Locating public bads in neighbouring countries

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1 Introduction

We consider two neighbouring countries represented by the real intervals $[-1, 0]$ and $[0, 1]$. Each country has to locate one public bad within its boundaries, a joint decision resulting in an element of the set $\mathscr{A} = [-1, 0] \times [0, 1]$. For example, consider the coast line shared by India and Pakistan. It is of interest to study how they can jointly decide upon the location of a windmill park for each country along the coast line. The location problem at hand concerns public bads, therefore residents' preferences are considered to be single dipped[1] on $\mathscr{A}$. This means that a resident has a single point, say his dip, in his own country being his worst possible location for such a bad. Further, preference increases with the minimal distance between his dip and each of the locations in a pair in $\mathscr{A}$. Note that only the minimal distance is taken in to account here. So, if at two pairs in $\mathscr{A}$ these distances are equal, then the pairs of locations are indifferent to the resident.

We assume that there are finitely many residents (say n) in the two countries combined. In this situation, a social choice function will take the dips of all the residents of the two countries (called a profile) as input, and give a pair in $\mathscr{A}$ as output. In this paper we characterise the class of all social choice functions that simultaneously satisfy strategy proofness, non-corruptibility, Pareto optimality and the far away condition. The notion of strategy proofness is exactly same as introduced by Gibbard (1973) and Satterthwaite (1975). We use a version of Pareto optimality that is stronger than the usual notion. More precisely, we impose Pareto optimality specific to each country.

The non-corruptibility property is based on the similar notion introduced by Ritz (1984). We modify this notion based on the literature on non-bossy condition. This property ensures that after a unilateral deviation of a resident, if he remains indifferent between the old outcome and the new outcome, according to his true preference and also according to his deviated preference, then both of these outcomes must be the same. This means that a unilaterally deviating resident cannot change the outcome and remain unaffected by this change; i.e.; a resident by unilateral deviation cannot make some other residents better off or worse off without affecting himself.

The far away condition is defined in such a way that in case all the agents are indifferent, the rule should select $(-1, 1)$ as the outcome. More precisely, this condition states that if no resident strictly opposes the decision of placing one bad at the extreme end of the corresponding country, irrespective of the location of the other bad, then that bad should be placed at the extreme end. This condition can be thought of as a tie breaking condition that ensures that the outcome remains one of the corner points of $\mathscr{A}$.

We show that the range of a social choice function satisfying these four properties consists of the corner points of $\mathscr{A}$; i.e. ; such a function never places a public bad of any country in the interior of that country. Now we provide an example of a non-dictatorial social choice function that satisfies all the four properties. For simplicity, assume that agents cannot have dips at -0.5 or at 0.5; and the total number of agents (n) is odd. In such a scenario, the rule will select $(-1, 1)$ if there is a decisive coalition for $(-1, 1)$. For simplicity assume that decisiveness of a coalition is determined by simple majority. As we did not assume anonymity, the decisiveness of a coalition may be defined in many different ways also. For example, a coalition could also be decisive if it includes some specific agents. So, in this scenario, the rule will select $(-1, 1)$ if there is a majority of agents who prefer $(-1, 1)$ to all other corner points of $\mathscr{A}$. Now consider the case where there is no decisive coalition for $(-1, 1)$. In such a case, if there are agents with dips in $[-1, -0.5)$, but there are no agents with dips in $(0.5, 1]$, then the rule selects $(0, 1)$. For the symmetrically opposite case where there are no agents with dips in $[-1, -0.5)$, but there are agents with dips in $(0.5, 1]$, the rule selects $(-1, 0)$. In all other cases, the rule will select $(0, 0)$. Note that this

[1] More precisely, this is a single trenched preference. For example, suppose the dip of an agent in country A be at -0.5. Then $\{-0.5\} \times B$ is the trench in his preference.

rule will satisfy all the four condition under the simplified scenario only. But we can generalise this rule to consider all possible cases. The class of rules that satisfy these four properties is characterised on the basis of a family of pairs of coalitions; one coalition consisting of agents who strictly prefer $(-1, 1)$ to all the other corner points of $\mathscr{A}$, the other coalition consisting of agents who are indifferent among all the corner points of $\mathscr{A}$. This family can be described in terms of two properties. The main property is monotonicity, which states that if a pair of coalitions (say (S, T)) is in this family, then, any other pairs of coalitions, which are larger than (S, T), will also be in this family. The notion of a pair of coalition being larger than another pair of coalition is defined later in the paper. The other condition is a technical condition that enforces the stronger Pareto optimality and the far away condition. We show that if there exists a social choice function that satisfies strategy proofness, non-corruptibility, Pareto optimality and the far away condition, then we can construct a family of pairs of coalitions satisfying these two properties. On the other hand, given any arbitrary family of pairs of coalitions satisfying these two properties, we construct a social choice function that satisfies strategy proofness, non-corruptibility, Pareto optimality and the far away condition.

This is a positive result as compared to the seminal impossibility theorem of Gibbard (1973) and Satterthwaite (1975) which says that if there are three or more alternatives, then it is impossible to find a non-dictatorial social choice function which is also strategy proof and pareto optimal. One way out from this impossibility result is to consider restricted preference domains. One possible restricted domain is the single dipped preference domain. Peremans and Storcken (1999) has shown the equivalence between individual and group strategy proofness in sub domains of single dipped preferences. Manjunath (2009) have characterised the class of all non dictatorial, strategy proof and Pareto optimal social choice functions when preferences are single dipped over an interval. Barberà, Berga and Moreno (2012) has characterised the class of all non dictatorial, group strategy proof and Pareto optimal social choice functions when preferences are single dipped over on a line. But there are impossibility results in this domain as well. Öztürk, Peters and Storcken (2012) has shown two impossibility results. They have shown that there does not exist any non dictatorial social choice function that is strategy proof and Pareto optimal when preferences are single dipped over a disk, and over convex polytopes[2]. Also there are a lot of literature regarding auction designs that determines the location of a noxious facility. They assume side payments which we do not. The non dictatorial rule we described is similar to the rule described by Manjunath and Barberà et al.

This paper is organised as follows. In Section 2, we formally introduce our model. In Section 3, we introduced notations that will be used throughout this paper. In Section 4, we formally define the four properties. In Section 5, we prove that the range of a social choice function satisfying these four properties consists of the corner points of $\mathscr{A}$. In Section 6, we characterise the class of social choice functions satisfying these four properties in terms of families of pairs of coalitions. In Section 7, we show, by means of examples of social choice functions, that the four properties are independent of each other. In Section 8, we conclude by perturbing our model in three aspects and try to see whether our results still holds or not.

[2]In this case, for some particular polytopes, there exists non dictatorial, strategy proof, Pareto optimal social choice functions.

2 Model

Let $A = [-1, 0]$ and $B = [0, 1]$.
Set of alternatives - $\mathscr{A} = A \times B$. We shall denote any alternative $(\alpha, \beta) \in \mathscr{A}$ as $\alpha\beta$ if there are no confusion among the two.
Set of agents - N = $N_A \cup N_B$, where,

- N_A is the set of agents in country A and $|N_A| = n_A$
- N_B is the set of agents in country B and $|N_B| = n_B$
- $N_A \cap N_B = \phi$

Each agent $i \in N_c$ has a single dipped preference $R_{x(i)}$ over $\mathscr{A}$ which is characterised by a unique point (dip) $x(i)$ in c (for all $c = A, B$) defined as follows -
Any alternative $(\alpha_1, \beta_1) \in \mathscr{A}$ is at least as good as another alternative $(\alpha_2, \beta_2) \in \mathscr{A}$ according to the dip at $x(i)$ iff

- $\min\{|\alpha_1 - x(i)|, |\beta_1 - x(i)|\} \geq \min\{|\alpha_2 - x(i)|, |\beta_2 - x(i)|\}$

$P_{x(i)}$ denotes the strict part of the preference and $I_{x(i)}$ denotes the indifference part of the preference.
A profile of preferences is defined as follows -

- $z = (z(1), z(2), \ldots, z(n_A + n_B))$, where
  - $z(i) \in A \cup B$ is the dip of agent $i \in N$.

Note that $z \in \mathscr{R} := A^{n_A} \times B^{n_B}$. For a profile z, let $z|_{N-S}$ be the dips of all agents in $N - S$ for any $S \subsetneq N$.

3 Notation

A **social choice function** f is defined as follows -
$f : \mathscr{R} \longrightarrow \mathscr{A}$
Define two functions $f_1 : \mathscr{R} \longrightarrow A$ and $f_2 : \mathscr{R} \longrightarrow B$ such that -

- For any $z \in \mathscr{R}, f(z) = (\alpha, \beta) \Leftrightarrow f_1(z) = \alpha$ and $f_2(z) = \beta$

Take any two points $x, y \in \mathbb{R}$. Define $x_y \in \mathbb{R}$ as $x_y := 2y - x$. In other words, for any two points $x, y \in \mathbb{R}$, the point $x_y \in \mathbb{R}$ is defined to be that unique point on the other side of y compared to x, such that the distance between x and y is same as that between y and x_y. Note that, given any $x, y \in A \cup B$, x_y may not belong to $A \cup B$.

Take two points $x, y \in A \cup B$. Define $\mu(x, y) := \frac{x+y}{2}$ as the middle point of the line joining x and y.
A social choice function f is **monotone** if $f(z) = f(z^{'})$ for any two profiles z and $z^{'}$, such that for some $i \in N$
$z^{'}(i) \leq z(i) \leq f_1(z) \leq f_2(z)$ or
$f_1(z) \leq z(i) \leq z^{'}(i) \leq \mu(f_1(z), f_2(z)) \leq f_2(z)$ or
$f_1(z) \leq \mu(f_1(z), f_2(z)) \leq z^{'}(i) \leq z(i) \leq f_2(z)$ or
$f_1(z) \leq f_2(z) \leq z(i) \leq z^{'}(i)$ holds,
while $z^{'}(j) = z(j)$ for all $j \neq i$.
So, f is monotone if outcomes do not change when agents increase their minimal distance to these bads while not changing the minimum one and not jumping across any of these bads.

