# “FAIR AND LOVELY”: SOME THEORETICAL CONSIDERATIONS IN THE EQUITABLE ALLOCATION OF RESOURCES

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## 1. Introduction

How should resources be divided “fairly” or “equitably” among members of a group or society? This is a question that human beings have wrestled with since antiquity and it is one that remains central in the contemporary world. For instance, should there be quotas in private sector jobs or in university student positions for ethnic, religious or caste minorities? If such quotas are ethically justified, then how should we decide on their quantity? How should the tax revenues of the Indian Government be distributed amongst the various States? How should the assets of a bankrupt firm be divided amongst its creditors? How should the stock of kidneys obtained from donors be allocated amongst potential recipients? How should property and assets be divided amongst claimants after death or divorce?

A fundamental aspect of this question is that it is *ethical or normative* in character. We cannot hope to obtain insights into it by analyzing how such decisions are or have been made in practice. It will not be sufficient to examine the consequences on resource allocation of the operation of institutions such as the “market” or “tradition” and “convention” or the existing legal framework. Instead, we need to proceed axiomatically by directly attempting to define what we mean by “equity” and “fairness” and then critically examining the consequences of adopting such a definition. In this essay, I shall briefly review and discuss a large literature on the problem of dividing resources when

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agents differ in their *preferences* over these resources. This severely restricts the scope of the analysis and precludes discussion of several of the issues we I have raised earlier. However, the merit of this approach is that it uses the basic model of exchange in economic theory and can therefore be integrated into standard Welfare Economics. For an extensive introduction into the range and complexity of fairness issues, the reader is referred to the elegant books of Moulin [17] and Young [30].

It is important to emphasize here that our discussion will be confined to the question of identifying and achieving fair *outcomes*. It can be argued very reasonably that fairness is not just a matter of outcomes but of the *procedures* that are used to determine these outcomes. For instance, a procedure where a dictator or a cabal of "wise" men determine what everyone gets can be justifiably regarded as unfair irrespective of the actual allocations obtained. I shall not consider the issue of procedural fairnesss at all; however I shall discuss the formulation of procedures where fair outcomes are achieved in environments where decision making is *decentralized* and agents are *self-interested*. This is an area of game theory known as mechanism design theory. The basic model is as follows. There does not exist a disinterested central agent who has the information required to select a fair outcome because this information is dispersed amongst the agents themselves. The challenge is therefore to devise ways for these agents to communicate their private information and undertake actions so that *equilibrium outcomes* when agents fully recognize their strategic power, are precisely outcomes that are fair.

This essay is organized as follows. In Section 2, I introduce some theoretical considerations underlying fairness. In Section 3, I discuss various definitions for fair outcomes that have been proposed for classical exchange economies. In Section 4, I extended the discussion to a model with indivisible objects. In Section 5, I discuss mechanisms for implementing fair outcomes and Section 6 concludes.

## 2. General Principles

One of the most celebrated principles of the normative theory of justice is Aristotle's equity principle: [1]

*"Equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences" - Aristotle, Nicomachean Ethics.*

There appear to be two different aspects of Aristotle's principle. The first, reflected in the requirement that "equals should be treated equally" is an *anonymity or symmetry* principle across agents. Thus agents who are identical in all respects must be treated identically. An immediate implication is that all kinds of arbitrary or whimsical discrimination is unfair. This seems very reasonable and few would disagree with it. The second requirement, "unequals (should be treated) unequally, in proportion to relevant similarities and differences" is however, fraught with difficulties, both philosophical and practical.

One way to interpret this requirement is to regard each individual as comprising a list of *characteristics*. This list could include the person's physical characteristics, whether he likes fish, is talented at chess, whether he works hard, drives safely, has a criminal record and so on. From this list one would have to identify a set of *relevant* characteristics. One can then further divide the set of relevant characteristics into those one whose account individuals should not suffer adversely or gain undue advantage (let us call this set of relevant characteristics, set $A$) from and those characteristics on the basis of which discrimination is justified or legitimate (let us call this set of relevant characteristics, set $B$). In the allocation of resources across individuals, variations in the characteristics belonging to $A$ must be sought to be equalized. On the other hand, variations across characteristics in set $B$ can be used as the basis for compensating agents differently but these differences

[1] I shall make no effort to review the literature on normative theories of justice, such as those of Rawls [19], Kolm [14] [15], Sen [21] and so on.

in compensation must be *proportionate* to the differences in characteristics.

It is clear that formidable conceptual difficulties are involved here. One of the most fundamental is the way in which we partition the set of relevant characteristics into sets $A$ and $B$. A natural and widely accepted principle is to regard $A$ as the set of *immutable or involuntary* characteristics and the set $B$ as the set of characteristics *over which the agent exercises her decision-making capacity.* Involuntary characteristics definitely include physical characteristics such as gender, race and ethnicity. In India, it would include social characteristics such as caste. It appears to be obvious that agents should not benefit or suffer on account of differences in these characteristics. However the equity principle can be pushed further to argue for instance, that those with physical handicaps should receive further compensation (such as allocations for building special infrastructure appropriate for people with disabilities) than those without such disabilities. Similarly, a person who is poor because of the accident of birth (as a result of which he remains illiterate) can legitimately be treated differently from an individual who is well-off. On the other hand, it may be perfectly legitimate to pay more to people who work harder or to charge higher insurance premia from those who drive recklessly because the decisions to work hard and drive recklessly are decisions for which the individual can reasonably be held responsible for.

There are several characteristics which are hard to classify as belonging to $A$ or $B$. Perhaps the most contentious one is ownership. Should an agent receive more because he owns more? Moulin [17] usefully identifies four principles of fair allocation, as Compensation, Reward, Exogenous Rights and Fitness. In determining the allocation of a good, the principles of Compensation, Reward and Fitness correspond roughly to the questions: who needs the good the most? who deserves it the most? and who will make the best use of it? respectively. Moulin illustrates these principles with reference to a classical story about a flute that must be given to one of four children. One child has the fewest toys; he therefore needs the flute the most and gets it by the Compensation principle. Another child takes the best care of the flute; he deserves it the most and gets it by the Reward principle. The father of the third child owns the flute. This child gets the flute because he can claim a right to it. The fourth child is the most talented flute player and gets it by the Fitness principle. In terms of our earlier discussion, the basis of the four principles, viz. need, effort, ownership (or an exogenous right) and the ability to use an object can be thought of as the relevant characteristics which can be taken into account while determining whether an allocation is fair.

In this essay, I shall only discuss fair division problems of a very simple kind. There are $n$ agents who have divide a given amount of resources. Two different kinds of models will be considered. One will be the classical exchange economy and the other will be a model where a finite set of objects has to be divided amongst the agents. As we shall see, these models will be considered separately because they differ in a significant respect. However, in both models, the only relevant characteristic of the agents will be assumed to their *preferences* over the resources to be divided. It is natural to assume that preference is a characteristic which is exogenous for the agent for which she is not "responsible". Fairness in these settings is therefore the issue of ensuring that agents do not benefit or suffer as a consequence of their preferences. The next two sections discusses ways in which this may be done.

## 3. Exchange Economies

Suppose there are $n$ agents who have to share a quantity of a single infinitely divisble resource, say money. Let us also assume quite reasonably that all agents like more money to less. What is a fair allocation in this case? The answer is quite obvious. Observe that all agents have identical preferences (they all like "more" to "less"), so they are identical with respect to all relevant characteristics. Aristotle's equity principle requires all agents to be treated identically in such a situation

("Equals should be treated equally.."). Hence they must receive a $\frac{1}{n}$ share of the resource.

Now suppose that there are two (infinitely divisible) goods, say bread and water to be divided amongst the $n$ agents. However we now allow for the possibility that agents differ in their valuations of different bundles of bread and water. Suppose for instance, that some agents like "bread more than water" while the others like "water more than bread" [2]. Does fairness still require each agent to get $\frac{1}{n}^{th}$ of the total amount of bread and water available? This no longer seems necessary. Perhaps we could give a little less bread to the agents who like bread less and compensate them with more water while doing exactly the reverse for the other agents and still be fair. But what are the general principles with which we can evaluate such decisions? [3]

A natural criterion for fairness would appear to be the equalization of the well-being of all agents. This seems very attractive but it founders on a major conceptual obstacle. Making this notion operational would require comparisons of the well-being of one agent with that of another. However, well-being or utility in economic theory is an *ordinal* concept; this renders comparisons of well-being across agents meaningless. A utility function is simply a representation of preferences. If agent $i$ prefers a bundle of commodities $x$ to a bundle $y$, her utility function $u$ will have the property that $u(x) > u(y)$. However any monotone transformation of $u$ will also represent the same preferences. For example, we could construct a new utility function $w$ by multiplying the "original" utility of every bundle $x$, $u(x)$ by the number $10^{27}$. The utility function $w$ will represent the same preferences as $u$ in the sense that whenever $u(x) > u(y)$, we will have $w(x) > w(y)$. Now suppose that agents $i$ and $j$ get bundles $x_i$ and $x_j$ which "equalize" their utility, i.e $u_i(x_i) = u_j(x_j)$ where $u_i$ and $u_j$ are the utility functions of $i$ and $j$ respectively. This equalization of utilities is non-robust because new utility functions $w_i$ and $w_j$ for agents $i$ and $j$ could be constructed

[2] I shall be more formal in due course.

