} p^{e_H};$$

Lemma 3.6.:- Let G be a graph with $e_G > 0$, then

$$\operatorname{Var}(X_G) \asymp (1-p) \sum_{H \subseteq G,\, e_H > 0} n^{2v_G - v_H} \cdot p^{2e_G - e_H}$$

$$\asymp (1-p) \max_{H \subseteq G,\, e_H > 0} \frac{(\mathbb{E} X_G)^2}{\mathbb{E} X_H} = (1-p) \cdot \frac{(\mathbb{E} X_G)^2}{\Phi_G}.$$

Lemma 3.7.:- For any graph G with $e_G > 0$, TFAE:

i) $np^{m(G)} \to \infty$;

ii) $n^{v_H} p^{e_H} \to \infty$ for every $H \subseteq G$ with $v_H > 0$,

iii) $\mathbb{E} X_H \to \infty$ for every $H \subseteq G$ with $v_H > 0$,

iv) $\Phi_G \to \infty$.

— to complete the proof of Thm. 3.5, we observe that if $p \gg n^{-1/m(G)}$, $\Phi_G \to \infty$ by Lemma 3.7. Hence by the second moment method & Lemma 3.6,

$$P_{\mathcal{G}(n,p)}(X_G = 0) \leq \frac{\mathrm{Var}(X_G)}{(\mathbb{E}X_G)^2} = \frac{(\mathbb{E}X_G)^2}{(\mathbb{E}X_G)^2} \cdot O(1/\Phi_G) = o(1).$$

Thus, $\lim_{n\to\infty} P(\mathcal{G}(n,p) \supset G) = 0$ for $p \gg n^{-1/m(G)}$. ▨

<u>Proof</u> (Lemma 3.6.):- As defined in Lemma 3.1., let

$$I_{G_t} = 1(\text{all } e_G \text{ edges of } G_t \text{ appear in } \mathcal{G}(n,p))$$

Hence, $X_G = \sum_t I_{G_t} \Rightarrow \mathrm{Var}(X_G) = \sum_{t,t'} \mathrm{Cov}(I_{G_t}, I_{G_{t'}})$.

Now, the edges of G_t & $G_{t'}$ are disjoint would imply I_{G_t} & $I_{G_{t'}}$ are indep. Therefore, if $E(G_t)$ & $E(G_{t'})$ are disjoint, $\mathrm{Cov}(I_{G_t}, I_{G_{t'}}) = 0$.

Now we consider G_t & $G_{t'}$ s.t. $H = G_t \cap G_{t'} \neq \phi$; so H is a subgraph of G with $e_H > 0$. Such G_t & $G_{t'}$ may be chosen by the following way:

one chooses v_H vertices for H. Then chooses $v_G - v_H$ vertices for G_t, from $n - v_H$ many vertices. And finally chooses $v_G - v_H$ many vertices from $n - v_G$ many remaining vertices for $G_{t'}$. This can be done in:

$$\binom{n}{v_H} \cdot \binom{n-v_H}{v_G - v_H} \cdot \binom{n - v_G}{v_G - v_H} \asymp n^{v_H} \cdot n^{2(v_G - v_H)} \text{ ways.}$$

The constants used in $\asymp$ doesn't depend on n, but on H. And the probability for the edges to appear in $G_t \cup G_{t'}$ is $p^{e_H} \cdot p^{2(e_G - e_H)} = p^{2e_G - e_H}$, and to appear in G_t or $G_{t'}$ is p^{e_G}.