Remark 1. *In this context, the lower contour set of an agent i with dip at $z(i)$, given an out come $(\alpha, \beta) \in \mathscr{A}$ is defined as $L((\alpha, \beta), z(i)) = \{(\gamma, \delta) \in \mathscr{A} : (\alpha, \beta) R_{z(i)} (\gamma, \delta)\}$, and a social choice function f is said to be Maskin monotone if $f(z) = f(z^{'})$ for all $z, z^{'} \in \mathscr{R}$ such that $L(f(z), z(i)) \subseteq L(f(z), z^{'}(i))$ for all $i \in N$.*
The monotonicity condition introduced in this paper is weaker than Maskin monotonicity. For example consider a profile z, such that there exists atleast one agent $i \in N_A$ with $z(i) = -1$. Suppose for an arbitrary social choice function f, we have $f(z) = (\alpha, \beta)$, where $\alpha \in (-1, 0)$ and $\beta \in [0, 1]$. Now consider another profile $z^{'}$ such that $z^{'}(i) = \mu(-1, \alpha)$ but $z^{'}(j) = z(j)$ for all $j \in N - \{i\}$. Notice that in this deviation, agent i decreases the distance to his closest bad. So imposing our monotonicity condition on f, do not put any restriction on $f(z^{'})$. Now notice that $L(f(z), z(i)) = \{(\gamma, \delta) \in \mathscr{A} : \gamma \in [-1, \alpha],\ \delta \in [0, 1]\} = L(f(z), z^{'}(i))$. For all $j \in N - \{i\}$, we have $L(f(z), z(j)) = L(f(z), z^{'}(j))$. So, if we assume that f satisfies Maskin monotonicity, then we get $f(z^{'}) = f(z)$.

Remark 2. *Suppose f is a monotone social choice function and $f(z) = (\alpha, \beta)$ for some profile $z \in \mathscr{R}$. Without loss of generality, assume that $\mu(\alpha, \beta) \in A$. Now consider another profile $z^* = (-1^S, \mu(\alpha, \beta)^{N_A - S}, 0^T, 1^{N_B - T}) \in \mathscr{R}$, where*

- $S \coloneqq \{i \in N_A : z(i) \leq \alpha\}$.
- $T \coloneqq \{i \in N_B : z(i) \leq \beta\}$.

As f is monotone, it follows that $f(z^*) = (\alpha, \beta)$.

4 Properties

For a profile $z \in \mathscr{R}$, we define another profile $z^{'} \in \mathscr{R}$ as a unilateral deviation from z if $z^{'} = (z^{'}(i), z|_{N - \{i\}})$ for some agent $i \in N$, where $z(i) \neq z^{'}(i)$. We denote this agent i as the unilaterally deviating agent.

1. **Strategy proofness :** A social choice function f is strategy proof if for all profile $z \in \mathscr{R}$ and for any unilateral deviation from z, say $z^{'} \in \mathscr{R}$, with any unilaterally deviating agent $i \in N$; we have $f(z) R_{z(i)} f(z^{'})$.

2. **Non-corruptibility :** We say that f is non-corruptible if for all profile $z \in \mathscr{R}$ and any unilateral deviation from z, say $z^{'} \in \mathscr{R}$, with any unilaterally deviating agent $i \in N$ we have that $f(z) = f(z^{'})$ whenever both $f(z) I_{z(i)} f(z^{'})$ and $f(z) I_{z^{'}(i)} f(z^{'})$.

3. **Country specific Pareto optimality :**
A social choice function f is Pareto optimal for country A if for any profile $z \in \mathscr{R}$, there does not exist any $\alpha \in A$ such that, $(\alpha, f_2(z)) R_{z(i)} f(z)$ holds for all $i \in N_A$ with $(\alpha, f_2(z)) P_{z(k)} f(z)$ for at least one $k \in N_A$.
A social choice function f is Pareto optimal for country B if for any profile $z \in \mathscr{R}$, there does not exist any $\beta \in B$ such that, $(f_1(z), \beta) R_{z(j)} f(z)$ holds for all $j \in N_B$ with $(f_1(z), \beta) P_{z(k)} f(z)$ for at least one $k \in N_B$.
A social choice function f is country specific Pareto optimal if it is both Pareto optimal for country A and Pareto optimal for country B.

4. **Far away condition :** We say that f satisfies far away condition if for all profiles z
   (a) If $(f_1(z), 1) R_{z(i)} f(z)$ for all $i \in N$, then $f_2(z) = 1$
   (b) If $(-1, f_2(z)) R_{z(i)} f(z)$ for all $i \in N$, then $f_1(z) = -1$

5 Implication of Strategy Proofness, Non-corruptibility, Country Specific Pareto Optimality and the Far Away Condition

Fix a profile z. Let f be a social choice function and suppose $f(z) = (\alpha, \beta)$. We are going to prove that if a social choice function satisfies strategy proofness, non-corruptibility, country specific Pareto optimality and the far away condition, then the outcome of that function must be one of the corner points of $\mathscr{A}$. Formally,

Theorem 1. *If f is a strategy proof, country specific Pareto optimal, non-corruptible social choice function that satisfies the far away condition, then for any profile $z \in \mathscr{R}$,*
$f(z) \in \{(-1, 1), (0, 0), (-1, 0), (0, 1)\}$.

We prove this theorem with the help of the following four lemmas.
The first lemma shows that monotonicity is implied by strategy proofness and non-corruptibility.

Lemma 1. *If f is strategy proof and non-corruptible, then f is monotone.*

Proof. Let $z^{'}$ be an unilateral deviation from z and suppose the unilaterally deviating agent is i. That is $z^{'} = (z^{'}(i), z|_{N-\{i\}})$ and either $z^{'}(i) < z(i) \leq \alpha \leq \beta$ or $\alpha \leq z(i) < z^{'}(i) \leq \mu(\alpha, \beta) < \beta$. It is sufficient to prove $f(z) = f(z^{'})$. Let $f(z^{'}) = (\alpha^{'}, \beta^{'})$. Now consider the two cases as follows

- **Case 1 :** $z^{'}(i) < z(i) \leq \alpha \leq \beta$
  Note that, in this case, $|\alpha - z^{'}(i)| < |\beta - z^{'}(i)|$. Then strategy proofness implies that

  1. Either $\alpha^{'}$ or $\beta^{'}$ must be in $[\alpha_{z(i)}, \alpha]$. This corresponds to strategy proofness restriction for deviation from z to $z^{'}$.
  2. Neither $\alpha^{'}$ nor $\beta^{'}$ can be in $(\alpha_{z^{'}(i)}, \alpha)$. This corresponds to strategy proofness restriction for deviation from $z^{'}$ to z.

  Note that, since $z^{'}(i) < z(i) \leq \alpha \leq \beta$, so we can say that $[\alpha_{z(i)}, \alpha) \subsetneq (\alpha_{z^{'}(i)}, \alpha)$. So this yields that either $\alpha^{'} = \alpha$ or $\beta^{'} = \alpha$.

  - **Sub Case 1 :** $\alpha^{'} = \alpha$
    Then, using the condition from strategy proofness, we can say that $\beta^{'} \in [0, 1] - (\alpha_{z^{'}(i)}, \alpha)$. This means that $\alpha^{'}$ is the closest to both $z(i)$ and $z^{'}(i)$ in $\{\alpha^{'}, \beta^{'}\}$. Since, we have assumed that α is the closest bad to $z(i)$, so we get $f(z) I_{z(i)} f(z^{'})$ and $f(z) I_{z^{'}(i)} f(z^{'})$ and non-corruptibility implies $f(z) = f(z^{'})$.
  - **Sub Case 2 :** $\beta^{'} = \alpha$
    Then, using the conditions from strategy proofness, we can say that $\alpha^{'} \in [-1, 0] - (\alpha_{z^{'}(i)}, \alpha)$. This means that $\beta^{'}$ is the closest to both $z(i)$ and $z^{'}(i)$ in $\{\alpha^{'}, \beta^{'}\}$. Since, we have assumed that α is the closest bad to $z(i)$, so we get $f(z) I_{z(i)} f(z^{'})$ and $f(z) I_{z^{'}(i)} f(z^{'})$ and non-corruptibility implies $f(z) = f(z^{'})$.

- **Case 2 :** $\alpha \leq z(i) < z^{'}(i) \leq \mu(\alpha, \beta) < \beta$
  Note that, in this case, $|\alpha - z^{'}(i)| \leq |\beta - z^{'}(i)|$. Then strategy proofness implies that

  1. Either $\alpha^{'}$ or $\beta^{'}$ must be in $[\alpha, \alpha_{z(i)}]$. This corresponds to strategy proofness restriction for deviation from z to $z^{'}$.

2. Neither $\alpha^{'}$ nor $\beta^{'}$ can be in $(\alpha, \alpha_{z^{'}(i)})$. This corresponds to strategy proofness restriction for deviation from $z^{'}$ to z.

Note that, since $\alpha \leq z(i) < z^{'}(i) \leq \mu(\alpha, \beta) < \beta$, so we can say that $(\alpha, \alpha_{z(i)}] \subsetneq (\alpha, \alpha_{z^{'}(i)})$. So this yields that either $\alpha^{'} = \alpha$ or $\beta^{'} = \alpha$.