[3] A survey of some of these issues can be found in Thomson and Varian [26].

where $w_i = 2u_i$ and $w_j = 5u_j$. These new utility functions represent the same preferences as $u_i$ and $u_j$ but it is not true that $w_i(x_i) = w_j(x_j)$. So the idea of equalizing utilities is completely unworkable.

One way around this impasse was proposed in Foley [11]. [4] An allocation is said to be *envy free* if no agent prefers his allocated share to that of any other agent. The idea is that no agent would like to exchange places with or be in the shoes of any other agent. This notion avoids the difficulties mentioned in the previous paragraph because no interpersonal comaprisons of utility are being made. Instead, the allocation of agent $i$ is being compared with the allocations of all agents $j, j \neq i$ using $i$'s utiliy function. We now make this notions more precise by formally describing the model.

There are $L$ commodities and $n$ agents with $L, n \geq 1$. The set of agents is denoted by $N$ with typical element $i$. The space of commodities is $\Re^L_+$ and elements of this space will be called commodity bundles or simply bundles. Each agent $i$ has a preference ordering $R_i$ defined over $\Re^L_+$. The ordering $R_i$ ranks every pair of bundles $x_i, y_i \in \Re^L_+$. The statement $x_i R_i y_i$ will be interpreted as "$x_i$ is at least as good as $y_i$ according to $R_i$". Since $R_i$ is an ordering, it satisfies the properties of completeness, reflexivity and transitivity. [5] We let $P_i$ and $I_i$ denote respectively the asymmetric and symmetric components of $R_i$. [6] We shall say that an ordering satisfies the *classical assumptions* if it is monotone, continuous and convex. [7]

[4]Young [30] attributes a related concept to Tinbergen [24].

[5]We say $R_i$ is complete if for all bundles $x_i$ and $y_i$, either $x_i R_i y_i$ or $y_i R_i x_i$ holds. We say that $R_i$ is reflexive if for all bundles $x_i$, $x_i R_i x_i$ holds. We say that $R_i$ is transitive if for all bundles $x_i, y_i$ and $z_i$, $x_i R_i y_i$ and $y_i R_i z_i$ implies $x_i R_i z_i$.

[6]We say that $x_i P_i y_i$ if $x_i R_i y_i$ but not $y_i R_i x_i$. In other words, $x_i P_i y_i$ implies that $x_i$ is "strictly preferred to $y_i$ according to $R_i$". We say that $x_i I_i y_i$ if $x_i R_i y_i$ and $y_i R_i x_i$ both hold. Thus $x_i I_i y_i$ implies that $x_i$ and $y_i$ are "indifferent to each other according to $R_i$".

[7]We say that $R_i$ is monotone, if for all bundles $x_i$ and $y_i$ such that $y_i$ is strictly larger in every component relative to $x_i$, we have $y_i P_i x_i$. Thus "more is better". We say that $R_i$ is continuous if for all bundles $x_i$, the sets $\{z_i \in \Re^L_+ | z_i R_i x_i\}$ and

There is an aggregate endowment $\Omega \in \Re^L$ of the $L$ commodities which have to be shared amongst the $n$ agents. We assume that $\Omega >> 0$, i.e. every component of the $L$ dimensional vector $\Omega$ is strictly greater than 0. An allocation $x \equiv (x_1, .., x_N) \in \Re_+^{Ln}$ is an $n$ collection of $\Re_+^L$ dimensional vectors. An allocation is feasible if it satisfies the restriction $\sum_{i \in N} x_i \leq \Omega$. A feasible allocation $x$ is simply a way to divide the aggregate endowment amongst the $n$ agents. Here $x_i$ is the bundle of $L$ commodities allocated to agent $i$.

**Definition 1.** *A feasible allocation $x$ is envy-free if, for all $i, j \in N$, we have $x_i R_i x_j$.*

An allocation is envy-free if no agent prefers the bundle allocated to another agent more than her own. No agent *envies* another agent and would not like to switch places with her.

Does an envy-free feasible allocation exist? Consider the allocation $\underline{0} \equiv (0, .., 0)$ where each agent gets 0. Observe that this allocation is feasible because the aggregate resource constraint is satisfied with strict inequality. Moreover the allocation is envy-free because all agents are getting identical bundles. The answer to the question is thus, yes, albeit trivially. However an allocation of 0 for everyone is clearly unsatisfactory. Goods are being thrown away which could have been used to make all agents better-off. A more appropriate question is whether there exist envy-free allocations which are also efficient.

**Definition 2.** *A feasible allocation $x$ is Pareto-efficient (or simply efficient) if there does not exist another feasible allocation $y$ such that $y_i P_i x_i$ for all $i \in N$.*

A feasible allocation is efficient if it is not possible to make all agents better-off by a reallocation of resources. It is well-known that under classical assumptions on preferences, this definition of efficiency is

$\{z_i \in \Re_+^L | x_i R_i z_i\}$ are closed. Finally, $R_i$ is convex if for all bundles $x_i, y_i$ and $z_i$ and $\lambda \in (0,1)$, $y_i R_i x_i$ and $z_i R_i x_i$ implies $(\lambda y_i + (1-\lambda) z_i) R_i x_i$.

equivalent to the one where it is not possible to make at least a single agent strictly better-off with all agents remaing at least as well-off as before. Clearly an efficient allocation cannot involve wastage of resources. It is also well-known that it imposes additional restrictions (i.e. more than non-wastage) on allocations.

We will demonstrate the existence of envy-free and efficient feasible allocations by explicit construction. A critical notion is that of a *competitive equilibrium allocation.* An endowment vector $\omega \equiv (\omega_1, ..., \omega_N)$ is a vector in $\Re_+^{Ln}$ satisfying the restriction $\sum_{i \in N} \omega_i = \Omega$.

**Definition 3.** *A feasible allocation $x^*$ is a competitive equilibrium allocation with respect to the endowment vector $\omega$ if there exists a vector $p \in \Re_+^L$ (called a price vector) such that for all $i \in N$*
*(i) $p.x_i^* = p.\omega_i$ and*
*(ii) for all $x_i \in \Re_+^L$, if $x_i P_i x_i^*$, then $p.x_i > p.\omega_i$.*

A feasible allocation $x$ is a competitive equilibrium allocation if his bundle $x_i$ maximizes his preference ordering $R_i$ over all bundles $x_i$ which he can afford to purchase with income $p.\omega_i$.

**Remark 1.** If preferences satisfy the classical assumptions, a competitive equilibrium allocation always exists. Moreover the competitive equilibrium allocation is efficient. These are standard results in general equilibrium theory - details can be found in Mas-Colell, Whinston and Green [16].

We now state an important existence result.

**Proposition 1.** *Assume that preferences satisfy the classical assumptions. An envy-free and efficient feasible allocation always exists.*

**Proof**: Let $x^*$ be a competitive equilibrium allocation with respect to the endowment vector $(\frac{\Omega}{n}, .., \frac{\Omega}{n})$. It follows from Remark 1 that $x^*$ exists and is efficient. It only remains to show that $x^*$ is envy-free. Suppose to the contrary that it violates envy-freeness. Then there exists $i$ and $j$ such that $x_j^* P_i x_j^*$. Since $x^*$ is a competitive equilibrium allocation with respect to $(\frac{\Omega}{n}, ..\frac{\Omega}{n})$, we must have $p.x_i^* = p.x_j^* = p.\frac{\Omega}{n}$. However since $x^*$ is a competitive equilibrium allocation, $x_j^* P_i x_i^*$ also implies $p.x_j^* > p.\frac{\Omega}{n}$ whch contradicts our earlier conclusion that $p.x_j^* = p.\frac{\Omega}{n}$. Hence $x^*$ is envy-free.

□

The proof of Proposition 1 demonstrates that the competitive equilibrium allocation from equal division of the aggregate endowment, is efficient and envy-free. Its envy-freeness follows from the property that all agents face identical budget sets, so that they can afford the bundles chosen by all other agents. Observe that the equal division of the aggregate endowment is envy-free but not efficient in general.

We assumed that preferences satisfy classical assumptions in order to demonstrate the existence of efficient and envy-free allocations. However these allocations can be shown to exist under weaker assumptions (see, for instance, Svensson [23] and Varian [28]). In general there are many efficient and envy-free allocations as shown diagrammatically in Young [30], Appendix A.8 and Thomson and Varian [26]. How should one choose the best amongst these allocations? Is there an allocation which is "fairest amongst them all"? One approach is that of Baumol [2] who proposed an iterative procedure which selects an envy-free allocation. Unfortunately the procedure cannot ensure that the allocation selected is also efficient. Recently, Domínguez and Nicolò [8] propose an iterative procedure which successfully deals with this issue, at least in the case of two agents. Their idea is to define the *minimal rights* of each agent as the minimal amounts of commodities she receives at an envy-free and efficient allocation. Once minimal rights have been assigned to each agent, residual resources can be assigned by the same procedure in the reduced economy and so on. The authors demonstrate this procedure leads to the selection of an envy-free and efficient allocation.

