Final Examination; Game Theory I

Instructor: **Debasis Mishra**; Max marks: **50 marks**; Time: **3+ hours**

November 03, 2025

- The question paper has three questions with a maximum total mark of 50. All questions are compulsory. Please complete the three questions in three hours.
- You may refer to hard copy of your class notes.
- Read all questions carefully at the beginning and ask for clarifications, if any.
- Do ALL parts of a question together.
- Start each question on a fresh page.
- Label all the figures/diagrams and make them large enough so that they are readable.

1. Two firms $\{1, 2\}$ sell the same product in a market by setting competing prices. If the prices are (p_i, p_j) where $p_i < p_j$, then firm i sells 2 units of the product and firm j sells zero units. If $p_i = p_j$, then both firms sell 1 unit each. The utility of each firm is the number of units sold by that firm times its price (zero cost of production). Suppose there are three possible prices that firms can set $\{10, 20, 30\}$.

   (a) Suppose the firms set prices for one period and die. What are the Nash equilibrium of this game? What are the min-max payoffs? **(4 marks)**

   Answer. There is are two Nash equilibria: firms set prices $(10, 10)$ and $(20, 20)$. They give payoffs $(10, 10)$ and $(20, 20)$ (since each sell one unit). The min-max payoff is $(10, 10)$.

   (b) Suppose the firms set prices for two periods and then die. Is there a subgame perfect Nash equilibrium (SPNE) of this 2-period repeated game (assuming payoff at the end of 2 periods is sum of payoffs in each period), where firms set $(30, 30)$ prices in the first period? Clearly describe the strategies used by the firms. **(4 marks)**

   Answer. Consider the strategy: set price 30 in the first period. If price $(30, 30)$ is observed in the first period, set price 20 in the second period. Else, set price 10 in the second period. Clearly, we are setting Nash equilibrium prices in the second period. In the first period, if strategy is followed, payoff is $30 + 20 = 50$. If there is a deviation to set price 20 (best possible deviation), payoff is $2 \times 20 + 10 = 50$. Hence, this strategy is an SPNE.

   (c) Suppose the firms set prices for infinite number of periods and use discounting criteria to evaluate payoffs from infinite utility streams. Consider the following actions by the two players (call it path A): Firm 1 sets price 30 in odd periods (first period is an odd period) and sets price 20 in even periods; firm 2 sets price 30 in all periods. Can A be supported in SPNE as the equilibrium path of the game for appropriate discount rates of the players? If yes, find some range of discounts where this can happen. **(10 marks)**

Answer. Yes. The payoffs from these strategies for firm 1:

$$(1-\delta)(30+40\delta+30\delta^2+40\delta^3+\dots) = (1-\delta)\Big(\frac{30}{1-\delta^2}+\frac{40\delta}{1-\delta^2}\Big)$$
$$= \frac{30+40\delta}{1+\delta}$$

for firm 2: she gets 30 in odd periods (when she sells one unit) and zero in even periods:

$$(1-\delta)(30+30\delta^2+30\delta^4+\dots) = (1-\delta)\frac{30}{1-\delta^2} = \frac{30}{1+\delta}$$

For all values of δ, these payoffs are higher than the worst Nash equilibrium payoff of stage game. Hence, a reversion to worst Nash equilibrium action profile strategy (whenever the history consists of the prescribed path actions, continue along the path; else play (10,10)) can ensure this is sustained in SPNE for some large enough δ.

To see this, suppose Player 1 deviates. She should not deviate in even period because she gets the maximum possible payoff (given strategy of Player 2) in even period. So, she can only deviate in odd periods. She does so by setting price 20 to sell 2 units. She gets payoff of 40 by this one shot deviation. But subsequent payoffs are 10. So, discounted sum is:

$$(1-\delta)(40+10\delta+10\delta^2+\dots) = 40(1-\delta)+10\delta = 40-30\delta$$

Her payoff by following strategy in any odd period is $\frac{30+40\delta}{1+\delta}$. Hence, for equilibrium we need

$$\frac{30+40\delta}{1+\delta} \geq 40-30\delta$$
$$\iff 30+40\delta \geq 40+10\delta-30\delta^2$$
$$\iff 3\delta^2+3\delta-1 \geq 0$$

This is possible for a large range of δ: in particular for $\delta \geq \frac{1}{3}$.