Therefore,

$$\mathrm{Var}(X_G) = \sum_{t,t'} \mathrm{Cov}(I_{G_t}, I_{G_{t'}})$$

$$= \sum_{E(G_t)\cap E(G_{t'}) = \phi} \{\mathbb{E}(I_{G_t}\cdot I_{G_{t'}}) - \mathbb{E}(I_{G_t})\cdot \mathbb{E}(I_{G_{t'}})\}$$

$$\asymp \sum_{H\subseteq G,\, e_H>0} n^{v_H}\cdot n^{2(v_G - v_H)}\cdot \left[p^{2e_G - e_H} - p^{2e_G}\right]$$

$$\asymp \sum_{H\subseteq G,\, e_H>0} n^{2v_G}\cdot p^{2e_G}\cdot n^{-v_H}\cdot p^{-e_H}\cdot \left[1 - p^{e_H}\right]$$

$$\asymp (1-p)\sum_{H\subseteq G,\, e_H>0} n^{2v_G - v_H}\cdot p^{2e_G - e_H}$$

— the last relation follows from the fact that,

$$(1-p^{e_H}) \geqslant (1-p)$$

$$\& \quad (1-p^{e_H}) = (1-p)(1+p+\cdots+p^{e_H-1})$$

$$\leqslant (1-p)(1+p+\cdots+p^{e_G-1}) \leqslant e_G\cdot(1-p)$$

Hence,

$$\mathrm{Var}(X_G) \asymp (1-p)\sum_{H\subseteq G,\, e_H>0} n^{2v_G - v_H}\cdot p^{2e_G - e_H}$$

$$= (1-p)\sum_{H\subseteq G,\, e_H>0} (n^{v_G}p^{e_G})^2 / n^{v_H}p^{e_H}$$

$$\asymp (\mathbb{E}X_G)^2\cdot (1-p)\max_{H\subseteq G,\, e_H>0} \frac{1}{\mathbb{E}X_H} = (1-p)\frac{(\mathbb{E}X_G)^2}{\mathbb{E}X_H}.$$

— the penultimate relation follows from the fact that

$\mathbb{E}X_G \asymp n^{v_G}p^{e_G}$ i.e $\exists\, c_1, c_2$ s.t, ~~$c_1\, n^{v_G}\cdot p^{e_G} \leq \mathbb{E}$~~

$$c_1 \cdot n^{v_G} \cdot p^{e_G} \leq \mathbb{E}X_G \leq c_2 \cdot n^{v_G} \cdot p^{e_G} \text{; for large } n.$$

And also from the fact that, for large n

$$\max_{H \subseteq G, e_H > 0} \frac{1}{n^{v_H} p^{e_H}} \leq \sum_{H \subseteq G, e_H > 0} \frac{1}{n^{v_H} p^{e_H}} \leq 2^{e_G} \cdot \max_{H \subseteq G, e_H > 0} \frac{1}{n^{v_H} \cdot p^{e_H}}$$

— since there are at most 2^{e_G} many terms in the sum.

Hence the proof. ▨

<u>Proof</u> (Lemma 3.7.):-

"(i) $\Longleftrightarrow$ (ii)" $\rightarrow$ Let $H' \subseteq G$ be s.t. $d(H') = m(G)$. If $n^{v_H} p^{e_H} \to \infty$ for all $H \subseteq G$, it is true for $H = H'$. So.

$$n^{v_{H'}} p^{e_{H'}} = (n p^{e_{H'}/v_{H'}})^{v_{H'}} \to \infty \text{ as } n \to \infty \text{ i.e } n p^{m(G)}$$

$\to \infty$ as $n \to \infty$. So ii) $\Rightarrow$ i).

For $p = 1$, i) $\Rightarrow$ ii) is trivial. If, $0 \leq p < 1$, it follows that for any $H \subseteq G$, $n^{v_H} p^{e_H} = (n p^{d(H)})^{v_H} \geq (n p^{m(G)})^{v_H}$ $\to \infty$ as $n \to \infty$. So i) $\Rightarrow$ ii).

"(ii) $\Longleftrightarrow$ (iii)" $\rightarrow$ We have, $\mathbb{E}X_H = {}^nC_{v_H} \cdot a_H \cdot p^{e_H}$, from Lemma 3.1; since ${}^nC_{v_H} \cdot a_H \asymp n^{v_H}$, $\mathbb{E}X_H \asymp n^{v_H} \cdot p^{e_H}$. This gives the both side implications.