Although the set of envy-free and efficient allocations may be large, there is an important sense in which the competitive equilibrium allocation from equal division, is salient. Varian [29] shows that in a economy with a continuum of agents where utility functions satisfy certain regularity properties, an allocation which satisfies a certain regularity property, efficiency and envy-freeness must be a competitive equilibrium allocation from equal division. An interpretation of this result is that if preferences are sufficiently "diverse", then the set of efficient and envy-free allocations is "approximately" the set of competitive equilibrium allocations from equal division.

There are several alternatives to envy-freeness that have been proposed. One of these is the notion of *egalitarian equivalence* introduced by Pazner and Schmeidler [18].

**Definition 4.** *A feasible allocation* $x$ *is egalitarian equivalent if there exists some reference bundle* $z_0 \in \Re^L_+$ *such that* $x_i I_i z_0$ *for all* $i \in N$.

An allocation is egalitarian equivalent if all agents are indifferent between it and some reference bundle that is common to all agents. Thus agents' well-being are being equalized in the sense that they are all indifferent to a common reference bundle. Like envy-freeness, egalitarian equivalence is an ordinal notion because no interpresonal comparisons of utility are made. Pazner and Schmeidler [18] demonstrate that efficient egalitarian equivalent allocations exist under very general conditions.

To see this, consider the following line of reasoning. Fix an economy with an endowment vector and utility functions for each agent. We can define a *utility possibility set* for the economy which consists of utility vectors (each component of such a vector denotes the utility of an agent) which are *feasible* for this economy. The "north-east" frontier of this set is the set of utility vectors which are efficient, i.e. the utility of an agent cannot be increased without diminishing that of another agent. If preferences are classical, this frontier is continuous, i.e cannot have "holes" or "jumps". Consider an *arbitrary* commmodity bundle $z_0$ with strictly positive amounts of all goods. The following two observations are critical (i) for $\bar{r}$ positive and large enough, the utility vector generated by the commmodity bundle vector $(\bar{r}z_0, ...., \bar{r}z_0)$ lies outside the utility possibility set (ii) for $\hat{r}$ positive and large enough, the utility vector generated by the commmodity bundle vector $(\hat{r}z_0, ...., \hat{r}z_0)$ lies inside the utility possibility set. Now trace the path in utility space obtained by varying $r$ from $\hat{r}$ to $\bar{r}$. Under classical assumptions, this path is continuous. Since it starts from within the utility possibility set and terminates at a point outside it, it *must* intersect the frontier somewhere. Let $\bar{x}$ be the allocation in the economy which corresponds to this point of intersection. Since it lies on the frontier of the utility possibility set, it is efficient by definition. In addition each agent at $\bar{x}$ is indifferent to some commodity bundle vector $(r^*z_0, .., r^*z_0)$. Hence, $\bar{x}$ is an efficient and egalitarian equivalent allocation. I present this argument a little more formally below.

**Proposition 2.** *Assume that preferences satisfy the classical assumptions. Let $z_0 \in \Re^L$ be such that $z_0 >> 0$. Then there exists a strictly positive real number $\bar{r}$ such that an efficient and egalitarian equivalent allocation exists with $\bar{r}z_0$ as the reference bundle. Moreover, for every such reference bundle, there is at most one efficient allocation.*

**Proof**: I shall give a proof in the case of $n = 2$ and the argument is essentially the same for the general case. Consider arbitrary utility representations $u_i$ of the preferences $R_i$, i.e. for all bundles $x_i, y_i \in \Re^L_+$, $u_i(x_i) \geq u_i(y_i)$ if and only if $x_i R_i y_i$. Let $\mathcal{U} = \{(a_1, a_2)$ there exists a feasible allocation $(x_i, x_2)$ such that $u_i(x_i) = a_i, i = 1, 2\}$. Thus $\mathcal{U}$ is the utility possibility set or the set of utilities which can be generated by allocating the aggregate endowment $\Omega$ to the two agents. Since preferences are monotone increasing and continuous, the frontier of this set must be downward sloping and continuous. Moreover the set $\mathcal{U}$ is closed, i.e. it contains its boundary.

Now consider the point $z_0$ in commodity space and let $r_0$ be a positive integer small enough so that $r_0 z_0 \leq \Omega$. This is always possible because $\Omega >> 0$. Consider the utility vector $\alpha$ generated by the allocation

$(\frac{r_0 z_0}{2}, \frac{r_0 z_0}{2})$. Since this allocation is feasible, $\alpha \in \mathcal{U}$. Let $r_1$ be a positive integer such that $r_1 z_0 > \Omega$. We can find such an $r_1$ because $z_0 >> 0$. Let $\beta$ be the utility vector generated by the allocation $(\frac{r_1 z_0}{2}, \frac{r_1 z_0}{2})$. Clearly $\beta \notin \mathcal{U}$ and $\beta > 0$ since preferences are monotone. Consider the path in utility space obtained by giving the two agents the same commodity bundle $\frac{r z_0}{2}$ as $r$ varies in the interval $[r_0, r_1]$. This path originates at $\alpha$ inside the set $\mathcal{U}$ and terminates at $\beta$ outside the set $\mathcal{U}$. Since preferences are continuous, this path is continuous. Hence, it must intersect the boundary of $\mathcal{U}$. Moreover, since the boundary of $\mathcal{U}$ is downward sloping and the path is upward sloping (due to the monotonicity of preferences), this intersection can occur only once. Suppose $\gamma \in \mathcal{U}$ is the point on the frontier of $\mathcal{U}$ and $\bar{r}$, the appropriate value of $r$ where the intersection occurs. Let $\bar{x}$ be the allocation such that $u_i(x_i) = \gamma_i$ with $i = 1, 2$ (such an allocation must exist since $\gamma \in \mathcal{U}$). Note that $\bar{x}$ is efficient and egalitarian equivalent with respect to the reference bundle $\bar{r} z_0$.

□

It is important to appreciate that the reference bundle $\bar{r} z_0$ may not be feasible. For instance suppose $z_0 = (\frac{\Omega}{2}, \frac{\Omega}{2})$. Typically, this allocation will not be efficient, i.e. the associated utility vector will lie strictly in the interior of the utility possibility set. In order to find the reference bundle $\bar{r} z_0$ we have to move along the ray (in commodity space) joining the origin to the point $(\frac{\Omega}{2}, \frac{\Omega}{2})$. Since the latter point is not efficient, we have to move further north-east along this ray which will imply that the reference bundle is not feasible.

According to Proposition 2, an arbitrary consumption bundle can be used to generate an efficient and egalitarian equivalent allocation. It follows that the set of efficient and egalitarian efficient allocations (like the set of efficient and envy-free allocations) can be large. Pazner and Schmeidler [18] show that efficient and egalitarian equivalent allocations exist quite generally in production economies where sometimes efficient and envy-free allocations do not.

We now turn to some notions of equity formulated by Thomson [25] based on the notion of *opportunities*. The fundamental idea is that an allocation is equitable if all agents face equal or equivalent opportunities.

A natural condition following Kolm [14] is to require all agents to maximize their preferences with respect to the *same* budget or opportunity set. An allocation obtained in this manner can be considered as equitable since differences in the bundles obtained by various agents can be attributed solely to differences in tastes (i.e. preferences). The difficulty with this idea (and indeed, with the entire approach) is that the opportunity sets cannot be determined independently of preferences. To see this, suppose that all agents faced the same budget set $\Re^L_+$ irrespective of their preferences. Then one could always find preferences for each agent such that maximizing choices do not constitute an allocation (there will be excess demand for some commodity).

Thomson [25] deals with this issue by postulating a *family* of budget sets $\mathcal{B}$ where each $B \in \mathcal{B}$ is a non-empty subset of $\Re^L_+$. It is convenient to think of $\mathcal{B}$ as the collection of common budget sets to be given to the agents for different preferences of the agents.

**Definition 5.** *The feasible allocation $x$ is an equal opportunity allocation relative to the family $\mathcal{B}$ if there exists $B \in \mathcal{B}$ such that for each $i \in N$, $x_i$ maximizes $R_i$ in $B$.*

In an equal opportunity allocation (with respect to a family $\mathcal{B}$) all agents are maximizing their preferences over a common budget set. The bundles chosen by all other agents are therefore feasible for any particular agent. An immediate consequence of this observation is that an equal opportunity allocation is envy-free. Moreover, if $\frac{\Omega}{n} \in B$ for all $B \in \mathcal{B}$, then the equal opportunity allocation with respect to $\mathcal{B}$, say $x$ will have the property that $x_i R_i \frac{\Omega}{n}$ for all $i \in N$.