If Player 2 deviates in odd period, she can do so by setting price 20 and getting payoff 40. Then she gets 10 afterwards. So, the discounted sum is again $40 - 30\delta$. Her payoff by following the strategy from an odd period is $\frac{30}{1+\delta}$. So, for equilibrium, we need

$$\frac{30}{(1+\delta)} \geq 40 - 30\delta$$
$$\iff 30 \geq 40 + 10\delta - 30\delta^2$$
$$\iff 3\delta^2 - \delta - 1 \geq 0$$

This is feasible for a large range of δ: for instance, $\delta \geq \frac{4}{5}$.

If Player 2 deviates in even period, she can do so by setting price 10 (sells 2 units) or price 20 (sells 1 unit) to get payoff 20 in both cases, but gets 10 from there on. So, discounted payoff is

$$20(1-\delta) + 10\delta = 20 - 10\delta$$

By following the strategy from an even period, her payoff is $(0, 30, 0, 30, 0, \ldots)$ whose discounted sum is $(1-\delta)(30\delta + 30\delta^3 + 30\delta^5 + \ldots) = \frac{30\delta}{1+\delta}$. Hence, for equilibrium, we need

$$\frac{30\delta}{1+\delta} \geq 20 - 10\delta$$
$$\iff 30\delta \geq 20 + 10\delta - 10\delta^2$$
$$\iff \delta^2 + 2\delta \geq 2$$
$$\iff \delta \geq \sqrt{3} - 1$$

Combining all the bounds on δ, we see some interval of δs where the SPNE can be sustained.

2. Two agents $\{m, w\}$ play the following game in Table 1. But there is uncertainty about who is the ROW player and who is the COLUMN player. There are two states of the world: $\{M, W\}$. State M corresponds to m agent being the ROW player and w agent being the COLUMN player. State W corresponds to w agent being the ROW player and m agent being the COLUMN player.

   Probability of State M is $(1-\epsilon)$ and State W is $\epsilon \in (0,1)$. These probabilities are common knowledge.

   The game goes as follows. Agent w observes the state (i.e., knows who is the ROW player and who is the COLUMN player) but agent m does not. Then, both the agents simultaneously take one of the actions in $\{a, b\}$. Depending on who is the ROW player and who is the COLUMN player, players realize their payoffs according to Table 1, where ROW player payoff is written first and then the COLUMN player payoff.

	a	b
a	$(2,1)$	$(0,0)$
b	$(0,0)$	$(1,2)$

Table 1: Normal form game

   (a) Describe this as a Bayesian game (i.e., describe the types of agents, actions available etc.). **(3 marks)**

   Answer. Agent m has no type. It has two actions available $\{a, b\}$. But by playing any of these actions, agent m does not know for sure what payoffs will be obtained.

   Agent w has two types (corresponding the two states): $\{M, W\}$. For each type same actions are available: $\{a, b\}$. The probabilities of M and W types are $1-\epsilon$ and ϵ respectively.

   If agent w is of type M, then agent m gets the row payoff; else agent w gets row payoff in Table 1. See Figure 1.

   (b) Is there a Bayesian equilibrium of this game when agent w uses the following strategy s_w: $s_w(M) = b, s_w(W) = a$ and agent m uses a pure strategy? **(4 marks)**

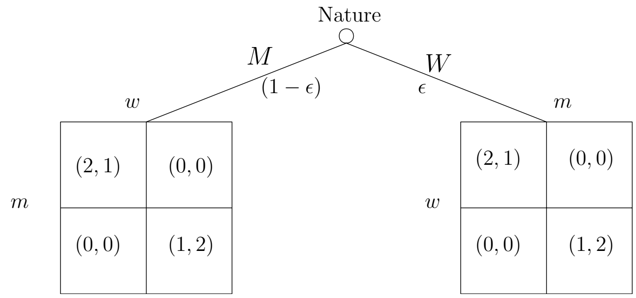

Figure 1: A Bayesian game

Answer. Consider an equilibrium where agent w uses the following strategy: $s_w(M) = b, s_w(W) = a$

Suppose agent m plays a in this equilibrium. Then, payoff of agent w is zero when type/state is M. This can be improved if agent w chooses a. So, in no such equilibrium agent m can play a.

