Final Examination; Theory of mechanism design

Instructor: **Debasis Mishra**; Max marks: **50 marks**; Time: **3 hours**

May 02, 2026

- The question paper has four questions with a maximum total mark of 50. All questions are compulsory. Please complete the questions in three hours.
- You may refer to hard copy of your class notes.
- Read all questions carefully at the beginning and ask for clarifications, if any.
- Do ALL parts of a question together.
- Start each question on a fresh page.
- Label all the figures/diagrams and make them large enough so that they are readable.

1. Consider the asymmetric auction environment for allocating the good with type space $[0, 1]$ for each of the n bidders. The distribution of values of bidder i is given by the CDF

$$F_i(x) = (x)^i \qquad \forall\, x \in [0, 1]$$

   (a) Verify if the virtual valuation function is monotone for each bidder. **(3 marks)**

   Answer. The density of bidder i is $f_i(x) = ix^{i-1}$, which is weakly increasing in x. As a result, $\frac{1-F_i(x)}{f_i(x)}$ is weakly decreasing in x. Hence, $x - \frac{1-F_i(x)}{f_i(x)}$ is strictly increasing in x.

   (b) In the optimal auction in this setting, what is the reserve price for each bidder (i.e., what is the minimum value each bidder must have non-negative virtual value)? Can you rank the reserve prices of the bidders? **(5 marks)**

   Answer. The reserve price of bidder i is r_i that solves

$$x = \frac{1 - x^i}{ix^{i-1}} \quad \text{or,} \quad r_i = \Big(\frac{1}{i+1}\Big)^{1/i}$$

   But then

$$\ln r_i = -\frac{1}{i}\ln(1+i)$$

   is an increasing function. To see this consider the function:

$$h(x) = -\frac{1}{x}\ln(1+x)$$

   Its derivative is

$$h'(x) = -\frac{1}{x(x+1)} + \frac{1}{x^2}\ln(1+x) = \frac{1}{x^2}\Big(\ln(1+x) - \frac{x}{x+1}\Big)$$

   If we let $\psi(x) = \ln(1+x) - \frac{x}{x+1}$, we see that $\psi'(x) = \frac{1}{x+1} - \frac{1}{(x+1)^2} > 0$ for all $x > 0$. Hence, ψ is strictly increasing and $\psi(0) = 0$. Hence, $h'(x) > 0$ for all $x > 0$. This establishes h is strictly increasing and hence, r_i is strictly increasing in i. This shows that $r_1 < r_2 < \ldots < r_n$.

(c) Suppose $n = 2$. Then, compute the allocation and payment of the bidders in the optimal auction when both the bidders have (same) value equal to $\frac{3}{4}$. **(4 marks)**

Answer. Virtual value of bidder 1 is $\frac{3}{4} - (1 - \frac{3}{4}) = \frac{1}{2}$. That of bidder 2 is $\frac{3}{4} - \frac{1-\frac{9}{16}}{2\times\frac{3}{4}} = \frac{3}{4} - \frac{7}{24} = \frac{11}{24}$. Both virtual values are non-negative. Hence, bidder 1 wins and pays the least value at which his virtual value becomes $\frac{11}{24}$. That happens when $2x - 1 = \frac{11}{24}$ or $x = \frac{35}{48}$. So, bidder 1 wins and pays $\frac{35}{48}$.

(d) Suppose each of the n bidders go to a separate market where there is a single good being sold. The seller in each of these n markets use an optimal mechanism. What is the optimal mechanism in each market i (where bidder i goes)? Note that in each of these markets there is only one bidder. Which market will generate the highest expected revenue? **(4 marks)**

Answer. The optimal mechanism in market i is to set a reserve price equal to r_i. Hence, expected revenue in market i is $r_i(1 - F_i(r_i)) = (r_i)^2(f_i(r_i)) = i(r_i)^{1+i}$, where the first equality is due to $r_i f_i(r_i) = (1 - F_i(r_i))$ and the second equality is due to $f_i(r_i) = i(r_i)^{i-1}$. By (b), r_i is increasing in i. Hence, expected market revenue of market n is the highest.