Is it possible to select the family $\mathcal{B}$ in order that equal opportunity allocations exist and are efficient? The answer is yes under classical assumptions on preferences. If these assumptions are satisfied, then competitive allocations exist (Remark 1). Then $\mathcal{B}$ can be defined to be the collection of linear budget sets obtained from the equal division of the endowment bundle, $\frac{\Omega}{n}$, for all competitive equilibrium price vectors as the preferences of agents vary. This family $\mathcal{B}$ will generate the competitive equilibrium allocation from equal division of the endowment which is efficient. Thomson [25] provides additional axioms under which equal opportunity allocations *coincide* with the competitive equilibrium allocations from the equal division of the endowment. He also demonstrates the existence of other families $\mathcal{B}$ which generate different solutions when these additional axioms do not hold.

Thomson introduces two other notions of equitable allocations based on opportunities.

**Definition 6.** *The feasible allocation $x$ is equal-opportunity equivalent relative to the family $\mathcal{B}$ if there exists $B \in \mathcal{B}$ such that for each $i \in N$, $x_i$ is indifferent to the maximizer of $R_i$ on $B$.*

This notion is clearly inspired by egalitarian equivalence. Each agent $i$ is indifferent between his bundle $x_i$ and the bundle that he would obtain by maximizing his preferences over a common budget set $B$ in the family $\mathcal{B}$. If $\mathcal{B} = \{\{z_0\} | z_0 \in \Re^L_+\}$, then equal opportunity equivalence reduces to egalitarian equivalence. If $\mathcal{B}$ is chosen to be set of linear budget sets obtained from equal division of the endowment using competitive equilibrium prices (as described previously), then equal opportunity equivalence reduces (like equal opportunity equivalence allocations) to the competitive equilibrium allocation from equal division of the endowment (Thomson [25] Lemma 2).

**Definition 7.** *The feasible allocation $x$ exhibits no-envy of opportunities relative to the family $\mathcal{B}$ if for each $i \in N$, there exists $B_i \in \mathcal{B}$ such that $x_i$ maximizes $R_i$ on $B_i$ and there does not exist $i, j \in N$ such that $i$ strictly prefers some bundle in $B_j$ to $x_i$.*

If an allocation exhibits no-envy of opportunities, agents can have different opportunity sets but no agent envies the opportunity set of another agent. Since we can always pick $B_1 = B_2 = .. = B_N$, it follows that if an allocation is an equal opportunity allocation with respect to a family $\mathcal{B}$, then it exhibits no-envy of opportunities with respect to the same family. Moreover if an allocation exhibits no-envy with respect to some family $\mathcal{B}$, then it must be envy-free. By choosing $\mathcal{B} = \{\{z_o\}|z_o \in \Re^L_+\}$, it follows that every envy-free allocation exhibits no-envy of opportunities relative to $\mathcal{B}$.

Thomson investigates the interrelationships between these concepts and those of envy-freeness and egalitarian equivalence in a variety of settings including those with production and public goods. Interested readers may consult his paper for details.

## 4. Indivisible Objects

In this section I consider the problem of equitably allocating a finite number of indivisible goods amongst agents. Once again, I shall assume that agents differ only in respect of their *preferences* over these goods. In other words, preferences are the only relevant characteristic of agents. I shall assume for simplicity that there are exactly $n$ goods to be divided amongst $n$ agents. The set of goods is denoted by $A = \{a_1, ..., a_n\}$ and the set of agents $N = \{1, .., n\}$ as before. Each agent $i$ requires only one good and likes any good more than getting nothing. Her preferences are characterized by an ordering $R_i$ over the elements of $A$. It is assumed further that $R_i$ is *antisymmetric* [8] which implies that indifference between distinct elements of $A$ is ruled out. This assumption is quite natural in an environment with a finite number of goods. However, in exchange economies it is restrictive and is incompatible with classical assumptions on preferences [9] which is why it was not assumed in the previous section.

An allocation $x$ is an assignment of objects to agents. It is an $n$-dimensional vector whose $i^{th}$ component $x_i$ is the object assigned to agent $i$. The possibility that an agent does not get any object will never

[8] We say that $R_i$ is antisymmetric if for all $a_j, a_k \in A$, $a_j R_i a_k$ and $a_k R_i a_j$ imply $a_j = a_k$.

[9] It is not compatible with continuity of preferences.

be considered [10] so that $x_i \in A$ for all $i \in N$. A feasible allocation is an allocation $x$ with the property that $x_i \neq x_j$ for all $i, j \in N$. This restriction ensures that the same object is not assigned to two agents.

The definition of envy-freeness introduced earlier carries over to this setting without any modifications. Thus, a feasible allocation $x$ is envy-free, if $x_i R_i x_j$ for all $i, j \in N$. Since $R_i$ is anti-symmetric, a feasible allocation is envy-free if the object received by agent $j$ is strictly worse than the object received by agent $i$ according to $i$'s preferences $R_i$. The notion of efficiency, however requires strengthening.

**Definition 8.** *The feasible allocation $x$ is strongly (Pareto) efficient if there does not exist another feasible allocation $y$ such that $y_i R_i x_i$ for all $i \in N$ and $y_j P_j x_j$ for some $j \in N$.*

If a feasible allocation is strongly efficient, it is efficient. In exchange economies, the converse is also true so that the two notions coincide. In finite good economies however, these notions are distinct as the following example demonstrates.

**Example 1.** Let $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3\}$. Preferences are as follows: $a_1 P_1 a_2 P_1 a_3$, $a_2 P_2 a_1 P_2 a_3$ and $a_3 P_1 a_1 P_1 a_2$. The feasible allocation $(a_3, a_1, a_2)$ [11] is not efficient because $(a_1, a_2, a_3)$ is a feasible allocation where $a_1 P_1 a_3$, $a_2 P_2 a_1$ and $a_3 P_3 a_2$. The feasible allocation $(a_2, a_1, a_3)$ is efficient because agent 3 cannot be made better-off relative to $a_3$. However it is not strongly efficient because $(a_1, a_2, a_3)$ is a feasible allocation where agents 1 and 2 are strictly better-off while 3 is no worse-off relative to $(a_2, a_1, a_3)$.

Do envy-free and strongly efficient allocations exist? Unfortunately, no and this is the reason why the indivisible goods model is different from that of the exchange economy. In fact envy-free allocations may not exist. To see this let $N = \{1, 2\}$ and $A = \{a_1, a_2\}$. Suppose

[10] There is no loss of generality here because we shall always be concerned with efficient allocations.

[11] Here $(a_3, a_1, a_2)$ is the allocation where agents 1, 2 and 3 get $a_3$, $a_1$ and $a_2$ respectively.

preferences $(P_1, P_2)$ are such that $a_1 P_1 a_2$ and $a_1 P_2 a_2$. For instance, an estate consisting of two paintings have to be divided between two heirs and both consider the painting $a_1$ to be more valuable than the other $a_2$. Or to consider a situation from Test Cricket, let $a_1$ and $a_2$ denote respectively the "good" which is the right to bat first and second respectively on a flat wicket that is expected to crumble on the last day. Both teams will prefer $a_1$ to $a_2$. Now, whatever method is used for allocating $a_1$ and $a_2$, the agent/team which gets $a_2$ will envy the other agent/team. Clearly, an envy-free allocation does not exist. Non-existence is a consequence of the "lumpiness" of the goods being divided. If they could be cut "finely", one could hope to divide them in manner satisfactory to both agents.

There are several ways to proceed from this point. In the case of the warring heirs, it seems to natural to introduce monetary side-payments. Thus one might allow one of the agents to take the more desirable object but to pay some money to the other agent. If monetary transfers are permitted, then we have an exchange economy model once again where efficient and envy-free allocations exist under weak assumptions. Another approach is to acknowledge that the problem cannot be solved satisfactorily and to try to solve it "as best as possible". This might involve, for instance, finding allocations which minimize the number of envious agents, or perhaps minimizing the extent of envy of the most envious agent. One way to operationalize the latter idea would be the following. The envy of an agent associated with an allocation could be the ordinal rank of the object received by the agent in her preference ordering. Thus an agent who receives an object which is $10^{th}$ ranked has an envy of "10". This makes sense because some agent is getting her first ranked object. The envy associated with an allocation could then be the maximum of envies experienced by various agents. The most equitable allocation would then be one which minimizes this maximal envy over all possible allocations. This procedure is equivalent to maximizing a Rawlsian welfare function where the utility of agent $i$

getting good $a_j$ is taken to be the negative of the ordinal ranking of $a_j$ according to $R_i$.

A third approach is something which Young [30] terms "Rotation". Consider a round-robin [12] chess tournament with $n$ players. It is well-known both statistically and theoretically that in games between top-level players, playing with the white pieces is significantly more advantageous than playing with black. In every individual game therefore, one player envies the other. However if $n$ is odd, it is possible to arrange the tournament schedule in a manner such that each player plays an equal number of games with white and black pieces. If $n$ is even, it can be ensured that for each player, the difference between white and black games is one. Thus, even though envy exists in every game, a version of envy-freeness in the aggregate can be ensured.