- **Sub Case 1 :** $\alpha^{'} = \alpha$
  Then, using the condition from strategy proofness, we can say that $\beta^{'} \in [0, 1] - (\alpha, \alpha_{z^{'}(i)})$. This means that $\alpha^{'}$ is the closest to both $z(i)$ and $z^{'}(i)$ in $\{\alpha^{'}, \beta^{'}\}$. Since, we have assumed that α is the closest bad to $z(i)$, so we get $f(z)I_{z(i)}f(z^{'})$ and $f(z)I_{z^{'}(i)}f(z^{'})$ and non-corruptibility implies $f(z) = f(z^{'})$.
- **Sub Case 2 :** $\beta^{'} = \alpha$
  Then, using the conditions from strategy proofness, we can say that $\alpha^{'} \in [-1, 0] - (\alpha, \alpha_{z^{'}(i)})$. This means that $\beta^{'}$ is the closest to both $z(i)$ and $z^{'}(i)$ in $\{\alpha^{'}, \beta^{'}\}$. Since, we have assumed that α is the closest bad to $z(i)$, so we get $f(z)I_{z(i)}f(z^{'})$ and $f(z)I_{z^{'}(i)}f(z^{'})$ and non-corruptibility implies $f(z) = f(z^{'})$.

□

From here onwards, we shall assume that f satisfies strategy proofness, non-corruptibility, country specific Pareto optimality and the far away condition.

The second lemma shows that if one of the two bads is located at 0, then the other one cannot be located at an interior point of its country.

Lemma 2. $\alpha = 0$ *implies* $\beta \in \{0, 1\}$ *and* $\beta = 0$ *implies* $\alpha \in \{-1, 0\}$.

Proof. Due to symmetry, it is sufficient to prove that $\alpha = 0$ implies $\beta \in \{0, 1\}$.
Suppose $\alpha = 0$ and $\beta > 0$. It is sufficient to prove that $\beta = 1$. Because of monotonicity, we may assume that $z(i) = -1$ for all agents $i \in N_A$ and $z(j) \geq \mu(0, \beta)$ for all agents $j \in N_B$. If there were agents $j \in N_B$ such that $z(j) > \mu(0, \beta)$, then we get, $(0, 0)P_{z(j)}(\alpha, \beta)$, which is a violation of country specific Pareto optimality. So $z(j) = \mu(0, \beta)$ for all agents $j \in N_B$. Hence the far away condition yields $\beta = 1$. □

The third lemma shows that if one of the two bads is located at the extreme end, then the other one cannot be located at an interior point of its country.

Lemma 3. $\alpha = -1$ *implies* $\beta \in \{0, 1\}$ *and* $\beta = 1$ *implies* $\alpha \in \{-1, 0\}$.

Proof. Due to symmetry, it is sufficient to prove that $\alpha = -1$ implies $\beta \notin (0, 1)$.
Suppose $\alpha = -1$, but to the contrary, $\beta \in (0, 1)$.
Notice that, in this case, $\mu(-1, \beta) \in (-0.5, 0)$.
Using Remark 2, we may assume that - $z = (\mu(-1, \beta)^{N_A}, 0^T, 1^{N_B - T})$, where $T \subseteq N_B$.
Notice that -

- $N_B - T \neq \phi$
  If $N_B - T = \phi$, then we would get that $T = N_B$ and then at the profile z, we get $(-1, 1)P_0(-1, \beta)$. This violates country specific Pareto optimality.
- $T \neq \phi$
  If $T = \phi$, we get $N_B - T = N_B$ and then at the profile z, we get $(-1, 0)P_1(-1, \beta)$. This violates country specific Pareto optimality.

Consider profile $z^* \in \mathscr{R}$, where $z^*(i) = z(i) = \mu(-1, \beta)$ for all $i \in N_A$, arbitrary otherwise.

Claim 1. $f_1(z^*) = -1$.

Proof. Suppose $f_1(z^*) \neq -1$; i.e. ; $f_1(z^*) > -1$. Now consider the following cases.

1. $|-1 - \mu(-1, \beta)| < |\mu(-1, \beta) - f_2(z^*)|$.
   In this case, $f_2(z^*) \in (-1_{\mu(-1,\beta)}, 1]$. Then $(-1, f_2(z^*))P_{z^*(i)}f(z^*)$ for any $f_2(z^*) \in (-1_{\mu(-1,\beta)}, 1]$ for all $i \in N_A$. This is a violation of Pareto optimality for country A.

2. $|-1 - \mu(-1, \beta)| \geq |\mu(-1, \beta) - f_2(z^*)|$.
   In this case, $f_2(z^*) \in [0, -1_{\mu(-1,\beta)}]$. Then $(-1, f_2(z^*))R_{z^*(i)}f(z^*)$ for any $f_2(z^*) \in (-1_{\mu(-1,\beta)}, 1]$ for all $i \in N$. This is a violation of the far away condition.

Combining these two cases, we conclude the proof of Claim 1. □

Now we are going to construct $z^{'} = (\mu(-1, \beta)^{N_A}, \mu(0, \beta)^T, \mu(\beta, 1)^{N_B - T})$ in two ways.

- In this method, first we are going to construct $z_1 = (\mu(-1, \beta)^{N_A}, \mu(0, \beta)^T, 1^{N_B - T})$ from z by moving all the agents in T from 0 to $\mu(0, \beta)$. Let $f(z_1) = (\alpha_1, \beta_1)$. We know from Claim 1, that $\alpha_1 = -1$. Then strategy proofness implies that $\beta_1 \in \{0, \beta\}$.
  Suppose $\beta_1 = \beta$. But this violates country specific Pareto optimality as $(-1, 0)P_1(-1, \beta)$ and $(-1, \beta)I_{\mu(0,\beta)}(-1, 0)$. So $\beta_1 = 0$ and $f(z_1) = (-1, 0)$.
  Next we move all the agents in $N_B - T$ from 1 in the profile z_1 to $\mu(\beta, 1)$ to get $z^{'}$. Let $f(z^{'}) = (\alpha^{'}, \beta^{'})$. From Claim 1, we know that $\alpha^{'} = -1$. Then strategy proofness implies $\beta^{'} = 0$. So we get $f(z^{'}) = (-1, 0)$.

- In this method, first we are going to construct $z_2 = (\mu(-1, \beta)^{N_A}, 0^T, \mu(\beta, 1)^{N_B - T})$ from z by moving all the agents in $N_B - T$ from 1 to $\mu(\beta, 1)$. Let $f(z_2) = (\alpha_2, \beta_2)$. From Claim 1, we know that $\alpha_2 = -1$. Then strategy proofness implies $\beta_2 \in \{\beta, 1\}$.
  Suppose $\beta_2 = \beta$. But, this violates the far away condition as $(-1, 1)R_{z_2(i)}(-1, \beta)$ for all $i \in N$. So $\beta_2 = 1$ and $f(z_2) = (-1, 1)$.
  Now we move all the agents in T from 0 in the profile z_2 to $\mu(0, \beta)$ to get $z^{'}$. Suppose $f(z^{'}) = (\alpha^{'}, \beta^{'})$. From Claim 1, we know $\alpha^{'} = -1$. Then strategy proofness would imply that $\beta^{'} = 1$. So, we get $f(z^{'}) = (-1, 1)$.

These two ways of construction yields contradictory outcome. □

The fourth lemma shows that none of the two bads can be in the interior of their respective countries simultaneously.

Lemma 4. $(\alpha, \beta) \notin (-1, 0) \times (0, 1)$

Proof. Suppose not. So, $\alpha \in (-1, 0)$ and $\beta \in (0, 1)$. Without loss of generality, assume that $\mu(\alpha, \beta) \in A$. Using Remark 2, we may assume that, $z = (-1^S, \mu(\alpha, \beta)^{N_A - S}, 0^T, 1^{N_B - T})$, where $S \subseteq N_A$ and $T \subseteq N_B$.
Note that -

- $S \neq \phi$.
  If $S = \phi$, then $(-1, \beta)R_{\mu(\alpha,\beta)}(\alpha, \beta)$, $(-1, \beta)R_0(\alpha, \beta)$ and $(-1, \beta)R_1(\alpha, \beta)$. So $f(z) = (\alpha, \beta)$ would violate the far away condition.

- $N_B - T \neq \phi$.
  If $N_B - T = \phi$, then $(\alpha, 1)R_{\mu(\alpha,\beta)}(\alpha, \beta)$, $(\alpha, 1)R_0(\alpha, \beta)$ and $(\alpha, 1)R_{-1}(\alpha, \beta)$. So $f(z) = (\alpha, \beta)$ would violate the far away condition.

- $N_A - S \neq \phi$
  If $N_A - S = \phi$, then we get $(0, \beta)P_{-1}(\alpha, \beta)$. So $f(z) = (\alpha, \beta)$ would violate Pareto optimality for country A.

- $T \neq \phi$
  If $T = \phi$, then we get $(\alpha, 0)P_1(\alpha, \beta)$. So $f(z) = (\alpha, \beta)$ would violate Pareto optimality for country B.

Now, from z, we are going to construct the following profiles-

1. $z_1 = (\mu(-1, \alpha)^S, \mu(\alpha, \beta)^{N_A - S}, 0^T, \mu(\beta, 1)^{N_B - T})$.
2. $z_2 = (-1^S, \mu(\alpha, 0)^{N_A - S}, \mu(0, \beta)^T, 1^{N_B - T})$.
3. $z_3 = (\mu(-1, \alpha)^S, \mu(\alpha, 0)^{N_A - S}, \mu(0, \beta)^T, \mu(\beta, 1)^{N_B - T})$

Firstly, we shall construct z_1 and z_2 from z. Then we shall construct z_3 from z_1 and z_2, which will lead to a contradiction with respect to outcome at profile z_3.
Construction of z_1 from z -
We are going to prove that $f(z_1) = (-1, 1)$. Consider the following two methods.