2. Consider the problem of allocating a single object to a buyer with private values. But now the type space is finite:

$$T := \{0, \epsilon, 2\epsilon, \ldots, K\epsilon\}$$

For simplicity, we will denote 0 as x_0, ϵ as x_1, ..., and $k\epsilon$ as x_k etc. Suppose probability of buyer having type x_i is $\pi_i > 0$.

(a) Write down the IC and IR constraints for this problem for a mechanism (q, p) - both in (q, p) space and (q, u) space. **(2 marks)**

Answer. It is the standard IC and IR constraints.

$$xq(x) - p(x) \geq xq(y) - p(y) \qquad \forall\ x, y \in T$$
$$xq(x) - p(x) \geq 0\ \forall\ x \in T$$

In terms of utility function $u(x) \equiv xq(x) - p(x)$, it translates to

$$u(x) \geq u(y) + (x - y)q(y) \qquad \forall\ x, y \in T$$
$$u(x) \geq 0 \qquad \forall\ x \in T$$

(b) Call a mechanism **locally IC** if for every $x_k, x_{k+1} \in T$, the pair of IC constraints between x_k and x_{k+1} holds (but other IC constraints are not required to hold).

i. If (q, p) is locally IC, show that q is weakly increasing. **(2 marks)**
Answer. By adding the locally IC constraints of any pair (x_k, x_{k+1}), we get $q(x_k) \geq q(x_{k+1})$.

ii. Denote by the $x \to y$ the IC constraint when true type is x and manipulation is prevented of misreport to y. Show that if (q, p) is locally IC, then for any k, we have $x_k \to x_{k+2}$ and $x_{k+2} \to x_k$. Use this to show that a locally IC mechanism is IC. **(6 marks)**

Answer. By local IC of $x_k \to x_{k+1}$ and $x_{k+1} \to x_{k+2}$, we get

$$\begin{aligned} u(x_k) &\geq u(x_{k+1}) + (x_k - x_{k+1})q(x_{k+1}) \\ &\geq u(x_{k+2}) + (x_{k+1} - x_{k+2})q(x_{k+2}) + (x_k - x_{k+1})q(x_{k+1}) \\ &\geq u(x_{k+2}) + (x_{k+1} - x_{k+2})q(x_{k+2}) + (x_k - x_{k+1})q(x_{k+2}) \\ &= u(x_{k+2}) + (x_k - x_{k+2})q(x_{k+2}) \end{aligned}$$

where the last inequality follows since q is weakly increasing and $x_k < x_{k+1}$. A similar proof works for $x_{k+2} \to x_k$.

Now, one can use induction. Suppose all types in distance $m\epsilon$ of each other satisfy IC for some positive integer m. Then, do it for $(m+1)\epsilon$. So, pick x_k, x_ℓ in $m\epsilon$ distance and assume $x_k \to x_\ell$ and $x_\ell \to x_{\ell+1}$ hold. We need to show $x_k \to x_{\ell+1}$ holds. This can be done by repeating the earlier proof:

$$\begin{aligned} u(x_k) &\geq u(x_\ell) + (x_k - x_\ell)q(x_\ell) \\ &\geq u(x_{\ell+1}) + (x_\ell - x_{\ell+1})q(x_{\ell+1}) + (x_k - x_\ell)q(x_\ell) \\ &\geq u(x_{\ell+1}) + (x_\ell - x_{\ell+1})q(x_{\ell+1}) + (x_k - x_\ell)q(x_{\ell+1}) \\ &= u(x_{\ell+1}) + (x_k - x_{\ell+1})q(x_{\ell+1}) \end{aligned}$$

where the last inequality follows since q weakly increasing and $x_k < x_\ell < x_{\ell+1}$.