"(iii) $\Longleftrightarrow$ (iv)" $\rightarrow$ Say $\Phi_G \to \infty$. Since $\Phi_G = \min_{H \subseteq G} \mathbb{E}X_H$, any larger expectation would go to infinity as well. So iv) $\Rightarrow$ iii).

And, iii) $\Rightarrow$ iv) is trivial since Φ_G is also expectation of some H' and hence $\Phi_G \to \infty$ as a special case. ▨

<u>Remark 3.8.</u> :- If $\Phi_G(n,p) \to \infty$, $X_G / \mathbb{E} X_G \xrightarrow{P} 1$.

<u>Proof</u> :- $P_{\mathcal{G}(n,p)}\left(|X_G - \mathbb{E}X_G| \geq \varepsilon \cdot \mathbb{E}X_G\right)$

$$= P_{\mathcal{G}(n,p)}\left(\left|\frac{X_G}{\mathbb{E}X_G} - 1\right| \geq \varepsilon\right)$$

$$\leq \frac{\mathrm{Var}(X_G)}{\varepsilon^2 \cdot \mathbb{E}^2 X_G} \qquad \text{[Chebyshev's ineq.]}$$

$$= O(1/\Phi_G) \qquad \text{[Lemma 3.6.]}$$

$$= o(1).$$

∎

<u>Theorem 3.9.</u> :- For a fixed graph G with at least one edge,

$$\lim_{n\to\infty} P(\mathcal{G}(n,M) \supset G) = \begin{cases} 0, & \text{if } M \ll n^{2-1/m(G)} \\ 1, & \text{if } M \gg n^{2-1/m(G)} \end{cases}$$

<u>Proof</u> :- Containing a fixed subgraph is a monotone property. For $N = \binom{n}{2}$, we take $p = M/N$.

By Thm. 3.5. we have,

$$\lim_{n\to\infty} P(\mathcal{G}(n,M/N) \supset G) = \begin{cases} 0, & \text{if } \frac{M}{N} \ll n^{-1/m(G)} \\ 1, & \text{if } \frac{M}{N} \gg n^{-1/m(G)} \end{cases}$$

For sufficiently large n, and by Corollary 2.5.,

$P(\mathcal{G}(n,M) \supset G) \to 0$ if $\frac{M}{N} \ll n^{-1/m(G)}$

i.e. $P(\mathcal{G}(n,M) \supset G) \to 0$ for $M \ll n^{2-1/m(G)}$.

Similarly, $P(\mathcal{G}(n,M) \supset G) \to 1$ for $M \gg n^{2-1/m(G)}$.

[Both of them follow from the fact that $N \asymp n^2$]

∎

Example 3.10. :-

(i) For $k \geq 3$, a k-cycle is balanced. Hence the threshold for $\mathcal{G}(n,p)$ to contain a k-cycle is $1/n$, since $m(G) = n/n = 1$ for G to be any cycle.

(ii) For $k \geq 2$, the complete graph K_k is balanced, so $m(K_k) = {}^{k}C_2 / k = \frac{k-1}{2}$. So the threshold for $\mathcal{G}(n,p)$ to a.a.s. contain K_k is $n^{-2/(k-1)}$.

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4. SUBGRAPH COUNT AT THRESHOLD

We have already seen that for a fixed graph G, the threshold for containment of G in $\mathcal{G}(n,p)$ is $n^{-1/m(G)}$. ~~Then~~ The question that follows is, whether this is sharp.

This section presents results regarding the asymptotics of Φ_G and $np^{m(G)}$, and then shows that the threshold in Thm. 3.5. is coarse. For $p = \Theta(\hat{p})$, the computation of $\lim_{n\to\infty} P(\mathcal{G}(n,p) \supset G)$ is somewhat difficult in general.

But then, a result, proved independently by Bollobás (1981); and Karoński and Rucinski (1983), for the class of strictly balanced graphs makes it easier. This computes the limiting distribution of X_G.