Similarly, if agent m plays b, then payoff of agent w is zero when type is W. This can be improved if agent w chooses b.

Hence, there is no Bayesian equilibrium where agent m plays a pure strategy.

(c) Suppose $\epsilon \neq \frac{1}{2}$. Is there a Bayesian equilibrium of this game where agent w uses the following strategy s_w: $s_w(M) = b, s_w(W) = a$ and agent m mixes between a and b ? **(5 marks)**

Answer. Suppose agent m mixes: a with probability p and b with probability $(1 - p)$. For agent m to mix, he should be indifferent between a and b. His payoff by playing a is calculated as follows:

- With probability $(1 - \epsilon)$, state is M, in which case agent w (Column player) plays b. This gives zero payoff to both the players.
- With probability ϵ, state is W, in which case agent w plays a. This gives a payoff 1 to the m (Column player).

Hence, expected payoff is ϵ. Similarly, payoff by playing b is calculated as follows:

- With probability $(1 - \epsilon)$, state is M, in which case agent w (Column player) plays b. This gives $(1 - \epsilon)$ payoff to m.
- With probability ϵ, state is W, in which case agent m plays a. This gives a payoff 0 to both the players.

Hence, expected payoff is $1-\epsilon$. So, for agent m to be randomize, we need $1-\epsilon = \epsilon$, which is $\epsilon = 0.5$. So, there is no Bayesian equilibrium if $\epsilon \neq 0.5$.

3. Consider the two extensive form games shown in Figure 2. Each vertex been labeled with names of players in case of decision vertex and payoff vectors in case of payoff vertex.

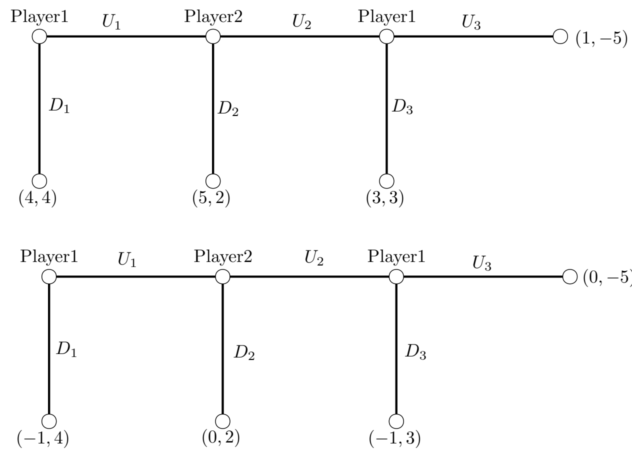

Figure 2: Two extensive form games

(a) Find all subgame perfect equilibria of both the games. **(4 marks)**

Answer. Backward induction can be used to solve the games. In the top game, the SPE is (D_1, U_2, D_3). In the bottom game, the SPE is (U_1, D_2, U_3).

(b) Notice that in both the games the payoff of Player 2 is the same. Now, suppose Player 2 does not know whether the top game is played or the bottom game is played, but Player 1 knows which game is being played. Player 2 knows that the top game is being played with probability 0.9 and the bottom game is being played with probability 0.1.

i. Describe the underline extensive form game of incomplete information. A neatly labeled game tree will be sufficient. **(2 marks)**

Answer. See Figure 3.

ii. Is there a perfect Bayesian equilibrium of this game where Player 2 plays U_2? **(5 marks)**

Answer. Sequential rationality implies Player 1 plays D_3 at the top game and U_3 at the bottom. Suppose μ_t is the belief of Player 1 that she is in