- Method 1 :
  First move all agents in S from -1 to $\mu(-1, \alpha)$.
  Let $z_1^1 = (\mu(-1, \alpha)^S, \mu(\alpha, \beta)^{N_A - S}, 0^T, 1^{N_B - T})$, and let $f(z_1^1) = (\alpha_1^1, \beta_1^1)$. Due to strategy proofness, we can say that $\alpha_1^1 \in \{-1, \alpha\}$. Now if $\alpha_1^1 = \alpha$, then we get $(\alpha, \beta)I_{-1}(\alpha, \beta_1^1)$ for all $\beta_1^1 \in [0, 1]$, and $(\alpha, \beta)I_{\mu(-1,\alpha)}(\alpha, \beta_1^1)$ for all $\beta_1^1 \in [0, 1]$. So, non-corruptibility would imply that $f(z_1^1) = (\alpha, \beta)$. But in the profile z_1^1, we have $(-1, \beta)R_{z_1^1(i)}(\alpha, \beta)$ for all $i \in N$. So $f(z_1^1) = (\alpha, \beta)$ violates the far away condition. So, $\alpha_1^1 = -1$. By Lemma 3 we get $\beta_1^1 \in \{0, 1\}$. So, $f(z_1^1) = (-1, \beta_1^1)$, where $\beta_1^1 \in \{0, 1\}$.
  Now we move all agents in $N_B - T$ from 1 in z_1^1 to $\mu(\beta, 1)$ to get the profile z_1. Let $f(z_1) = (\alpha_1, \beta_1)$. Now if $\beta_1^1 = 1$, then due to monotonicity, we have $f(z_1) = (-1, 1)$. If $\beta_1^1 = 0$, then strategy proofness implies $\beta_1 = 0$. Then we get $(-1, 0)I_1(\alpha_1, 0)$ for all $\alpha_1 \in [-1, 0]$, and $(-1, 0)I_{\mu(\beta,1)}(\alpha_1, 0)$ for all $\alpha_1 \in [-1, 0]$. So, non-corruptibility implies $f(z_1) = (-1, 0)$.
  So, in this method, we get $f(z_1) \in \{(-1, 0), (-1, 1)\}$.

- Method 2 :
  In this method, first move all the agents in $N_B - T$ from 1 to $\mu(\beta, 1)$. Then we move all the agents in S from -1 to $\mu(-1, \alpha)$. Because of similar reasons as described in method 1, we can say that $f(z_1) \in \{(0, 1), (-1, 1)\}$.

So, combining these two methods, we conclude that $f(z_1) = (-1, 1)$.

Construction of z_2 from z -
We are going to prove that $f(z_2) = (0, 0)$ in two steps as follows -
First move all the agents in T from 0 to $\mu(0, \beta)$.
Let $z_2^1 = (-1^S, \mu(\alpha, \beta)^{N_A - S}, \mu(0, \beta)^T, 1^{N_B - T})$, and suppose $f(z_2^1) = (\alpha_2^1, \beta_2^1)$. Strategy proofness implies one of these two bads must be in $\{\beta\} \cup [-\beta, 0]$. Now we consider the following cases based on the location of α_2^1 and β_2^1 in $\{\beta\} \cup [-\beta, 0]$.

1. Suppose $\beta_2^1 = \beta$.
   Then, for all $\alpha_2^1 \in [-1, 0]$ we get $(\alpha_2^1, 0) I_{z_2^1(i)} (\alpha_2^1, \beta)$ for all $i \in T$ and $(\alpha_2^1, 0) P_{z_2^1(i)} (\alpha_2^1, \beta)$ for all $i \in N_B - T$. This violates Pareto optimality for country B.

2. Suppose $\beta_2^1 = 0$.
   Using Lemma 2, we can say that, $\alpha_2^1 \in \{0, -1\}$. So Pareto optimality for country A implies $\alpha_2^1 = 0$.

3. Suppose $\alpha_2^1 = 0$.
   Using Lemma 2, we can say that, $\beta_2^1 \in \{0, 1\}$. So, Pareto optimality for country B implies $\beta_2^1 = 0$.

4. Suppose $\alpha_2^1 \in [-\beta, 0)$.
   Suppose $\beta_2^1 = 0$. As $\alpha \leq -\beta$, so for any $\alpha_2^1 \in [-\beta, 0)$, we have $(0, 0) P_{-1} (\alpha_2^1, 0)$ and $(0, 0) R_{\mu(\alpha,\beta)} (\alpha_2^1, 0)$. This violates Pareto optimality for country A. So $\beta_2^1 \in (0, 1]$. Now suppose $\alpha_2^1 \in [0_{\mu(\alpha,\beta)}, 0)$. Then for any $\beta_2^1 \in (0, 1]$, we have $(0, \beta_2^1) P_{-1} (\alpha_2^1, \beta_2^1)$ and $(0, \beta_2^1) R_{\mu(\alpha,\beta)} (\alpha_2^1, \beta_2^1)$. Clearly, this is a violations of Pareto optimality for country A. So $\alpha_2^1 \in [-\beta, 0_{\mu(\alpha,\beta)})$. Now if $\beta_2^1 \in (0, \beta]$, then for any $\alpha_2^1 \in [-\beta, 0_{\mu(\alpha,\beta)})$, we have $(\alpha_2^1, 0) P_1 (\alpha_2^1, \beta_2^1)$ and $(\alpha_2^1, 0) R_{\mu(0,\beta)} (\alpha_2^1, \beta_2^1)$. This is a violation of Pareto optimality for country B. So, ${\beta_2}^1 \in (\beta, 1]$.

So, $f(z_2^1) = (0, 0)$ or $\alpha_2^1 \in [-\beta, 0_{\mu(\alpha,\beta)})$ and $\beta_2^1 \in (\beta, 1]$. Now we will prove that $f(z_2) = (0, 0)$. Now consider the following cases -

1. $f(z_2^1) = (0, 0)$.
   Now we move all the agents in $N_A - S$ from $\mu(\alpha, \beta)$ in z_2^1 to $\mu(\alpha, 0)$ to get the profile z_2. Due to monotonicity, we can say that $f(z_2) = 00$.

2. $f(z_2^1) = (\alpha_2^1, \beta_2^1)$, where $\alpha_2^1 \in [-\beta, 0_{\mu(\alpha,\beta)})$ and $\beta_2^1 \in (\beta, 1]$.
   Notice that, α_2^1 is closest to both $\mu(\alpha, \beta)$ and $\mu(\alpha, 0)$ in $\{\alpha_2^1, \beta_2^1\}$. Now we move all the agents in $N_A - S$ from $\mu(\alpha, \beta)$ in z_2^1 to $\mu(\alpha, 0)$ to get the profile z_2. Let $f(z_2) = (\alpha_2, \beta_2)$. Then strategy proofness implies that one of the bads must be in either $[\alpha_2^1, {\alpha_2^1}_{\mu(\alpha,\beta)}]$ or $\{\alpha_2^1\} \cup [{\alpha_2^1}_{\mu(\alpha,0)}, {\alpha_2^1}_{\mu(\alpha,\beta)}]$ depending on whether $\alpha_2^1 \in [\mu(\alpha, 0), 0_{\mu(\alpha,\beta)})$ or $\alpha_2^1 \in [-\beta, \mu(\alpha, 0))$. Notice that, both of these sets are contained in the interval $[\alpha, \beta]$ as $\alpha \leq -\beta$ and $\mu(\alpha, \beta) \leq 0$. Now if α_2 is in one of these sets, then using country specific Pareto optimality, we can say that $\alpha_2 = 0$. Then using Lemma 2 we get $\beta_2 \in \{0, 1\}$. Then country specific Pareto optimality implies $\beta_2 = 0$. Similar arguments follow for the case when β_2 is in one of these sets. So, in this case also, we get $f(z_2) = (0, 0)$.

Combining all the cases, we can conclude that $f(z_2) = (0, 0)$.

Construction of z_3 -

- From $z_2 = (-1^S, \mu(\alpha, 0)^{N_A - S}, \mu(0, \beta)^T, 1^{N_B - T})$ -
  We know $f(z_2) = (0, 0)$.
  Now we move all agents in S from -1 in profile z_2 to $\mu(-1, \alpha)$. Let this profile be
  $z_2^2 = (\mu(-1, \alpha)^S, \mu(\alpha, 0)^{N_A - S}, \mu(0, \beta)^T, 1^{N_B - T})$. Let $f(z_2^2) = (\alpha_2^2, \beta_2^2)$. Strategy proofness implies $\alpha_2^2 = 0$. Then $(0, 0) I_{-1} (0, \beta_2^2)$ for all $\beta_2^2 \in [0, 1]$ and $(0, 0) I_{\mu(-1,\alpha)} (0, \beta_2^2)$ for all $\beta_2^2 \in [0, 1]$. So non corruptibility implies that $f(z_2^2) = (0, 0)$.
  Now we move all the agents in $N_B - T$ from 1 in the profile z_2^2 to $\mu(\beta, 1)$ to get the profile z_3. Let $f(z_3) = (\alpha_3, \beta_3)$. Strategy proofness implies $\beta_3 = 0$. Then $(0, 0) I_1 (\alpha_3, 0)$ for all $\alpha_3 \in [-1, 0]$ and $(0, 0) I_{\mu(\beta,1)} (\alpha_3, 0)$ for all $\alpha_3 \in [-1, 0]$. So non corruptibility implies that $f(z_3) = (0, 0)$.