Theorem 4.1.:- Let G be a fixed graph with at least one edge. Then, for every sequence $p = p(n) < 1$,

$$\exp\left\{-\frac{1}{1-p}\,\Phi_G\right\} \leq P(\mathcal{G}(n,p) \not\supset G) \leq \exp\{-\Theta(\Phi_G)\}$$

– proof of this theorem involves the FKG inequality, hence that has been skipped.

Result 4.2. :- When $p = \Theta(n^{-1/m(G)})$, we have $\Phi_G = \Theta(1)$ for a graph G.

Proof:- We have already noticed in the proof of Thm. 3.5. that,

$$\Phi_G \asymp \min_{H \subseteq G,\, e_H > 0} n^{v_H} p^{e_H}.$$

For $p = \Theta(n^{-1/m(G)})$,

$$\Phi_G \asymp \min_{H \subseteq G,\, e_H > 0} n^{v_H}\, n^{-e_H/m(G)}$$

$$\asymp \min_{H \subseteq G,\, e_H > 0} n^{\{v_H \cdot m(G) - e(H)\}/m(G)} \quad \ldots *)$$

Since, $m(G) = \max_{H \subseteq G} \frac{e_H}{v_H}$, $\forall H \subseteq G$, $v_H \cdot m(G) - e_H \geq 0$.

So in $*)$, the quantity in RHS is minimized when the exponent of n is zero i.e. $v_H / e_H = 1/m(G)$. Therefore –

$$\Phi_G \asymp n^0 = 1 \Rightarrow \Phi_G = \Theta(1). \quad \blacksquare$$

Result 4.3. :- If $p = \Theta(n^{-1/m(G)})$, we have

$$0 < \liminf_{n\to\infty} P(\mathcal{G}(n,p) \supset G) \leq \limsup_{n\to\infty} P(\mathcal{G}(n,p) \supset G) < 1.$$

Proof:- Using Theorem 4.1. & Result 4.2., as $n \to \infty$

when $p = \Theta(n^{-1/m(G)})$,

$$0 < \liminf_{n\to\infty} P(\mathcal{G}(n,p) \not\supset G) \leq \limsup_{n\to\infty} P(\mathcal{G}(n,p) \not\supset G) \leq 1.$$

This completes the proof. ∎

Result 4.4. :- The threshold in Thm. 3.5 cannot be sharpened.

Proof:- Consider $\eta = 1$. then for $\hat{p} = n^{-1/m(G)}$, set $p = (1+\eta)\hat{p} = 2n^{-1/m(G)} = \Theta(n^{-1/m(G)})$.

By Result 4.3., though $p \geq (1+\eta)\hat{p}$ we have

$$\limsup_{n\to\infty} P(\mathcal{G}(n,p) \supset G) < 1$$

Therefore the threshold in 3.5. is not sharp from above.

Similarly, we take $\eta = \frac{1}{2}$; then for $p = (1-\eta)\hat{p}$, we have $p = \Theta(n^{-1/m(G)})$. Albeit $p \leq (1-\eta)\hat{p}$, we have from Result 4.3.

$$\liminf_{n\to\infty} P(\mathcal{G}(n,p) \supset G) > 0$$

So the threshold cannot be sharpened from below. Hence the proof. ∎

Theorem 4.5. :- If G is a strictly balanced graph and $np^{m(G)} \to c > 0$, then $X_G \xrightarrow{d} \mathrm{Poi}(\lambda)$, the Poisson distribution with mean $\lambda = c^{v_G}/aut(G)$.

Proof:- The techinique of 'Method of Moments' shall be used to prove this theorem. Let X_G denotes the # of distinct copies of G in $\mathcal{G}(n,p)$. Let G_t be the distinct copies of G in K_n for $t = 1, \dots, {}^{n}C_{v_G} \cdot a_G$.