(c) Suppose q is weakly increasing. Show that if $x_k \to x_{k+1}$ binds (that is an equality), then $x_{k+1} \to x_k$ holds. Similarly, if $x_{k+1} \to x_k$ binds, then $x_k \to x_{k+1}$ holds. **(5 marks)**

Answer. Suppose $x_k \to x_{k+1}$ binds. Then,

$$u(x_k) = u(x_{k+1}) + (x_k - x_{k+1})q(x_{k+1})$$

But then,

$$u(x_{k+1}) - u(x_k) = (x_{k+1} - x_k)q(x_{k+1}) \geq (x_{k+1} - x_k)q(x_k)$$

where the inequality holds since $x_{k+1} > x_k$ and q is weakly increasing. Hence,

$x_{k+1} \to x_k$.

Similarly, suppose $x_{k+1} \to x_k$ binds. Then,

$$u(x_{k+1}) = u(x_k) + (x_{k+1} - x_k)q(x_k)$$

But then,

$$u(x_k) - u(x_{k+1}) = (x_k - x_{k+1})q(x_k) \geq (x_k - x_{k+1})q(x_{k+1})$$

where the inequality follows since $x_k < x_{k+1}$ and q is weakly increasing.

(d) Show that for any IC mechanism, if IR holds for x_0, it holds for all x_k. Use this to show that any optimal mechanism must set $u(x_0) = 0$. **(5 marks)**

Answer. u is weakly increasing: $u(x_{k+1}) \geq u(x_k) + (x_{k+1} - x_k)q(x_k) \geq u(x_k)$, where the last inequality follows since $x_{k+1} \geq x_k$. Hence, $u(x_0) \geq 0$ implies $u(x_k) \geq 0$ for all x_k.

Now, assume for contradiction $u(x_0) > 0$. Then, define $p'(x) := p(x) + u(x_0)$. Since (q, p) is IC, we have (q, p') is also IC. New utility is $u'(x) = u(x) - u(x_0) \geq u(x_0) - u(x_0) = 0$, where the inequality follows since u is weakly increasing. Hence, IR holds for (q, p'). But $p'(x) > p(x)$ for all x implies (q, p') is a new IC and IR mechanism with higher expected revenue, contradicting optimality of (q, p).

3. Consider the two-sided matching problem with equal no of men and women. Suppose each man ranks woman w_1 the lowest and woman w_2 the second lowest at a preference profile. Show that in every stable matching, the men matched to w_1 and w_2 are the same. **(6 marks)**

Answer. We already know from rural hospital theorem, in every stable matching w_1 is matched to say m_1. Remove, (m_1, w_1) from economy and consider the reduced problem. A stable matching in the original problem when we remove (m_1, w_1) matching must give a stable matching of this reduced problem. As a result, applying the rural hospital theorem again, we get in all stable matchings w_2 is matched to the same men.

4. In the combinatorial auctions model, a seller is selling two goods a and b. It incurs

cost for selling goods:

$$c(\{a\}) = 5, c(\{b\}) = 8, c(\{a,b\}) = 10$$

Depending on which goods are sold the cost is incurred. If both goods are sold, a cost of 10 is incurred. These costs are common knowledge among buyers.

There are two buyers and their values of goods are given:

$$v_1(\{a\}) = 6, v_1(\{b\}) = 9, v_1(\{a,b\}) = 12$$
$$v_2(\{a\}) = 9, v_2(\{b\}) = 8, v_2(\{a,b\}) = 15$$

These values are private information. The seller wishes to use an "efficient" allocation and a DSIC mechanism. Efficiency means an allocation rule that maximizes value of buyers minus the cost of selling. What will be the allocation and payment at this profile of valuations by using an analogue of the VCG mechanism (with this new definition of efficiency)? What is the payoff of the seller? **(8 marks)**

Answer. Efficiency: $\{a\}$ goes to 2 and $\{b\}$ goes to 1 with a total surplus of $18-10=8$. Marginal contribution of 1 is 3 and 2 is 6. Hence, payments are respectively $9-3=6$ and $9-6=3$. Seller payoff is $10-9=-1$.