- From $z_1 = (\mu(-1,\alpha)^S, \mu(\alpha,\beta)^{N_A-S}, 0^T, \mu(\beta,1)^{N_B-T})$ -
  We know that $f(z_1) = (-1,1)$.
  Now we move all the agents in $N_A - S$ from $\mu(\alpha,\beta)$ in the profile z_1 to $\mu(\alpha,0)$. Let this profile be $z_1^3 = (\mu(-1,\alpha)^S, \mu(\alpha,0)^{N_A-S}, 0^T, \mu(0,\beta)^{N_B-T})$. Let $f(z_1^3) = (\alpha_1^3, \beta_1^3)$. Strategy proofness implies that $\alpha_1^3 = -1$. Lemma 3 implies $\beta_1^3 \in \{0,1\}$. Strategy proofness implies $\beta_1^3 \in [-1_{\mu(\alpha,0)}, 1]$. As $-1_{\mu(\alpha,0)} > 0$, so $f(z_1^3) = (-1,1)$. Now we move all the agents in T from 0 in the profile z_1^3 to $\mu(0,\beta)$ to get the profile z_3 Let $f(z_3) = (\alpha_3, \beta_3)$. Strategy proofness implies that $\beta_3 = 1$. Lemma 3 implies $\alpha_3 \in \{0,-1\}$. Strategy proofness implies $\alpha_3 \in [-1, 1_{\mu(0,\beta)}]$. As $1_{\mu(0,\beta)} < 0$, so $f(z_3) = (-1,1)$.

This method contradicts $f(z_3) = (0,0)$. □

Proof of Theorem 1. Follows from Lemma 2, 3 and 4. □

6 Rules

In this section, we are going to characterise the class of rules that satisfy country specific Pareto optimality, strategy proofness, non-corruptibility and far away condition. Notice that, from Theorem 1 it follows that the set of possible alternatives is $\mathscr{B} = \{-11, -10, 01, 00\}$. Now consider two different profiles z and $z^{'}$ in $\mathscr{R}$. We say that $z|_{\mathscr{B}} = z^{'}|_{\mathscr{B}}$ iff the following four conditions hold.

1. $\{i \in N : z(i) \in [-1, -0.5)\} = \{i \in N : z^{'}(i) \in [-1, -0.5)\}$.
2. $\{i \in N : z(i) \in \{-0.5, 0.5\}\} = \{i \in N : z^{'}(i) \in \{-0.5, 0.5\}\}$.
3. $\{i \in N : z(i) \in (-0.5, 0.5)\} = \{i \in N : z^{'}(i) \in (-0.5, 0.5)\}$.
4. $\{i \in N : z(i) \in (0.5, 1]\} = \{i \in N : z^{'}(i) \in (0.5, 1]\}$.

In other words $z|_{\mathscr{B}} = z^{'}|_{\mathscr{B}}$ implies that the preference of every agent with respect to $\mathscr{B}$ are equal in z and $z^{'}$. Now restricted to $\mathscr{B}$, there are only four possible single dipped preferences, which are given in the following table.

Dips	Preferences
$z(i) \in [-1, -0.5) \Rightarrow i \in N_A$	$00I_{z(i)}01P_{z(i)} - 10I_{z(i)} - 11$
$z(i) \in \{-0.5, 0.5\}$	$00I_{z(i)}01I_{z(i)} - 10I_{z(i)} - 11$
$z(i) \in (-0.5, 0.5)$	$-11P_{z(i)} - 10I_{z(i)}00I_{z(i)}01$
$z(i) \in (0.5, 1] \Rightarrow i \in N_B$	$00I_{z(i)} - 10P_{z(i)}01I_{z(i)} - 11$

Lemma 5. *$f(z) = f(z^{'})$ for all social choice functions f that satisfies all the four properties and for all $z, z^{'} \in \mathscr{R}$, with $z|_{\mathscr{B}} = z^{'}|_{\mathscr{B}}$.*

Proof. Suppose $f(z) \neq f(z^{'})$. Without loss of generality, let $z^{'}$ be a unilateral deviation of an agent $i \in N$ from z. Then we have either $f(z)P_{z(i)}f(z^{'})$, or $f(z^{'})P_{z(i)}f(z)$, or $f(z)I_{z(i)}f(z^{'})$. Among these cases, $f(z^{'})P_{z(i)}f(z)$ violates strategy proofness for the deviation from z to $z^{'}$. Since $z|_{\mathscr{B}} = z^{'}|_{\mathscr{B}}$ and $f(z) \in \mathscr{B}$ for all $z \in \mathscr{R}$, so we can say that

- $f(z)P_{z(i)}f(z^{'}) \Rightarrow f(z)P_{z^{'}(i)}f(z^{'})$.
- $f(z)I_{z(i)}f(z^{'}) \Rightarrow f(z)I_{z^{'}(i)}f(z^{'})$.

Now $f(z)P_{z^{'}(i)}f(z^{'})$ violates strategy proofness for the deviation from $z^{'}$ to z. Also since $f(z)I_{z(i)}f(z^{'})$ and $f(z)I_{z^{'}}f(z^{'})$ holds, so $f(z) \neq f(z^{'})$ violates the non-corruptibility condition. $\square$

Remark 3. *Any one of these four preferences consists of at most two indifference classes. So, the single dipped preferences restricted to $\mathscr{B}$ constitutes a dichotomous preference domain.*

We are going to prove that the class of all social choice functions that satisfies the four properties are voting between -11 and 00 for all cases except when there is no winning coalition for -11 but selecting 00 as the outcome would violate the far away condition. In these cases the rule selects, because of the far away condition, either -10 or 01.

We define $\mathscr{W} \subset 2^N \times 2^N$ as decisive if it satisfies the following properties.

1. Monotonicity : Consider any $(U, V) \in 2^N \times 2^N$, if $(U, V) \in \mathscr{W}$, then $(U^{'}, V^{'}) \in \mathscr{W}$ for any other $(U^{'}, V^{'}) \in 2^N \times 2^N$, such that $U \subseteq U^{'}$ and $U \cup V \subseteq U^{'} \cup V^{'}$.
2. Boundary Conditions :
   (a) $(\phi, N) \in \mathscr{W}$.
   This property, along with monotonicity of $\mathscr{W}$ implies that
   $(X, N - X) \in \mathscr{W}$ for all $X \subseteq N$.
   (b) $(U, V) \notin \mathscr{W}$ for all $(U, V) \in 2^N \times 2^N$ such that -
      - Either $N_A \subseteq U \cup V$, $U \cap N_B = \phi$, $N_B \nsubseteq V$.
      - Or $N_B \subseteq U \cup V$, $U \cap N_A = \phi$, $N_A \nsubseteq V$.

   This property, along with monotonicity of $\mathscr{W}$ implies that
   $(\phi, X) \notin \mathscr{W}$ for all $X \subsetneq N$.
3. Disjoint : If $(U, V) \in \mathscr{W}$, then $U \cap V = \phi$

Let f be any social choice function. We define a pair $(X, Y) \in 2^N \times 2^N$ as a winning pair if there exists a profile $z \in \mathscr{R}$ such that -

- $X = \{i \in N : z(i) \in (-0.5, 0.5)\}$.
- $Y = \{i \in N : z(i) \in \{-0.5, 0.5\}\}$.
- $f(z) = -11$

We define the set of all winning pairs in $2^N \times 2^N$ as

- $\mathscr{W}_f \coloneqq \{(X, Y) \in 2^N \times 2^N : (X, Y) \text{ is a winning pair } \}$.

Lemma 6. *Let f satisfy strategy proofness, non-corruptibility, country specific Pareto optimality and the far away condition. Then $\mathscr{W}_f$ is decisive.*

Proof. Let $z, z^{'}$ be profiles such that $\{i \in N : z(i) \in (-0.5, 0.5)\} = \{i \in N : z^{'}(i) \in (-0.5, 0.5)\}$ and $\{i \in N : z(i) \in \{-0.5, 0.5\}\} = \{i \in N : z^{'}(i) \in \{-0.5, 0.5\}\}$. Then $z|_{\mathscr{B}} = z^{'}|_{\mathscr{B}}$, and as f satisfies all the four properties, so from Lemma 5, $f(z) = f(z^{'})$. So, it follows that $\mathscr{W}_f$ well defined.
Now we show that $\mathscr{W}_f$ satisfies the following properties -

- Proof of monotonicity -

  Proof. Let $(X, Y) \in \mathscr{W}_f$ and a underlying profile for the pair (X, Y) be z. Consider another pair $(X^{'}, Y^{'}) \in 2^N \times 2^N$ such that $X \subseteq X^{'}$ and $X \cup Y \subseteq X^{'} \cup Y^{'}$. Let a underlying profile of the pair $(X^{'}, Y^{'})$ be $z^{'}$. Then either $z|_{\mathscr{B}} = z^{'}|_{\mathscr{B}}$, or z and $z^{'}$ satisfy the monotonicity conditions as described in section 2. So, using Lemma 1 or Lemma 5, we can say $f(z^{'}) = -11$ and $(X^{'}, Y^{'}) \in \mathscr{W}_f$.

- Proof of boundary condition a -

  Proof. Consider the pair $(\phi, N) \in 2^N \times 2^N$. Note that, the underlying profile for this pair is unique : $z = (-0.5^{N_A}, 0.5^{N_B})$. As f satisfies the far away condition, we get $f(z) = -11$ and $(\phi, N) \in \mathscr{W}_f$. Monotonicity of $\mathscr{W}_f$ implies $(X, N - X) \in \mathscr{W}_f$ for all $X \subseteq N$.

- Proof of boundary condition b -

  Proof. Consider any pair $(X, Y) \in 2^N \times 2^N$ such that -

  - either $N_A \subseteq X \cup Y,\ X \cap N_B = \phi,\ N_B \not\subseteq Y$.
  - Or $N_B \subseteq X \cup Y,\ X \cap N_A = \phi,\ N_A \not\subseteq Y$.