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Let I_{G_t} denote the indicator random variable that the graph G_t is contained in $\mathcal{G}(n,p)$. ~~For~~ Therefore,

$$X_G = \sum_t I_{G_t}$$

We consider the factorial moments of X_G. If we can show that the factorial moments of X_G converges to that of Z, for $Z \sim Poi(\lambda)$, $\forall k \in \mathbb{N}$; the proof is done.

Define, $(X_G)_k = X_G(X_G - 1)\cdots(X_G - k + 1)$, for $k \geq 1$ – the k^{th} factorial moment of X_G.

Then, we have,

$$\mathbb{E}(X_G)_k = \mathbb{E}\left[X_G(X_G-1)\cdots(X_G-k+1)\right]$$

$$= \sum_{G_1,\ldots,G_k} P(I_{G_1} = \cdots = I_{G_k} = 1)$$

$$= E_k + E_k'$$

where E_k & E_k' respectively denote the partial sums when the vertex sets of $G_1,\ldots,G_k$ are disjoint and not disjoint respectively.

When the vertex sets of $G_1,\ldots,G_k$ are disjoint, $I_{G_1},\ldots,I_{G_k}$ are independent. Hence,

$$P(I_{G_1} = \cdots = I_{G_k} = 1) = P(I_{G_1} = 1, \ldots, I_{G_k} = 1)$$
$$= P(I_{G_1} = 1)\cdots P(I_{G_k} = 1)$$
$$= (p^{e_G})^k$$

Hence, $D_k = C_{G,k} \cdot (p^{e_G})^k$ – where $C_{G,k}$ denote the # of k-vertex disjoint copies of G in K_n.

Now,

$$C_{G,k} = \binom{n}{v_G} \cdot \binom{n-v_G}{v_G} \cdots \binom{n-(k-1)v_G}{v_G} \cdot \left[\frac{v_G!}{aut(G)}\right]^k$$

– since we may choose first v_G vertices out of n, then v_G vertices out of remaining $n - v_G$ many vertices and so on.

Since G is strictly balanced, $m(G) = e_G/v_G$. So,

$$C_{G,k} = \frac{n!}{(n-kv_G)!} \cdot \frac{1}{(aut(G))^k}$$

Since $p^{m(G)} \cdot n \to c$ as $n \to \infty$ we have,

$$p^{e_G} \cdot n^{v_G} \to c^{v_G} \text{ as } n \to \infty$$

$$\text{i.e.} \quad p^{e_G} \sim \frac{c^{v_G}}{n^{v_G}} \Rightarrow [p^{e_G}]^k \sim \frac{(c^{v_G})^k}{n^{kv_G}}.$$

And,

$$\frac{n!}{(n-kv_G)!} = n \cdot (n-1) \cdots (n-kv_G+1) \sim n^{kv_G}$$

$$\therefore E_K = C_{G,k} \cdot [p^{e_G}]^k \sim \left[\frac{c^{v_G}}{aut(G)}\right]^k = \lambda^k.$$

If we can show that E_K' is $o(1)$, the proof is completed. Consider a k not mutually vertex disjoint copies of G where are t many vertices. Let e_t denote the minimum # of edges in such t-vertex union.

<u>Claim 4.6.</u> :- For every $k \geq 2$ and $k \leq t < kv_G$, we have $e_t > t\, m(G)$.

<u>Proof</u> (Claim 4.6.) :- For a graph F, we define,

$f(F) = m(G) \cdot v_F - e_F$.