Instead of developing the three approaches described above further, I shall focus on a fourth method which is the method used in cricket, viz. coin tossing and randomization. This is a well-known way to resolve (minor!) conflicts of interest. The idea here is that it may be possible to achieve envy-freeness *ex-ante*, i.e. before the coin toss although there may be envy *ex-post*, i.e. after the realization of the coin toss. Ex-ante envy-freeness is possible because agents' prospects before the coin toss are identical.

Recall that a feasible allocation $x$ is an $n$-dimensional vector with $x_i \in A$ for all $i \in N$ and $x_i \neq x_j$ for all $i, j \in N$. Let $X$ denote the set of allocations. Since $X$ is the set of all one to one maps from the set $\{1, .., n\}$ to the set $\{a_1, .., a_n\}$, it follows that $|X| = n!$. Let the members of the set $X$ be enumerated in some way, $x^1, ..., x^{n!}$. A *randomized allocation* consists of a collection of $n!$ real numbers $\lambda \equiv (\lambda^1, ..., \lambda^{n!})$ such that $\lambda^j \geq 0$ for all $j = 1, .., n!$ and $\sum_{j=1}^{n!} \lambda^j = 1$. A randomized allocation is a probability distribution over $X$. Thus $\lambda^j$ is the probability with which allocation $x^j$, $j = 1, .., n!$ is chosen. For example, let $N = \{1, 2\}$ and $A = \{a_1, a_2\}$. The set $X$ consists of two

[12]A round-robin tournament is one where each player plays all other players once.

allocations $x^1 = (a_1, a_2)$ and $x^2 = (a_2, a_1)$. A randomized allocation is a pair $(\lambda^1, \lambda^2)$ with $\lambda^1, \lambda^2 \geq 0$ and $\lambda^1 + \lambda^2 = 1$ with the following interpretation: $\lambda^j$, $j = 1, 2$ is the probability with which allocation $j$ is picked. In the cricket example, a fair coin is used so that $\lambda^1 = \lambda^2 = 0.5$ so that the probability with which each team gets assigned $a_1$ (or gets to bat first) is 0.5.

The notions of envy-freeness and efficiency must now be defined with respect to randomized allocations. A natural requirement of efficiency in the context of randomized allocations is that of *ex-post efficiency.* [13]

**Definition 9.** *The randomized allocation $\lambda$ is ex-post efficient if $\lambda^j > 0$ implies $x^j$ is strongly efficient.*

A randomized allocation is ex-post efficient if every allocation which has a strictly positive probability of being chosen, is strongly efficient. Equivalently, every inefficient feasible allocation has zero probability of being chosen.

The appropriate definition of envy-freeness requires some discussion. Since agents are faced with lotteries over objects, some way of evaluating such lotteries must be incorporated into the model. The natural assumption is to assume that agents use the expected utility criterion of von-Neumann and Morgenstern but there is a further difficulty. Agents' preferences are specified in terms of an ordinal ranking over all possible objects rather than in terms of a vN-M utility function. I shall once again, follow the approach of Gibbard [12] and require every agent not to envy the lottery of any other agent for *any* utilty function which represents his preferences.

**Definition 10.** *The utility function $u : A \rightarrow \Re$ represents the preference ordering $R_i$ if, for all $a_j, a_k \in A$, $u(a_j) > u(a_k)$ if and only if $a_j P_i a_k$.*

**Definition 11.** *The randomized allocation $\lambda \equiv (\lambda^1, ..., \lambda^{n!})$ is ordinally envy-free if for all $i, j \in N$ and all utility functions $u$ which represent $R_i$, we have*

[13] See Gibbard [12].

$$\sum_{k=1}^{n!} u(x_i^k)\lambda^k \geq \sum_{k=1}^{n!} u(x_j^k)\lambda^k$$

Consider the randomized allocation $\lambda$. Then with probability $\lambda^k$, the allocation $x^k$ is realized $k = 1, .., n!$. Agents $i$ and $j$ therefore faced with objects $x_i^k$ and $x_j^k$ respectively with probability $\lambda^k$. If agent $i$'s preferences were represented by the utility function $u$, then $i$'s expected utility evaluation of her lottery over objects would be $\sum_{k=1}^{n!} u(x_i^k)\lambda^k$ while that of agent $k$'s lottery would be $\sum_{k=1}^{n!} u(x_j^k)\lambda^k$. Envy-freeness would require the former expression to be larger than the latter. Ordinal envy-freeness requires this inequality to hold for *all* utility functions which represent $R_i$. It is a demanding requirement in the sense that it guarantees the ex-ante absence of envy, independently of the cardinal representation that agents may have of their ordinal preferences.

There is an alternative way to define ordinal envy-freeness in terms of stochastic dominance. For all $a_t \in A$ and $R_i$ let $B(a_t, R_i) = \{a_r \in A | a_r R_i a_t\}$. Thus $B(a_t, R_i)$ consists of objects which are regarded at least as good as $a_t$ according to $R_i$. Note that this set includes $a_t$. For any randomized allocation $\lambda$ and $a_t \in A$, $j \in N$ and $R_i$ let

$$\lambda_j(B(a_t, R_i)) = \sum_{\{k \in \{1,..,n!\} | x_j^k \in B(a_j, R_i)\}} \lambda^k$$

Thus $\lambda_j(B(a_t, R_i))$ is the probability of agent $j$ getting an object in the randomized allocation $\lambda$, that is at least as good as object $a_t$ according to the ordering $R_i$.

**Definition 12.** *The randomized allocation is stochastically envy-free if for all $i, j \in N$ and $a_t \in A$, we have $\lambda_i(B(a_t, R_i)) \geq \lambda_j(B(a_t, R_i))$.*

Consider the following example. Suppose $A = \{a_1, a_2, a_3\}$ and let $i$ and $j$ be arbitrary agents. Assume $a_1 P_i a_2 P_i a_3$ and consider the randomized allocation $\lambda$. If $\lambda$ is stochastically envy free, then the following must hold:

(i) The probability that $i$ receives $a_1$ must be at least as great as the probability that $j$ gets it.

(ii) The probability that $i$ receives either $a_1$ or $a_2$ is at least as the probability that $j$ gets either of the two objects.

The condition also requires the probability of $i$ getting at least $a_1$ or $a_2$ or $a_3$ to be at least as the probability of $j$ getting these objects but this always holds trivially with both probabilities being equal to one. Observe that the definition of stochastic envy-freeness does not involve any expected utility calculation and thereby sidesteps the issue of utility representation of preferences entirely.

The following can be shown quite easily.

**Proposition 3.** *A randomized allocation is ordinally envy-free if and only if it is stochastically envy-free.*

Returning to the example with three players and three objects, suppose that $i$'s preferences over objects is $a_1 P_i a_2 P_i a_3$ (as before) and the randomized allocation $\lambda$ is such that $i$'s lottery over objects is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (i.e. all three objects with equal probability) while $j$'s lottery is $(\frac{2}{5}, 0, \frac{3}{5})$ (i.e. $a_1, a_2$ and $a_3$ with probabilities $\frac{2}{5}$, 0 and $\frac{3}{5}$ respectively). Then $\lambda$ is not envy-free because if $i$'s has a utilty function where $a_1$ has utility 1, $a_2$ has utility almost 0 and $a_3$ has utility 0, $i$'s expected utility from the utility is approximately $\frac{1}{3}$ while his expected utility from $j$'s lottery is approximately $\frac{2}{5}$ which leads $i$ to envy $j$. Suppose that $k$'s lottery were instead $(\frac{1}{5}, \frac{3}{5}, \frac{1}{5})$ so that sum of the probabilities of getting $a_1$ and $a_2$ for $j$ is greater than the same probability for $i$. Here $i$ will again envy $j$ for utility functions where the utility of $a_2$ is "almost equal" to the utility of $a_1$. This example demonstrates why ordinal envy-freeness implies stochastic envy-freeness. The converse implication is also easy to establish.

Consider the case where $N = \{1, 2\}$ and $A = \{a_1, a_2\}$. If the agents have different preferences [14] then there is a unique ex-post efficient deterministic allocation which involves giving each agent her most preferred object. Every randomized ex-post efficient allocation must therefore be this deterministic allocation. It is also ordinally envy-free because each agent is getting her most preferred object. The more interesting case arises where the two agents have the same preferences,

[14] Note that only two preferences are possible here, either $a_1$ is preferred to $a_2$ or its reverse.

say $a_1 P_i a_2$ for $i = 1, 2$. We have seen earlier that every deterministic allocation in this situation generates envy. Now consider the randomized allocation $\lambda$ which places probability $\lambda^1$ and $\lambda^2$ on the allocations $(a_1, a_2)$ and $(a_2, a_1)$ respectively (of course, $\lambda^1 + \lambda^2 = 1$). In order for 1 not to envy 2, we must have $\lambda^1 \geq \lambda^2$ while $\lambda^2 \geq \lambda^1$ must hold in order for 2 not to envy 1. Hence $\lambda = (0.5, 0.5)$ is the unique ordinally envy-free and ex-post efficient randomized allocation.