  In such a case, a underlying profile must be

  - either z, where $z(i) \geq -0.5$ for all $i \in N_A$ and $z(i) \geq 0.5$ for all $i \in N_B$ and $z(i) \neq 0.5$ for all $i \in N_B$.
  - or $z^{'}$, where $z^{'}(i) \leq 0.5$ for all $i \in N_B$ and $z(i) \leq -0.5$ for all $i \in N_A$ and $z(i) \neq -0.5$ for all $i \in N_B$.

  respectively. Due to country specific Pareto optimality, $f(z) \neq -11$ and $f(z^{'}) \neq -11$ and $(X, Y) \notin \mathscr{W}_f$. Consider $(X^{'}, Y^{'}) \in 2^N \times 2^N$ such that $X^{'} = \phi$ and $Y^{'} \neq N$. Note that, for any such pair, there exists another pair $(X, Y) \in 2^N \times 2^N$, not necessarily unique, with either $N_A \subseteq X \cup Y,\ X \cap N_B = \phi,\ N_B \not\subseteq Y$, or $N_B \subseteq X \cup Y,\ X \cap N_A = \phi,\ N_A \not\subseteq Y$, such that $X^{'} \subseteq X$ and $X^{'} \cup Y^{'} \subseteq X \cup Y$. Because $(X, Y) \notin \mathscr{W}_f$, monotonicity implies $(X^{'}, Y^{'}) \notin \mathscr{W}_f$.

- Proof of disjoint condition -

  Proof. For every $z \in \mathscr{R}, \{i \in N : z(i) \in (-0.5, 0.5)\} \cap \{i \in N : z(i) \in \{-0.5, 0.5\}\} = \phi$. So this condition is satisfied trivially.

So $\mathscr{W}_f$ is decisive. $\square$

Now suppose we have a $\mathscr{V} \subseteq 2^N \times 2^N$. For any profile $z \in \mathscr{R}$, define

- $S(z) \coloneqq \{i \in N : z(i) \in (-0.5, 0.5)\}$.
- $T(z) \coloneqq \{i \in N : z(i) \in \{-0.5, 0.5\}\}$.
- Using $\mathscr{V}$, we may define a social choice function as follows

$$g^{\mathscr{V}}(z) = \begin{cases} -11 & \text{if } (S(z), T(z)) \in \mathscr{V} \\ -10 & \text{if } (S(z), T(z)) \notin \mathscr{V} \text{ and } N_A \subseteq S(z) \cup T(z) \\ & \text{and } N_B \not\subseteq S(z) \cup T(z) \\ 01 & \text{if } (S(z), T(z)) \notin \mathscr{V} \text{ and } N_B \subseteq S(z) \cup T(z) \\ & \text{and } N_A \not\subseteq S(z) \cup T(z) \\ 00 & \text{otherwise} \end{cases}$$

Note that, $g^{\mathscr{V}}$ is well-defined, as the four cases are exclusive.

Remark 4. *Consider two profiles $z, z^{'} \in \mathscr{R}$ such that $z|_{\mathscr{B}} = z^{'}|_{\mathscr{B}}$. Then it follows that $S(z) = S(z^{'})$ and $T(z) = T(z^{'})$, so $g^{\mathscr{V}}(z) = g^{\mathscr{V}}(z^{'})$.*

Lemma 7. *Let $\mathscr{V}$ be decisive. Then $g^{\mathscr{V}}(z)$ is country specific Pareto optimal, strategy proof, non-corruptible and satisfies the far away condition for all $z \in \mathscr{R}$.*

Proof. First we show that $g^{\mathscr{V}}(z)$ satisfies strategy proofness for all $z \in \mathscr{R}$. Let z and $z^{'}$ be a unilateral deviation of an agent $i \in N$. We have to show $g^{\mathscr{V}}(z)R_{z(i)}g^{\mathscr{V}}(z^{'})$. Without loss of generality, suppose $i \in N_A$ and $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$. So, from Remark 4, we can say that $z|_{\mathscr{B}} \neq z^{'}|_{\mathscr{B}}$. Now consider the following cases.

- $z(i) = -0.5$
  Note that $g^{\mathscr{V}}(z) \in \mathscr{B}$ for all $z \in \mathscr{R}$. So, in this case, we can say that $g^{\mathscr{V}}(z)R_{z(i)}g^{\mathscr{V}}(z^{'})$.
- $z(i) \in [-1, -0.5)$
  Note that, as $z|_{\mathscr{B}} \neq z^{'}|_{\mathscr{B}}$, we have $S(z) \subseteq S(z^{'})$ and $S(z) \cup T(z) \subseteq S(z^{'}) \cup T(z^{'})$. As we have assumed that $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$ and $\mathscr{V}$ is monotone, so we can say that $(S(z), T(z)) \notin \mathscr{V}$. Also as $z(i) \in [-1, -0.5)$, so we can say $N_A \not\subseteq S(z) \cup T(z)$. So we can conclude $g^{\mathscr{V}}(z) \in \{(0, 0), (0, 1)\}$. So, in this case, we can say that $g^{\mathscr{V}}(z)R_{z(i)}g^{\mathscr{V}}(z^{'})$.
- $z(i) \in (-0.5, 0]$
  Note that, as $z|_{\mathscr{B}} \neq z^{'}|_{\mathscr{B}}$, we have $S(z^{'}) \subseteq S(z)$ and $S(z^{'}) \cup T(z^{'}) \subseteq S(z) \cup T(z)$. Now if $(S(z), T(z)) \in \mathscr{V}$, then we get $g^{\mathscr{V}}(z) = (-1, 1)$ and $g^{\mathscr{V}}(z)P_{z(i)}g^{\mathscr{V}}(z^{'})$. Otherwise $(S(z), T(z)) \notin \mathscr{V}$. As $\mathscr{V}$ is monotone, so we can say that $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$. So we can say $g^{\mathscr{V}}(z) \neq (-1, 1) \neq g^{\mathscr{V}}(z^{'})$, and hence we can conclude that $g^{\mathscr{V}}(z)R_{z(i)}g^{\mathscr{V}}(z^{'})$.

Combining these cases we can say that $g^{\mathscr{V}}$ satisfies strategy proofness.

Now we show that $g^{\mathscr{V}}(z)$ satisfies non-corruptibility for all $z \in \mathscr{R}$. Let z and $z^{'}$ be a unilateral deviation of an agent $i \in N$. Let $g^{\mathscr{V}}(z)I_{z(i)}g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z)I_{z^{'}(i)}g^{\mathscr{V}}(z^{'})$. To the contrary, suppose $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$. So, from Remark 4, we can say that $z|_{\mathscr{B}} \neq z^{'}|_{\mathscr{B}}$. Without loss of generality, let $i \in N_A$. As $z|_{\mathscr{B}} \neq z^{'}|_{\mathscr{B}}$, so we can say

- $z(i) \in [-1, -0.5) \Rightarrow z^{'}(i) \in [-0.5, 0]$.
- $z(i) = -0.5 \Rightarrow z^{'}(i) \in [-1, 0] - \{-0.5\}$.
- $z(i) \in (-0.5, 0] \Rightarrow z^{'}(i) \in [-1, -0.5]$.

Since we have assumed that $g^{\mathscr{V}}(z)I_{z(i)}g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z)I_{z^{'}(i)}g^{\mathscr{V}}(z^{'})$, so we get the following cases.

- Case 1 : $g^{\mathscr{V}}(z) = (-1, 1)$ and $g^{\mathscr{V}}(z^{'}) = (-1, 0)$
  In this case, we get $(S(z), T(z)) \in \mathscr{V}, (S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $N_A \subseteq S(z^{'}) \cup T(z^{'})$ and $N_B \not\subseteq S(z^{'}) \cup T(z^{'})$. Since $g^{\mathscr{V}}(z)I_{z(i)}g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z)I_{z^{'}(i)}g^{\mathscr{V}}(z^{'})$, so we must have either $z(i) \in [-1, -0.5)$ and $z^{'}(i) = -0.5$, or $z(i) = -0.5$ and $z^{'}(i) \in [-1, -0.5)$. Note that, the later case cannot happen as we have $N_A \subseteq S(z^{'}) \cup T(z^{'})$. The former case implies that $S(z) = S(z^{'})$ and $S(z) \cup T(z) \subsetneq S(z^{'}) \cup T(z^{'})$, which contradicts the monotonicity of $\mathscr{V}$.

- Case 2 : $g^{\mathscr{V}}(z) = (0,0)$ and $g^{\mathscr{V}}(z^{'}) = (0,1)$
In this case, we get $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $N_B \subseteq S(z^{'}) \cup T(z^{'})$ and $N_A \nsubseteq S(z^{'}) \cup T(z^{'})$. Since z and $z^{'}$ are unilateral deviation of an agent in N_A, so we can say that $N_B \subseteq S(z) \cup T(z)$. Now, if $(S(z), T(z)) \in \mathscr{V}$, then $g^{\mathscr{V}}(z) = (-1,1)$, a contradiction. Now suppose $(S(z), T(z)) \notin \mathscr{V}$. In this case, if $N_A \subseteq S(z) \cup T(z)$, then we can find a partition X of N such that $S(z) = X$ and $T(z) = N - X$. This is a violation of the the boundary condition (a) of $\mathscr{V}$. So in this case $N_A \nsubseteq S(z) \cup T(z)$. Then $g^{\mathscr{V}}(z) = (0,1)$, another contradiction.

- Case 3 : $g^{\mathscr{V}}(z) = (0,0)$ and $g^{\mathscr{V}}(z^{'}) = (-1,0)$
In this case, we get $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $N_A \subseteq S(z^{'}) \cup T(z^{'})$ and $N_B \nsubseteq S(z^{'}) \cup T(z^{'})$. Since $g^{\mathscr{V}}(z) I_{z(i)} g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z) I_{z^{'}(i)} g^{\mathscr{V}}(z^{'})$, so we must have either $z(i) \in (-0.5, 0]$ and $z^{'}(i) = -0.5$, or $z(i) = -0.5$ and $z^{'}(i) \in (-0.5, 0]$. Since z and $z^{'}$ are unilateral deviation of an agent in N_A, so we can say that $N_A \subseteq S(z) \cup T(z)$ and $N_B \nsubseteq S(z) \cup T(z)$. Now, if $(S(z), T(z)) \in \mathscr{V}$, then $g^{\mathscr{V}}(z) = (-1,1)$, a contradiction; and if $(S(z), T(z)) \notin \mathscr{V}$, then $g^{\mathscr{V}}(z) = (-1,0)$, another contradiction.