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We note that, $f(G)=0$ and since G is strictly balanced, $f(H) \cancel{<0} > 0$ for every $H \subsetneq G$. For $k=2$, let ~~$F = H_1 \cup H_2$ b~~ $F = G_1 \cup G_2$ be a not mutually vertex disjoint union of 2 copies of G. Since,

$$v_{G \cup H} = v_G + v_H - v_{G \cap H}$$

$$\& \; e_{G \cup H} = e_G + e_H - e_{G \cap H}$$ – for any graphs G & H; We see that

$$f(F_1 \cup F_2) = f(F_1) + f(F_2) - f(F_1 \cap F_2)$$ – for any graphs F_1, F_2. Now, $H = G_1 \cap G_2$ is a proper subgraph of G. Therefore,

$$f(F) = f(G_1 \cup G_2) = f(G_1) + f(G_2) - f(H) = -f(H) < 0.$$

Now, for some $k \geq 3$, let us assume for any not mutually vertex disjoint union of k-copies of G, $f\left(\bigcup_{i=1}^{k} G_i\right) < 0$. Say G_{k+1} is a copy of G s.t. $H = \left(\bigcup_{i=1}^{k} G_i\right) \cap G_{k+1} \neq \phi$. Then,

$$f\left(\bigcup_{i=1}^{k+1} G_i\right) = f\left(\bigcup_{i=1}^{k} G_i\right) + f(G_{k+1}) - f(H)$$

$$= f\left(\bigcup_{i=1}^{k} G_i\right) - f(H) < 0.$$

Therefore, $f(F) < 0$ for any such union, especially when the # edges is minimum for union of total t many vertices. Hence, $m(G) \cdot t - e_t < 0$. ▨

Having proven claim 4.6. we denote, $e_t = \delta_t + t m(G)$ where $\delta_t > 0$. Let us denote by $\mathcal{F}_k$, the family of all graphs obtained by taking unions of k not all pairwise vertex disjoint copies of G.

$$\therefore E_k' = \sum_{F \in \mathcal{F}_K} f(n,F) \cdot p^{e_F}$$ – where $f(n,F)$ denotes

the # of sequences $G_{i_1}, \dots, G_{i_K}$ of k distinct copies of G s.t.

$$V\left(\bigcup_{j=1}^{k} G_{i_j}\right) = \{1, \dots, v_F\} \quad \& \quad \bigcup_{j=1}^{K} G_{i_j} \cong F$$

for a total of ${}^{n}C_{v_F}$ many choices. Clearly,

$f(n,F) \asymp n^{v_F}$ and since e_t is the min. # of edges of a t-vertex union, we have

$$E_k' = \sum_{t=K}^{kv_G - 1} O(n^t p^{e_t})$$

$$= \sum_{t=K}^{kv_G - 1} O\left[(np^{m(G)})^t \cdot p^{\delta_t}\right]$$

Now, $np^{m(G)} \to c$ implies $p \to 0$ therefore, as $\delta_t > 0 \ \forall t$,

$$E_k' = \sum_{t=k}^{kv_G - 1} o(1) = o(1).$$ ▨

From Result 4.4., we find that $n^{-1/m(G)}$ is a coarse threshold for containment of G. When p is $\Theta(n^{-1/m(G)})$, the derivation of $\lim_{n \to \infty} P(\mathcal{G}(n,p) \supset G)$ may not be easy. But, for strictly balanced graphs, not only the precise value of the limit, but the entire limiting distribution can be calculated for X_G.

We now shall see the $\mathcal{G}(n,M)$ analogue of Thm. 4.5.

Theorem 4.7. :- Let G be a strictly balanced graph, and let X_G denote the # of copies of G in $\mathcal{G}(n,M)$. If $M \sim cn^{2-v/e}$, for some $c>0$, then $X_G \xrightarrow{d} \mathrm{Poi}(\lambda)$, the Poisson distribution with mean $\lambda = (2c)^{e_G}/\mathrm{aut}(G)$.

Proof: Let $(X_G)_k$ be defined as in Thm. 4.5. The thm. will be proved if we can show $\mathbb{E}(X_G)_k \to \lambda^k$ for every $k \geq 1$.

We observe,

$$\mathbb{E}(X_G)_k = \sum_{G_1,\dots,G_k} \mathbb{P}(I_{G_1} = \cdots = I_{G_k} = 1)$$

$$= E_k + E_k'$$

– where the partial sums E_k & E_k' respectively cover the cases when the vertex sets of the k-union are pairwise disjoint and not pairwise disjoint.