Let us now try and extend these ideas to the case where there the number of agents and objects is arbitrary. However in order to do that, a digression into deterministic allocation procedures is required.

Two common deterministic allocation methods for a finite number of objects, are the *Priority Method* and the *Top Trading Cycle Method*. [15] In the Priority Method (or PM), all agents are ordered in a queue which we call a priority. The first agent in the queue gets her best object; the second agent then gets his best object amongst the remaining objects and so on. In general, the $r^{th}$ agent in the queue gets his best object amongst the $n - r + 1$ unassigned objects. The last agent gets the object no one else has chosen.

In the Top Trading Cycle Method (or TTCM), agents start with an arbitrary initial assignment of objects to agents with each agent getting exactly one object. Each agent then points to the agent (which could be herself) who has her most preferred object. Since the number of agents and objects are finite, there must exist at least one cycle, i.e. a set of agents $i_1, i_2, .., i_K$ where $i_1$ points at $i_2$, $i_2$ points to $i_3$ and so on and $i_K$ points to $i_1$. Objects are then traded along these cycles which means that $i_1$ gets $i_2$'s object, $i_2$ gets $i_3$'s object and so on and $i_K$ gets $i_1$'s object. Agents in these cycles now withdraw with the objects they have been assigned. The process is now repeated with the remaining agents and objects. Since there must be a cycle at every stage, the number of agents who withdraw at every stage must be strictly positive. This

[15] Young [30] Chapter 2 and Moulin [17] contain extensive discussions of these methods.

ensures that the algorithm terminates after a finite number of stages or rounds.

Suppose $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3\}$. Preferences are as follows: $a_1 P_1 a_2 P_1 a_3$, $a_1 P_2 a_3 P_1 a_2$ and $a_1 P_3 a_2 P_3 a_3$. Consider a PM where the queue order is 1 followed by 2 followed by 3. Then 1 will get $a_1$, 2 will get $a_3$ while 3 will get $a_2$, i.e the allocation is $(a_1, a_3, a_2)$. Consider on the other hand, the TTCM where the initial assignment is $(a_2, a_1, a_3)$. Then, in the first round, 2 will get $a_1$, while in the second round 1 will get $a_2$ so that 3 will get $a_3$; hence the final allocation is $(a_2, a_1, a_3)$.

What is the relationship between the allocations generated by these procedures. The next proposition provides a partial answer.

**Proposition 4.** *Fix the preferences of all agents* $(P_1, ..., P_n)$*. Let* $x$ *be the allocation generated by PM for a particular priority. Then there exists an initial assignment of objects such that the TTCM with respect to this initial assignment is* $x$*. Conversely, let* $x'$ *be an allocation generated by the TTCM for a particular initial assignment of objects. Then there exists a priority such that the PM with respect to this priority is exactly* $x'$*.*

**Proof**: Suppose $x$ is the allocation generated by PM for some priority. Assume w.l.o.g that $x = (a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)})$ where $\sigma : N \to N$ is a one to one function. Thus agent $i$ agets object $a_{\sigma(i)}$, for $i = 1, .., n$. Now consider the TTCM from the initial assignment $(a_{\sigma(1)}, a_{\sigma(2)}, .., a_{\sigma(n)})$ and let denote $x'$ this allocation. Let $N_1$ be the set of agents who are getting their most preferred objects in $x$. Clearly $N_1$ includes the first agent in the priority. Observe that all these players will point to themseleves in the first stage of the TTCM and withdraw with these objects. Hence $x'_i = a_{\sigma(i)}$ for all $i \in N_1$. Repeating this argument with the remaining agents and object, we can conclude that $x'_i = a_\sigma(i)$ for all $i \in N$, i.e. $x = x'$.

Now let $x'$ be an allocation from a TTCM from some given initial assignment. Let $N_1$ be the set of agents who withdraw after the first round after getting their best objects. Generally let $N_k$, $k = 1, , .K$ be the set of agents who withdraw in round $k$. Now consider a priority where agents in $N_1$ precede those in $N_2$ who precede those in $N_3$ and so on. It is easy to verify that the PM with this priority will generate $x'$. □

According to Proposition 4 every PM for a given priority is equivalent to a TTCM for *some* initial distribution of objects and vice-versa. However, for a fixed priority and a given initial distribution of objects there exists preferences for each agent such that the PM and TTCM lead to different allocations. To see this, consider the case where $N = \{1, 2\}$ and $A = \{a_1, a_2\}$. Fix the priority, 1 followed by 2. In the case where preferences are $a_1 P_i a_2$, $i = 1, 2$, (let us call this profile (I)), PM leads to the allocation $(a_1, a_2)$ while in the case where preferences are $a_2 P_i a_1$, $i = 1, 2$, (let us call this profile (II)), this leads to the allocation $(a_2, a_1)$. Suppose the intial distribution is of objects is $(a_1, a_2)$. Then TTCM in both profiles (I) and (II) lead to the allocation $(a_1, a_2)$. If, on the other hand, the initial distribution is $(a_2, a_1)$, then the TTCM leads to the allocation $(a_2, a_1)$ in both profiles (I) and (II).

The PM and the TTCM also have important efficiency properties.

**Proposition 5.** *Fix preferences for all agents. For every priority, the PM leads to a strongly efficient allocation. Conversely, for every strongly efficient allocation $x$, there exists a priority for which PM leads to $x$. Analogous properties hold for the TTCM.*

**Proof**: Assume contrariwise that there exists a priority for which the PM leads to an allocation $x$ which is not strongly efficient. Hence there exists an allocation $y$ and a non-empty subset of agents, say $N_0$ such that all agents in this subset are strictly better-off in $y$ relative to $x$ and all agents in $N - N_0$ are no worse-off in $y$ relative to $x$. Observe that the first person in the queue, say agent 1 is getting her best object in $x$. Hence $1 \notin N_0$ and $x_1 = y_1$. But the second person in the queue, say 2 is getting her best object given that 1 is getting object $x_1$. Hence

$2 \notin N_0$ and $x_2 = y_2$. Proceeding in this manner, it follows that $x = y$ and $N_0 = \emptyset$, a contradiction.

Consider an arbitrary strongly efficient allocation $x$. We claim that $x$ satisfies the following property: fix a subset of agents, $N_0$ and consider the set of objects, say $A_0$ not allocated to $N_0$ under $x$, i.e. $A_0 = A - \cup_{j \in N_0} x_j$. Then there exists an agent $j \in N - N_0$ such that $x_j$ is $j$'s best object in $A_0$. Suppose that this property does not hold for some $N_0$. Then, none of the agents in $N - N_0$ are getting their best objects in $A_0$. Now let each agent in $N - N_0$ point to the agent in $N - N_0$ who has, according to $x$, her best object in $A_0$. Given finiteness, a cycle must exist. Let objects be reallocated from $x$ according to this cycle keeping the allocations of agents not belonging to the cycle, constant relative to $x$. Let this new allocation be denoted $y$. Agents belonging to the cycle are strictly better-off in $y$ relative to $x$ and the others are no worse-off. This contradicts the assumption that $x$ is strongly efficient.

Let $N_0 = \emptyset$. According to the claim just proved, there exists an agent who gets his best object. Construct a priority where this agent is first. After assigning this agent his best object, the claim can be used again to assert that there exists an agent who is getting her object amongst the remaining objects. This agent is second in the priority. Moreover, proceeding in this manner, a complete priority can be constructed. It is trivial to verify that the PM according to this priority leads to $x$.

The claims regarding the TTCM now follow from Proposition 4.

□

Proposition 5 clearly demonstrates that PM and TTCM are salient methods for allocation if strong efficiency is a desired property of the allocation. However they suffer from serious deficiencies from the standpoint of equity. In the case of PM, the agents who have positions earlier in the queue have advantages over those who follow them. In particular, the first agent in the queue gets her most preferred object while the last agent simply gets what is left over. The TTCM is also unsatisfactory because the initial (arbitrary) distribution of objects strongly influences the allocation. Can randomization be helpful in ameliorating these difficulties?

According to Proposition 5, every strongly efficient allocation can be obtained as a PM with respect to some priority or a TTCM with respect to some initial distribution of objects. It follows therefore that every ex-post efficient randomized allocation can be thought of either as the outcome of PMs from a *randomization over priorities* or as the outcome of TTCMs from a *randomization over the initial allocation of objects*. Moreover, Proposition 5 also ensures that every such randomized allocation will be ex-post efficient.

What randomizations over priorities or initial allocations are appropriate from the perspective of fairness? For priorities, the uniform randomization is a natural candidate. In a deterministic PM, one can imagine agents envying the priorities of the agents who precede them. A uniform randomization over priorities ought to eliminate such envy. Note that are $n!$ different priorities so that a uniform randomization would pick each of these $n!$ priorities with probability $\frac{1}{n!}$. I shall denote the PM from the uniform distribution over priorities as the *Uniform Priority Method* or UPM randomized allocation. For randomizations over the initial distributions of objects, a uniform distribution also seems the most natural. Once again there are $n!$ such initial distributions and the uniform distribution would pick each of these initial distributions with probability $\frac{1}{n!}$. I shall denote the TTCM from the uniform distribution over initial distribution over objects as the *Uniform Top Trading Cycle Method* or UTTCM.