Combining these cases we can say that $g^{\mathscr{V}}$ satisfies non-corruptibility.

Now we show that $g^{\mathscr{V}}(z)$ satisfies country specific Pareto optimality and the far away condition for all $z \in \mathscr{R}$. Consider the following cases.

- Case 1 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \in \mathscr{V}$
In this case, we have $g^{\mathscr{V}}(z) = (-1,1)$. So, as the outcome in this case is $(-1,1)$, we can say that in this case $g^{\mathscr{V}}$ satisfies the far away condition. Now consider the following sub cases.
  - Sub case 1 : $S(z) \neq \phi$
  Then for all agent in $S(z), (-1,1)$ is the unique best outcome.
  - Sub case 2 : $S(z) = \phi$
  Since $(S(z), T(z)) \in \mathscr{V}$, then using boundary condition a, we can say $T(z) = N$. Then notice that $(-1,1)$ is one of the country specific Pareto optimal outcome for this profile.

  Combining these sub cases, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition.

- Case 2 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_A \subseteq S(z) \cup T(z)$ and $N_B \nsubseteq S(z) \cup T(z)$
In this case, we have $g^{\mathscr{V}}(z) = (-1,0)$. Since $(S(z), T(z)) \notin \mathscr{V}$, so using boundary condition b, we can say that $X \coloneqq \{i \in N : z(i) \in (0.5, 1]\} \neq \phi$. Note that, $(-1,0)$ and $(0,0)$ are the best alternatives for all agents in X, but in this case, $(-1,0) R_{z(i)} (0,0)$ for all $i \in N$. So, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition and the far away condition.

- Case 3 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_B \subseteq S(z) \cup T(z)$ and $N_A \nsubseteq S(z) \cup T(z)$
Similar to case 2.

- Case 4 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_A \nsubseteq S(z) \cup T(z)$ and $N_B \nsubseteq S(z) \cup T(z)$
In this case, we have $g^{\mathscr{V}}(z) = (0,0)$ and $X \coloneqq \{i \in N : z(i) \in (0.5, 1]\} \neq \phi$ and $Y \coloneqq \{i \in N : z(i) \in [-1, -0.5)\} \neq \phi$. Note that, $(-1,0)$ and $(0,0)$ are the best alternatives for all agents in X; and $(0,1)$ and $(0,0)$ are the best alternatives for all agents in Y. So for all agent in $X \cup Y, (0,0)$ is the only common alternative that is the best for all such

agents. Notice that, as $X \neq \phi$ and $Y \neq \phi$, so we can say, $(-1,0)R_{z(i)}(0,0)$ do not hold for all $i \in N$, and $(0,1)R_{z(i)}(0,0)$ do not hold for all $i \in N$. So, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition and the far away condition.

Combining all these cases, we get the required proof. $\square$

Theorem 2. *A social choice function f satisfies strategy proofness, non-corruptibility, country specific Pareto optimality and the far away condition if and only if $f = g^{\mathscr{V}}$ for some decisive $\mathscr{V} \subset 2^N \times 2^N$.*

Proof. Suppose f is a social choice function that satisfies strategy proofness, non-corruptibility, country specific Pareto optimality and the far away condition. Then from Lemma 6, we get $\mathscr{W}_f$ is decisive, and it is easy to show that $f(z) = g^{\mathscr{W}_f}(z)$ for all $z \in \mathscr{R}$.

Now suppose $\mathscr{V} \subset 2^N \times 2^N$ is decisive. Then Lemma 7 shows that $g^{\mathscr{V}}$ satisfies all the four properties. $\square$

Remark 5. *Notice that $g^{\mathscr{V}}$ is not Maskin monotonic as defined in Remark 1. We show this by means of an example. First fix any decisive $\mathscr{V} \subset 2^N \times 2^N$. Now consider a profile z where $z(i) = -1$ for all $i \in N_A$ and $z(j) = 1$ for all $i \in N_B$. Then $(S(z), T(z)) = (\phi, \phi)$, and it follows that $g^{\mathscr{V}}(z) = 00$. Now consider another profile $z^{'}$, where $z^{'}(i) = -0.5$ for all $i \in N_A$ and $z^{'}(j) = 0.5$ for all $i \in N_B$. Then $(S(z^{'}), T(z^{'})) = (\phi, N)$, and it follows that $g^{\mathscr{V}}(z^{'}) = -11$. But $L(f(z), z(i)) = \mathscr{A} = L(f(z), z^{'}(i))$ for all $i \in N$. This is a violation of Maskin monotonicity. Also note that for any decisive $\mathscr{V}$, Lemma 7 shows that $g^{\mathscr{V}}$ is strategy proof and non-corruptible. So Lemma 1 would imply that $g^{\mathscr{V}}$ is monotone as defined in this paper.*

7 Examples

In this section, we provide four examples of social choice functions to show that the four assumptions are independent. In each example, only one assumption is violated, while all other assumptions are satisfied.

1. Example that violates only country specific Pareto optimality :
   The constant rule that assigns $(-1,1)$ to every profile violates only country specific Pareto optimality.

2. Example that violates only the far away condition :
   We define $\mathscr{V} \subset 2^N \times 2^N$ as follows

   - $\mathscr{V} := \{(X, N-X) : \phi \neq X \subseteq N\}$.

   Having such a $\mathscr{V}$, we define a social choice function as follows

   - For any profile $z \in \mathscr{R}$
     (a) $S(z) := \{i \in N : z(i) \in (-0.5, 0.5)\}$.
     (b) $T(z) := \{i \in N : z(i) \in \{-0.5, 0.5\}\}$.
   - For every profile $z \in \mathscr{R}$, we may define a social choice function as follows.

$$g^{\mathscr{V}}(z) = \begin{cases} -11 & \text{if } (S(z), T(z)) \in \mathscr{V} \\ \\ -10 & \text{if } (S(z), T(z)) \notin \mathscr{V} \text{ and } N_A \subseteq S(z) \cup T(z) \\ & \text{and } N_B \nsubseteq S(z) \cup T(z) \\ \\ 01 & \text{if } (S(z), T(z)) \notin \mathscr{V} \text{ and } N_B \subseteq S(z) \cup T(z) \\ & \text{and } N_A \nsubseteq S(z) \cup T(z) \\ \\ 00 & \text{otherwise} \end{cases}$$

   As the four cases are distinct, so $g^{\mathscr{V}}$ is well defined.

   Now we show that $g^{\mathscr{V}}(z)$ satisfies strategy proofness for all $z \in \mathscr{R}$. Let z and $z^{'}$ be a unilateral deviation of an agent $i \in N$. We have to show $g^{\mathscr{V}}(z) R_{z(i)} g^{\mathscr{V}}(z^{'})$. Without loss of generality, suppose $i \in N_A$ and $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$. Now consider the following cases.

   - $z(i) = -0.5$
     Note that $g^{\mathscr{V}}(z) \in \mathscr{B}$ for all $z \in \mathscr{R}$. So, in this case, we can say that $g^{\mathscr{V}}(z) R_{z(i)} g^{\mathscr{V}}(z^{'})$.
   - $z(i) \in [-1, -0.5)$
     In this case, $g^{\mathscr{V}}(z) \in \{(0,0), (0,1)\}$. So, in this case, we can say that $g^{\mathscr{V}}(z) R_{z(i)} g^{\mathscr{V}}(z^{'})$.
   - $z(i) \in (-0.5, 0]$
     Suppose $(S(z), T(z)) \in \mathscr{V}$. Then we get $g^{\mathscr{V}}(z) = (-1,1)$ and $g^{\mathscr{V}}(z) P_{z(i)} g^{\mathscr{V}}(z^{'})$. Otherwise $(S(z), T(z)) \notin \mathscr{V}$. As $z^{'}$ is a unilateral deviation of agent i, so, it follows that $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$. So we can say $g^{\mathscr{V}}(z) \neq (-1,1) \neq g^{\mathscr{V}}(z^{'})$, and hence we can conclude that $g^{\mathscr{V}}(z) R_{z(i)} g^{\mathscr{V}}(z^{'})$.

   Combining these cases we can say that $g^{\mathscr{V}}$ satisfies strategy proofness.

Now we show that $g^{\mathscr{V}}(z)$ satisfies non-corruptibility for all $z \in \mathscr{R}$. Let z and $z^{'}$ be a unilateral deviation of an agent $i \in N$. Let $g^{\mathscr{V}}(z) I_{z(i)} g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z) I_{z^{'}(i)} g^{\mathscr{V}}(z^{'})$. To the contrary, suppose $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$. As $g^{\mathscr{V}}$ is well defined, it follows that $S(z) \neq S(z^{'})$ or $T(z) \neq T(z^{'})$. Without loss of generality, let $i \in N_A$. As this is a unilateral deviation of agent i, so we can say

- $z(i) \in [-1, -0.5) \Rightarrow z^{'}(i) \in [-0.5, 0]$.
- $z(i) = -0.5 \Rightarrow z^{'}(i) \in [-1, 0] - \{-0.5\}$.
- $z(i) \in (-0.5, 0] \Rightarrow z^{'}(i) \in [-1, -0.5]$.

Since we have assumed that $g^{\mathscr{V}}(z) I_{z(i)} g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z) I_{z^{'}} g^{\mathscr{V}}(z^{'})$, so we get the following cases.