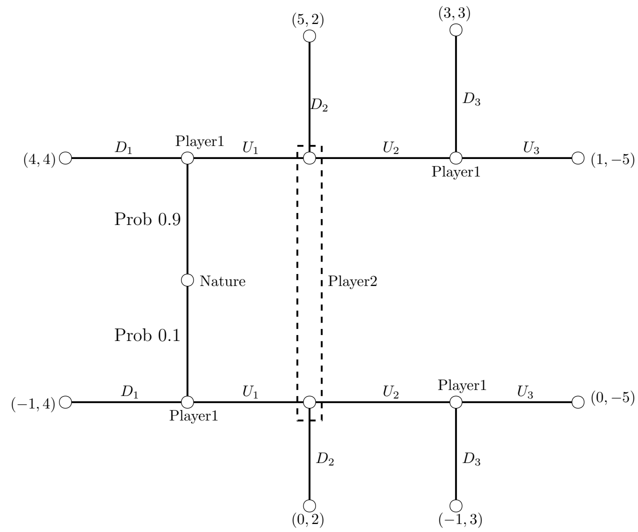

Figure 3: Extensive form game with incomplete information

the top game and $(1-\mu_t)$ is the probability that she is in the bottom game. Then, her expected payoff from playing U_2 is

$$3\mu_t + (-5)(1-\mu_t) = 8\mu_t - 5$$

and from playing D_2 is 2. So, U_2 is sequentially rational if $\mu_t \geq \frac{7}{8}$ and D_2 is sequentially rational if $\mu_t \leq \frac{7}{8}$.
If Player 2 plays U_2, then Player 1 finds sequentially rational to play D_1 in the top game and U_1 in the bottom game. As a result, Bayesian rationality of Player 2 implies $\mu_t = 0$. This implies that $\mu_t = 0 < \frac{7}{8}$, and it is sequentially rational to play D_2. So, there is no PBE where player 2 plays U_2.

iii. Is there a perfect Bayesian equilibrium of this game where Player 2 plays D_2? **(3 marks)**

Answer. Again, sequential rationality implies Player 1 plays D_3 at the top game and U_3 at the bottom. For Player 2 to play D_2, we must have $\mu_t \leq \frac{7}{8}$ by sequential rationality. Then, sequential rationality of Player 1 plays U_1 at both its initial decision vertices. Then, Bayesian rationality of Player 2 implies, $\mu_t = 0.9$. Hence, there is no PBE where player 2 plays D_2.

iv. Find all perfect Bayesian equilibria of this game. **(6 marks)**

Answer. From the previous two answers, the only possibility is Player 2 mixes between U_2 and D_2 at her information set: say $\alpha U_2 + (1-\alpha)D_2$. This means that Player 2 must be indifferent between U_2 and D_2, which implies $\mu_t = \frac{7}{8}$.

Now, in the bottom game, if Player 1 plays U_1, she gets zero (independent of the value of α) but gets -1 by playing D_1. Hence, by sequential rationality she will play U_1. In the top game, pure action U_1 will lead to (by Bayesian rationality of Player 2) $\mu_t = 0.9$ and pure action D_1 will lead to $\mu_t = 0$. But we need $\mu_t = \frac{7}{8}$. Hence, Player 1 must randomize between D_1 and U_1 at the top game. Then, she should be indifferent between D_1 and U_1, which means playing U_1 must give her payoff of 4. To do so, Player 2 should randomize $\alpha = \frac{1}{2}$ (so that $5 \times \frac{1}{2} + 3 \times \frac{1}{2} = 4$). Then, to generate $\mu_t = \frac{7}{8}$ (for Bayesian rationality of Player 2), Player 1 must randomize $(1-\beta)D_1 + \beta U_1$ such that

$$\frac{7}{8} = \frac{0.9\beta}{0.9\beta + 0.1}$$
$$\iff \beta = \frac{7}{9}$$

Hence, the unique PBE is:

$$\frac{2}{9}D_1 + \frac{7}{9}U_1, U_1, \frac{1}{2}D_2 + \frac{1}{2}U_2, D_3, U_3, \mu_t = \frac{7}{8}$$