But which of these two methods should we choose? Fortunately this is not a difficult decision because Abdulkadiroglu and Sönmez [1] have shown that the two randomized allocations are *identical* for all possible preference profiles of agents. A proof of this proposition is beyond the scope of this paper. However, an example is given below which illustrates the general result.

**Example 2.** Let $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3\}$. Preferences are as follows: $a_1 P_1 a_2 P_1 a_3$, $a_1 P_2 a_3 P_2 a_2$ and $a_2 P_1 a_1 P_1 a_3$. The priorities 123, 132, 213, 231, 312, 321 lead to the allocations $(a_1, a_3, a_2)$, $(a_1, a_3, a_2)$ $(a_2, a_1, a_3)$, $(a_3, a_1, a_2)$, $(a_1, a_3, a_2)$ and $(a_3, a_1, a_2)$ respectively. Since the priorities are each drawn with probability $\frac{1}{6}$, agent 1, 2 and 3's lotteries over $a_1$, $a_2$ and $a_3$ according to UPM are $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$, $(\frac{1}{2}, 0, \frac{1}{2})$ and $(0, \frac{5}{6}, \frac{1}{6})$ respectively. The possible initial distributions of objects are $(a_1, a_2, a_3)$, $(a_1, a_3, a_2)$, $(a_2, a_1, a_3)$, $(a_2, a_3, a_1)$, $(a_3, a_1, a_2)$ and $(a_3, a_2, a_1)$. The TTCM from these initial distributions lead to the allocations $(a_1, a_3, a_2)$, $(a_1, a_3, a_2)$, $(a_2, a_1, a_3)$, $(a_1, a_3, a_2)$, $(a_3, a_1, a_2)$ and $(a_3, a_1, a_2)$ respectively. Hence, UTTCM leads to lotteries $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$, $(\frac{1}{2}, 0, \frac{1}{2})$ and $(0, \frac{5}{6}, \frac{1}{6})$ respectively. But these are exactly the lotteries generated by UPM.

The only question which remains is: does the UPM (or UTTCM) lead to an ordinally envy-free randomized allocation? The somewhat surprising answer is no, as can be verified from Example 2. Observe that the probability of agent 1 getting one of his two best objects, $a_1$ and $a_2$ is $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$. However the probability of agent 3 getting one of these objects is $\frac{5}{6}$ which is strictly greater than $\frac{2}{3}$. Therefore when agent 1 has a utility function which gives $a_2$ a utility "close" to the utility of $a_1$, his most preferred object, he will envy agent 3. Hence the randomized allocation fails to satisfy ordinal envy-freeness.

Do ex-post and ordinally envy-free allocations exist for these preferences? Consider a randomized PM where $\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5$ and $\lambda^6$ denote the probabilities of picking the priorities 123, 132, 213, 231, 312 and 312 respectively. This generates the following lotteries over $a_1$, $a_2$ and $a_3$ for agents 1, 2 and 3 respectively: $(\lambda^1 + \lambda^2 + \lambda^5, \lambda^3, \lambda^4 + \lambda^6)$, $(\lambda^3 + \lambda^4 + \lambda^6, 0, \lambda^1 + \lambda^2 + \lambda^3)$ and $(0, \lambda^1 + \lambda^2 + \lambda^4 + \lambda^5 + \lambda^6, \lambda^3)$. It can be verified that a necessary and sufficient condition for ordinal envy-freeness to hold are the following two equations are satisfied.

(i) $\lambda^1 + \lambda^2 + \lambda^5 = \lambda^3 + \lambda^4 + \lambda^6 = \frac{1}{2}$

(ii) $\lambda^3 = \lambda^4 + \lambda^6$

A solution to these equations exists. For example, choose $\lambda^1 = \lambda^2 = \lambda^5 = \frac{1}{6}$, $\lambda^3 = \frac{1}{4}$ and $\lambda^4 = \lambda^6 = \frac{1}{8}$. This leads to lotteries $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $(\frac{1}{2}, 0, \frac{1}{2})$ and $(0, \frac{3}{4}, \frac{1}{4})$ for agents 1, 2 and 3 respectively. Moreover this solution for $\lambda$'s is not unique - in fact the set of solutions is a convex set.

Is it possible to find ex-post efficient and ordinally envy-free randomized allocations for all possible preferences of agents? Bogomolnaia and Moulin [3] demonstrate that is this indeed possible. In fact it is possible to reconcile ordinal envy-freeness with a notion of efficiency stronger than ex-post efficiency, which they call ordinal efficiency. A discussion of these issues is, unfortunately, beyond the scope of this essay.

## 5. Procedures Leading to Fair Outcomes

In this section, I briefly discuss some procedures whose outcomes are equitable. A procedure consists of decisions taken by the various agents and a rule which specifies an allocation depending on the decisions taken. Why is a procedure necessary at all? There are two reasons. The first is an assumption that though the collective goal of the agents is fairness, their behaviour as individuals is *self interested.* For instance consider the case of the classical exchange economy where an aggregate endowment $\Omega$ has to be divided amongst $n > 1$ agents. If a particular agent is asked to make the division and allocate various shares to everyone, then he is likely to keep the entire bundle $\Omega$ to himself. In that case, a natural solution might be to turn to a disinterested arbiter or referee [16] and ask her to make the decision. For instance, the arbiter could be asked to compute the competitive equilibrium allocation from equal division of $\Omega$ and implement the solution. This difficulty with this approach is that the arbiter (being an "outsider" or a computer) is unlikely to have the *information* regarding agent preferences required to compute the equitable allocation. Since these preferences are not known to the arbiter, they have to be solicited from the agents themselves. However rational agents will then realize that

[16] The arbiter need not be a "real" person - it could be a computer.

they it may be advantageous to misrepresent their true preferences. In the "equilibrium" that obtains when agents recognize their strategic possibilities, there is no guarantee that the allocation received by the agents is the competitive equilibrium allocation from equal division of $\Omega$ with respect to the agents *true* preferences. The informational asymmetry between the agents and the arbiter is the second reason why a procedure is required.

For most of this section I shall be concerned with the classical exchange economy where there are $n$ agents with preferences $(R_1, ..., R_n)$ who have to divide between themselves an aggregate resource endowment $\Omega \in \Re^L_{++}$. It will sometimes be convenient to consider utility representations $u_i$ of preferences $R_i$. The particular representation chosen will have no bearing on any of the results. It is assumed that $R_i$ is continuous and increasing for all $i = 1, .., n$.

One of the best-known procedures for dividing resources is the method of *Divide and Choose* (Crawford [5], Dubins and Spanier [9], Kolm [14], Steinhaus [22]; see also Brams and Taylor [4] for a survey.) This method applies in the special case where $n = 2$. One of the agents, say 1 is designated as the divider. She proposes a split $(x, \Omega - x)$ of $\Omega$ where $x \in \Re^L_+$. The other agent, 2 called the chooser, chooses one of the portions $x$ and $\Omega - x$ and the divider gets to keep the other portion.

How should the divider propose the split and which portion should the chooser pick? The Divide and Choose Game is a finite game of complete information [17]. A strategy for agent 1 is a split and a strategy for agent 2 is a function which allocates a portion to each agent for every possible split. If agent 2 is rational, she will pick the portion which gives her more utility, i.e. when faced with the split $(x, \Omega - x)$, she chooses $x$ if $u_2(x) > u_2(\Omega - x)$ and $\Omega - x$ otherwise. Agent 1 anticipating 2's rational behaviour will therefore solve the following problem:

[17] Details may be found in Gibbons [13], Chapter 2.

$$\max_x u_1(x)$$
$$\text{subject to } u_2(\Omega - x) \geq u_2(x)$$

Let $x^*$ be a solution to the problem above. The equilibrium of the game can be thought as follows. Agent 1 proposes the division $(x^*, \Omega - x^*)$ and asks agent 2 to pick $\Omega - x^*$. In view of the constraint $u_2(\Omega - x^*) \geq u_2(x^*)$, a rational agent 2 will choose $\Omega - x^*$. The solution described above is the *subgame perfect Nash equilibrium* of the Divide and Choose game and has been computed using the well-known Backwards Induction Algorithm of Kuhn [18].

The main interest in this game lies in the fact that the solution $(x^*, \Omega - x^*)$ is an envy-free allocation. (Note that here, $x^*$ is the portion received by agent 1 and $\Omega - x^*$, the share received by 2.) The constraint ensures that agent 2 does not envy 1. Now suppose that agent 1 envies agent 2, i.e. $u_1(\Omega - x^*) > u_1(x^*)$. First note that since preferences are continuous, it must be true that $u_2(\Omega - x^*) = u_2(x^*)$. Therefore the split $(\Omega - x^*, x^*)$ also satisfies the constraint and leads to a higher value of the maximand (since $u_1(\Omega - x^*) > u_1(x^*)$ by hypothesis). This contradicts our assumption that $x^*$ solves the maximization problem.