When the vertex sets of $G_1,\dots,G_k$ are pairwise disjoint, I_{G_t}'s are indep. $\forall t$. Therefore, we can choose the k many disjoint copies by choosing them one by one. Firstly from n vertices, then from $n-v_G$ vertices and so on;..., finally the remaining $M-ke_G$ many edges may be chosen from rest of $N-ke_G$ edges.

Therefore,

$$E_k = \binom{n}{v_G}\frac{v_G!}{\mathrm{aut}(G)}\cdot\binom{n-v_G}{v_G}\cdot\frac{v_G!}{\mathrm{aut}(G)}\times\cdots\times\frac{\binom{N-ke_G}{M-ke_G}}{\binom{N}{M}}$$

$$= {}^{n}C_{v_G}\times\cdots\times\binom{n-(k-1)v_G}{v_G}\cdot\left[\frac{v_G!}{\mathrm{aut}(G)}\right]^k\cdot\prod_{i=0}^{ke_G-1}\frac{M-i}{N-i}$$

Since, $\binom{n - r \cdot v_G}{v_G} \approx \frac{n^{v_G}}{v_G!}$ $\forall r$.

& $\frac{M}{N} \sim \frac{2c}{n^{v/e}}$, we get

$$E_K \sim \frac{\cancel{n^{v_G}}}{\cancel{v_G}} \left[\frac{n^{v_G}}{v_G!}\right]^K \cdot \left[\frac{v_G!}{aut(G)}\right]^K \cdot \left[\frac{2c}{n^{v_G/e_G}}\right]^{Ke_G}$$

$$= \left[\frac{(2c)^{e_G}}{aut(G)}\right]^K = \lambda^K.$$

All we need to show is now, that $E_K' = o(1)$. From Claim 4.6., $e_t > t\, m(G)$ for any non-vertex disjoint ~~copies~~ union of copies of G, for G strictly balanced.

Therefore,

$$E_k' = \sum_{t=v_G}^{kv_G-1} O\left(\binom{n}{t} \cdot \left[\binom{t}{v_G} \cdot \frac{v_G!}{aut(G)}\right]^k \cdot \binom{N-e_t}{M-e_t} \Big/ {}^N C_M\right)$$

$\left[\text{Since, } \binom{N-r_1}{M-r_1} > \binom{N-r_2}{M-r_2} \text{ for } r_2 > r_1\right]$

$$= \sum_{t=v_G}^{kv_G-1} O\left(n^t \cdot \prod_{i=0}^{e_t-1} \frac{M-i}{N-i}\right)$$

$$= \sum_{t=v_G}^{kv_G-1} O\left(n^t \cdot (M/N)^{e_t}\right)$$

$$= \sum_{t=v_G}^{kv_G-1} O\left[\left(n \cdot (M/N)^{m(G)}\right)^t \cdot (M/N)^{\delta_t}\right]$$

$[\text{where, } e_t = \delta_t + t\,m(G)]$

$\because n \cdot \left(\frac{M}{N}\right)^{m(G)} \to (2c)^{m(G)}$ hence $M/N \to 0$ as $n \to \infty$.

Therefore, $E_k' = \sum_{t=v_G}^{kv_G-1} o(1) = o(1)$. ▨

APPENDIX

Result:- For $k \geq 1$, $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$.

Proof:- $$^nC_k = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots(k-k+1)}$$

Now, $\dfrac{n-r}{k-r} \geq \dfrac{n}{k}$ for $0 \leq r < k$ hence the proof. ▨

Result:- For any r, k s.t. $n \geq r+k$, $\binom{n-r}{k} \asymp n^k$.

Proof:- $$\binom{n-r}{k} = \frac{(n-r)!}{k!\cdot(n-k-r)!}$$

$$= \frac{(n-r)(n-r-1)\cdots(n-r-(k-1))}{k\,(k-1)\cdots(k-(k-1))}$$

$$\asymp (n-r)^k \asymp n^k.$$ ▨

Corollary :- For any r, k s.t. $n \geq r+k$, $\binom{n-r}{k} \sim \dfrac{n^k}{k!}$.

