Assignment 5

Due date. 25 October, 2018.

1. Consider the max-flow problem in directed graphs (we studied in the first part of the course). It is defined by a directed graph with two vertices s and t. Each edge (i, j) has a capacity $c(i, j)$. So, the problem is described by the graph (N, E) and capacity constraints c. A flow $f : E \to \mathbb{R}_+$ is feasible if excess flow at every vertex $i \notin \{s, t\}$ is zero and sum of all excess flows is zero. The max-flow problem is to maximize the excess flow at terminal node.

   (a) Formulate the max-flow problem as a linear program.

   (b) Write down its dual.

   (c) Suppose capacities are integers. What can you say about the optimal solution of primal and dual problems? Prove max-flow and min-cut theorem using this.

2. We will prove the maximum matching and minimum vertex cover theorem (for bipartite graphs) using linear programming duality and total unimodularity.

   (a) Formulate the problem of finding a maximum matching as an integer program and argue that its linear relaxation gives integral optimal solution.

   (b) Formulate the problem of finding a minimum vertex cover as an integer program and argue that its linear relaxation gives integral optimal solution.

   (c) Show that these relaxed linear programs are dual of each other. This prove the max-matching equals min-vertex-cover theorem.

3. Convert the following optimization problem into a linear program:

$$
\begin{aligned}
Z = \min\ & |x| + |y| + |z| \\
\text{s.t.}\quad & x + y \leq 1 \\
& 2x + z = 3.
\end{aligned}
$$

4. Consider the following **fractional program**.

$$\max \frac{\sum_{j=1} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta}$$

$$\text{s.t.}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \ \forall\ i \in \{1, \ldots, m\}$$

$$x_j \geq 0 \ \forall\ j \in \{1, \ldots, n\}.$$

Assume that for all feasible x of this linear program $\sum_{j=1}^{n} d_j x_j + \beta > 0$. Show how to solve this as a single linear program.