The divide and choose solution $(x^*, \Omega - x^*)$ may not, however be efficient. In order to see this assume that the utility functions $u_i$, $i = 1, 2$ are twice continuously differentiable. Let $u_i^j(y_i)$, $j = 1, ., L$ and $i = 1, 2$ denote the $j^{th}$ partial derivative of the function $u_i$ evaluated at the consumption bundle $y_i$ for agent $i$. Assuming that the solution $(x^*, \Omega - x^*)$ is interior, it must satisfy

$$\frac{u_1^j(x^*)}{u_1^k(x^*)} = \frac{u_1^j(x^*) - u_2^j(\Omega - x^*)}{u_1^k(x^*) - u_2^k(\Omega - x^*)}$$

for all $j, k \in \{1, .., L\}$. This is clearly different from the necessary and sufficient condition for efficiency which is

$$\frac{u_1^j(x^*)}{u_1^k(x^*)} = \frac{u_2^j(\Omega - x^*)}{u_2^k(\Omega - x^*)}$$

[18]Details can again be found in Gibbons [13].

for all $j, k \in \{1, .., L\}$. Observe, however that the two conditions are equivalent in the special case where the two agents have identical preferences, i.e. $u_1 = u_2$. Therefore the equilibrium in the Divide and Choose method is not efficient in general unless further assumptions are made on preferences. Another important feature of this method is that its outcome is the envy-free allocation most preferred by the divider (Kolm [14], Crawford [5]).

There have been several generalizations of the Divide and Choose Method to $n$ players (see Brams and Taylor [4]). Here, I only present a method due to Thomson [27] which he calls the Divide and Permute method. Each player $i = 1, .., n$ proposes a permutation $\sigma_i$ of the set $N$. [19] In addition, two designated agents, say 1 and 2 also announce feasible allocations $x^1$ and $x^2$. The outcome of an announcement vector $((x^1, \sigma_1), (x^2, \sigma_2), \sigma_3, ..., \sigma_n)$ is completely described by the following two rules:
(i) if $x^1 \neq x^2$, then all agents get the 0 bundle and
(ii) if $x^1 = x^2 = x$, then the outcome is $\sigma_1 \circ \sigma_2 \circ ... \circ \sigma_n(x)$.

Agents 1 and 2 propose feasible allocations. If they differ in their proposals, then all agents get nothing. Suppose they propose the same feasible allocation $x$. Then the final allocation is $\sigma_1 \circ .... \circ \sigma_n(x)$ where $\sigma_i$ is the permutation announced by agent $i$, $i = 1, .., n$. In other words, each agent $i$ gets a component of the vector $x$, say $x_k$ where $k$ is the image of $i$ in the composed permutation $\sigma_1 \circ ... \circ \sigma_n$. An observation which is critical to the proof of the proposition which follows is that for every agent $i$, for every $n-1$ tuple $(\sigma_1, .., \sigma_{i-1}, \sigma_{i+1}, .., \sigma_n)$ and $k \in \{1, .., n\}$, there exists $\sigma_i$ such that $\sigma_1 \circ .... \circ \sigma_n(i) = k$. Thus no matter what permutations the other agents announce, agent $i$ can announce a permutation which will give him the $k^{th}$ component of $x$

[19] A permutation $\sigma_i$ of the set $N$ is a one-to-one map $\sigma_i : N \to N$. The composition of any two permutations $\sigma_i$ and $\sigma_j$ denoted by $\sigma_i \circ \sigma_j$ is defined by $\sigma_i(\sigma_j(k))$ for all $k = 1, .., n$. It can be easily verified that $\sigma_i \circ \sigma_j$ is also a permutation of the set $N$. The identity permutation is the one where each element of $N$ is mapped to itself.

for any $k$. This is an elementary fact regarding the composition of permutations and can be verified easily.

The rules of Divide and Permute in conjunction with a preference ordering for every agent $(R_1, .., R_n)$ consitutes a game in normal form [20]. The best-known and most widely used solution concept for such games is that of *Nash equilibrium.* In the present context, a Nash equilibrium is an $n$-tuple $((\bar{x}^1, \bar{\sigma}_1), (\bar{x}^2, \bar{\sigma}_2), \bar{\sigma}_3, .., \bar{\sigma}_n)$ such that no agent $i$ can be strictly better-off (with respect to her preference ordering $R_i$) by deviating unilaterally from it [21]. The set of Nash equilibium outcomes of the Divide and Permute game coincides with the set of envy-free allocations.

**Proposition 6.** *Fix an arbitrary $n$-tuple of preferences $(R_1, .., R_n)$. Every Nash equilibrium of the Divide and Permute game is envy-free with respect to $(R_1, ..., R_n)$. Conversely, every envy-free allocation with respect to $(R_1, .., R_n)$ can be supported as a Nash equilibrium of the Divide and Permute game.*

**Proof**: Let $((\bar{x}^1, \bar{\sigma}_1), (\bar{x}^2, \bar{\sigma}_2), \bar{\sigma}_3, .., \bar{\sigma}_n)$ be an arbitrary Nash equilibrium of the Divide and Permute game. It must be the case that $\bar{x}^1 = \bar{x}^2$. Suppose this was false. Then both agents 1 and 2 are getting the bundle 0. Agent 1 can deviate by proposing the same allocation $\bar{x}^2$ as agent 2. Moreover by announcing a suitable permutation, he can ensure that he obtains a strictly positive bundle (since $\bar{x}^2$ is an allocation, at least one of its component must be strictly positive). Since $R_1$ is increasing 1 will be strictly better-off by deviating which contradicts the hypothesis that $((\bar{x}^1, \bar{\sigma}_1), (\bar{x}^2, \bar{\sigma}_2), \bar{\sigma}_3, .., \bar{\sigma}_n)$ is a Nash equilibrium. Suppose therefore that $\bar{x}^1 = \bar{x}^2 = x$. The final allocation is a permutation of the components of $x$ which is denoted by $\sigma(x)$. Suppose that agent $i$

[20] A game in normal form is a collection $\langle N, S_1, ..., S_n, \pi_1, ...\pi_n \rangle$ where $N$ is the set of players, $S_i$, $i = 1, .., n$ is the strategy set for player $i$ and $\pi : S_1 \times S_2 \times ... \times S_n \rightarrow \Re$ is $i$'s payoff function. Details can be found in Gibbons [13] Chapter 1.

[21] More generally, $(\bar{s}_1, .., \bar{s}_n) \in S_1 \times ... \times S_n$ is a Nash equilibrium of $\langle N, S_1, ..., S_n, \pi_1, ...\pi_n \rangle$ if $\pi_i(\bar{s}_i, \bar{s}_{-i}) \geq \pi_i(s_i \bar{s}_{-i})$ for all $s_i \in S_i$ and $i = 1, .., n$.

is getting $x_k$ (determined by the permutations of all agents). By unilaterally deviating, $i$ can obtain any component of $x$. None of these deviations can make $i$ better-off by the definition of Nash equilibrium. It follows that $\sigma(x)$ is envy-free.

Now pick an envy-free allocation $x$. Consider the strategy profile where all agents 1 and 2 propose $x$ and all agents announce the identity permutation. Then the outcome according to the rules of Divide and Permute is $x$. It remains to show that these strategies constitute a Nash equilibrium, i.e. no agent can be strictly-off by deviating. Agents 1 and 2 by unilaterally deviating with respect to the announced allocation will only get 0 which will not make them better-off. By deviating with respect to the permutation each agent can get only a different component of $x$. By envy-freeness of $x$, $x_i R_i x_k$ for all $i$ and $k$ so that none of these deviations are worthwhile for any agent. Hence these strategies constitute a Nash equilibrium. □

As with Divide and Choose, Divide and Permute does not guarantee efficiency. Thomson [27] provides a more elaborate procedure where all Nash equilibrium allocations are efficient, in addition to being envy-free. Two papers which consider procedures which generate efficient egalitarian allocations are Crawford [6] and Demange [7].

The theory of designing procedures whose outcomes (or equilibria) satisfy some fairness and efficiency requirements is part of the more general theory of *implementation*. A survey of these issues can be found in the essay by Dutta [10] in the present volume.

## 6. Conclusion

In this essay I have attempted to discuss some concepts in the theory of fairness and equity in models where agents with different preferences have to share a fixed quantity of resources. I have pointed out that the scope of this theory is somewhat narrow because it only considers the case where agents differ only with respect to a single relevant charcteristic, viz. preferences. Nevertheless, it is a rich and elegant theory that explores the interaction between axioms relating to fairness, efficiency and incentives. There is a substantial literature which examines similar issues in other but related contexts. Some of this work has implications for public policy, for instance, the recent work on the allocation of kidneys amongst potential transplant patients (see Roth and Sönmez [20]). For a clear and stimulating discussion of many of these issues, the reader is again referred to Moulin [17] and Young [30].

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