Theorem:- When $X = \sum_{i \in I} I_i$ is a sum of indicators, then

$$\mathbb{E}(X)_r = \sum_{i_1,\dots,i_r \in I}^{*} P(I_{i_1} = \cdots = I_{i_r} = 1)$$

where $\sum^{*}_{i_1,\dots,i_r}$ denotes sum over distinct indices.

Result (Method of Moments):-

a) For a non(-ve) integer valued random variable X,

$$P(X>0) \leq \mathbb{E}X,$$

b) For any random variable X with $\mathbb{E}X > 0$,

$$P(X=0) \leq \frac{Var(X)}{(\mathbb{E}X)^2}.$$

Lemma:- For an increasing property Q –

a) $0 \leq p_1 \leq p_2 \leq 1$ implies

$$P(G(n, p_1) \in Q) \leq P(G(n, p_2) \in Q)$$

b) $0 \leq M_1 \leq M_2 \leq N$ implies

$$P(G(n, M_1) \in Q) \leq P(G(n, M_2) \in Q).$$

Asymptotic Notations:-

i) $a_n = O(b_n)$ as $\lim n \to \infty$, if $\exists C \in \mathbb{R}$ and $n_\varepsilon \in \mathbb{N}$ s.t. $|a_n| \leq C b_n$ $\forall n \geq n_\varepsilon$.

ii) $a_n = \Omega(b_n)$ as $\lim n \to \infty$, if $b_n = O(a_n)$.

iii) $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$.

iv) $a_n \asymp b_n$, if $a_n = \Theta(b_n)$

v) $a_n \sim b_n$ if as $\lim n \to \infty$, $a_n / b_n \to 1$.

vi) $a_n = o(b_n)$, if as $\lim n \to \infty$ $a_n / b_n \to 0$, i.e. $\forall \varepsilon > 0$ $\exists N_\varepsilon$ s.t. $\forall n \geq N_\varepsilon$ $|a_n / b_n| < \varepsilon$.

vii) $a_n \gg b_n$ or $b_n \ll a_n$ if $b_n = o(a_n)$

– We say, A_n, an event related to a random structure and depending on n, holds a.a.s i.e asymptotically almost sure if $P(A_n) \to 1$ as $n \to \infty$.

Notations:-

$\mathcal{G}(n,p) \rightarrow$ Binomial Random graph with edge prob. p.

$\mathcal{G}(n,M) \rightarrow$ Uniform Random graph with M edges.

$V(G) \rightarrow$ Vertex set of graph G

$E(G) \rightarrow$ Edge set of graph G

$|E(G)| / |G| \rightarrow$ # of edges of G [also e_G]

$|V(G)| \rightarrow$ # of vertices of G [also v_G]

$aut(G) \rightarrow$ size of automorphism group of G

$\mathbb{E}(X)_r \rightarrow r^{th}$ factorial moment of X

$d(G) \rightarrow$ density of graph G

$m(G) \rightarrow$ maximum density of graph G.

$\Gamma \rightarrow [n] \times [n]$, set of all possible edges of K_n.

$[n] \rightarrow \{1, 2, \ldots, n\}$

$2^{\Gamma} \rightarrow$ power set of Γ

$N \rightarrow {}^{n}C_2 = |\Gamma|$

Exercises:-

a) Lemma 2.1.

b) Claim 2.2.

c) Proposition 2.4

d) Corollary 2.5.

e) Claim 3.2.

f) Remark 3.8.

g) Result 4.2.

h) Result 4.4.

Bibliography :-

1. Random Graphs – Svante Janson, Tomasz Luczak, Andrej Ruciński.

2. Random Graphs – Bella Bollobás.

3. Threshold functions for small subgraphs – Bella Bollobás, 1981, Mathematical Proceedings of the Cambridge Philosophical Society.