- Case 1 : $g^{\mathscr{V}}(z) = (-1, 1)$ and $g^{\mathscr{V}}(z^{'}) = (-1, 0)$
  In this case, we get $(S(z), T(z)) \in \mathscr{V}, (S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $N_A \subseteq S(z^{'}) \cup T(z^{'})$ and $N_B \nsubseteq S(z^{'}) \cup T(z^{'})$. Since $g^{\mathscr{V}}(z) I_{z(i)} g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z) I_{z^{'}} g^{\mathscr{V}}(z^{'})$, so we must have either $z(i) \in [-1, -0.5)$ and $z^{'}(i) = -0.5$, or $z(i) = -0.5$ and $z^{'}(i) \in [-1, -0.5)$. Note that, the later case cannot happen as we have $N_A \subseteq S(z^{'}) \cup T(z^{'})$ and the former case contradicts $(S(z), T(z)) \in \mathscr{V}$.
- Case 2 : $g^{\mathscr{V}}(z) = (0, 0)$ and $g^{\mathscr{V}}(z^{'}) = (0, 1)$
  In this case, we get $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $N_B \subseteq S(z^{'}) \cup T(z^{'})$ and $N_A \nsubseteq S(z^{'}) \cup T(z^{'})$. Since z and $z^{'}$ are unilateral deviation of an agent in N_A, so we can say that $N_B \subseteq S(z) \cup T(z)$. Now, if $(S(z), T(z)) \in \mathscr{V}$, then $g^{\mathscr{V}}(z) = (-1, 1)$, a contradiction. Then $(S(z), T(z)) \notin \mathscr{V}$. In this case, if $N_A \subseteq S(z) \cup T(z)$, then we can find a partition X of N such that $S(z) = X$ and $T(z) = N - X$. This implies $(S(z), T(z)) \in \mathscr{V}$, another contradiction. So in this case $N_A \nsubseteq S(z) \cup T(z)$. Then $g^{\mathscr{V}}(z) = (0, 1)$, another contradiction.
- Case 3 : $g^{\mathscr{V}}(z) = (0, 0)$ and $g^{\mathscr{V}}(z^{'}) = (-1, 0)$
  In this case, we get $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $N_A \subseteq S(z^{'}) \cup T(z^{'})$ and $N_B \nsubseteq S(z^{'}) \cup T(z^{'})$. Since $g^{\mathscr{V}}(z) I_{z(i)} g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z) I_{z^{'}} g^{\mathscr{V}}(z^{'})$, so we must have either $z(i) \in (-0.5, 0]$ and $z^{'}(i) = -0.5$, or $z(i) = -0.5$ and $z^{'}(i) \in (-0.5, 0]$. Since z and $z^{'}$ are unilateral deviation of an agent in N_A, so we can say that $N_A \subseteq S(z) \cup T(z)$ and $N_B \nsubseteq S(z) \cup T(z)$. Now, if $(S(z), T(z)) \in \mathscr{V}$, then $g^{\mathscr{V}}(z) = (-1, 1)$, a contradiction; and if $(S(z), T(z)) \notin \mathscr{V}$, then $g^{\mathscr{V}}(z) = (-1, 0)$, another contradiction.

Combining these cases we can say that $g^{\mathscr{V}}$ satisfies non-corruptibility.

Now we show that $g^{\mathscr{V}}(z)$ satisfies country specific Pareto optimality for all $z \in \mathscr{R}$. Consider the following cases.

- Case 1 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \in \mathscr{V}$
  In this case, we have $g^{\mathscr{V}}(z) = (-1, 1)$. So, in this case $S(z) \neq \phi$ and for all agent in $S(z), (-1, 1)$ is the unique best outcome. So, in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition.
- Case 2 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_A \subseteq S(z) \cup T(z)$ and $N_B \nsubseteq S(z) \cup T(z)$.
  In this case, we have $g^{\mathscr{V}}(z) = (-1, 0)$. Since $N_B \nsubseteq S(z) \cup T(z)$, we can say that $X := \{i \in N : z(i) \in (0.5, 1]\} \neq \phi$. Note that, $(-1, 0)$ and $(0, 0)$ are the best alternatives for all agents in X. So, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition.

- Case 3 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_B \subseteq S(z) \cup T(z)$ and $N_A \not\subseteq S(z) \cup T(z)$. Similar to case 2.
- Case 4 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_A \not\subseteq S(z) \cup T(z)$ and $N_B \not\subseteq S(z) \cup T(z)$. In this case, we have $g^{\mathscr{V}}(z) = (0, 0)$ and $X := \{i \in N : z(i) \in (0.5, 1]\} \neq \phi$ and $Y := \{i \in N : z(i) \in [-1, -0.5)\} \neq \phi$. Note that, $(-1, 0)$ and $(0, 0)$ are the best alternatives for all agents in X; and $(0, 1)$ and $(0, 0)$ are the best alternatives for all agents in Y. So for all agent in $X \cup Y, (0, 0)$ is the only common alternative that is the best for all such agents. So, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition.

Combining all these cases, we get the required proof.

Notice that $g^{\mathscr{V}}(z) = (0, 0)$ when $(S(z), T(z)) = (\phi, N)$ is a violation of the far away condition.

3. Example that violates only non-corruptibility :
We define subsets of $2^N \times 2^N$ as follows

(a) $\mathscr{N}_1 := \{(U,V) \in 2^N \times 2^N : U = \phi,\ V \subsetneq N\}$

(b) $\mathscr{N} := (2^N \times 2^N) - \mathscr{N}_1$

(c)

$$\mathscr{V}_1 := \left\{ (U,V) \in \mathscr{N} : \begin{array}{l} \exists\, \phi \neq N_1 \subsetneq N_A \text{ and } \phi \neq N_2 \subsetneq N_B \text{ such that } N_1 \subset N_A - (U \cup V), \\ N_2 \subset N_B - (U \cup V) \text{ and } |U \cup V| > |N| - |U \cup V| \end{array} \right\}$$

(d)

$$\mathscr{V}_2 := \left\{ (U,V) \in \mathscr{N} : \begin{array}{l} N_A \subset U \cup V \text{ and } \exists\, \phi \neq N_2 \subsetneq N_B \text{ such that} \\ N_2 \subset N_B - (U \cup V) \text{ and } |U \cup V| > |N| - |U \cup V| \end{array} \right\}$$

(e)

$$\mathscr{V}_3 := \left\{ (U,V) \in \mathscr{N} : \begin{array}{l} N_B \subset U \cup V \text{ and } \exists\, \phi \neq N_1 \subsetneq N_A \text{ such that} \\ N_1 \subset N_A - (U \cup V) \text{ and } |U \cup V| > |N| - |U \cup V| \end{array} \right\}$$

(f) $\mathscr{V}_4 := \{(U,V) \in 2^N \times 2^N : U \cup V = N\}$

(g) $\mathscr{V} := \mathscr{V}_1 \cup \mathscr{V}_2 \cup \mathscr{V}_3 \cup \mathscr{V}_4$
Note that for all $i \neq j, \mathscr{V}_i \cap \mathscr{V}_j = \phi$ where $i, j \in \{1,2,3\}$.

Using this $\mathscr{V}$, we define a social choice function as follows

- For any profile $z \in \mathscr{R}$
  (a) $S(z) := \{i \in N : z(i) \in (-0.5, 0.5)\}$.
  (b) $T(z) := \{i \in N : z(i) \in \{-0.5, 0.5\}\}$.
- For every profile $z \in \mathscr{R}$, we may define a social choice function as follows.

$$g^{\mathscr{V}}(z) = \begin{cases} -11 & \text{if } (S(z),T(z)) \in \mathscr{V} \\ -10 & \text{if } (S(z),T(z)) \notin \mathscr{V} \text{ and } N_A \subseteq S(z) \cup T(z) \text{ and } N_B \nsubseteq S(z) \cup T(z) \\ 01 & \text{if } (S(z),T(z)) \notin \mathscr{V} \text{ and } N_B \subseteq S(z) \cup T(z) \text{ and } N_A \nsubseteq S(z) \cup T(z) \\ 00 & \text{otherwise} \end{cases}$$

As the four cases are distinct, so $g^{\mathscr{V}}(z)$ is well defined.

Now we show that $g^{\mathscr{V}}(z)$ satisfies strategy proofness for all $z \in \mathscr{R}$. Let z and $z^{'}$ be a unilateral deviation of an agent $i \in N$. We have to show $g^{\mathscr{V}}(z) R_{z(i)} g^{\mathscr{V}}(z^{'})$. Without loss of generality, let $i \in N_A$ and $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$. So, we may assume that $S(z) \neq S(z^{'})$ or $T(z) \neq T(z^{'})$. Now consider the following cases.

- $z(i) = -0.5$
  Note that $g^{\mathscr{V}}(z) \in \mathscr{B}$ for all $z \in \mathscr{R}$. So, in this case, we can say that $g^{\mathscr{V}}(z) R_{z(i)} g^{\mathscr{V}}(z^{'})$.

- $z(i) \in [-1, -0.5)$
  Let us define $Y := \{i \in N \colon z(i) \in (0.5, 1]\}$. In this case if $(S(z), T(z)) \in \mathscr{V}$ then $g^{\mathscr{V}}(z) = -11$. Otherwise $g^{\mathscr{V}}(z) = 01$ if $Y = \phi$, or $g^{\mathscr{V}}(z) = 00$ if $Y \neq \phi$. So, suppose $g^{\mathscr{V}}(z) = -11$ and $(S(z), T(z)) \in \mathscr{V}$. Then we can say $|S(z) \cup T(z)| > |N| - |S(z) \cup T(z)|$. Notice that $i \in N - (S(z) \cup T(z))$. As we have assumed that $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$, so it follows that $i \notin N - (S(z^{'}) \cup T(z^{'}))$. This implies that
  $|S(z^{'}) \cup T(z^{'})| > |S(z) \cup T(z)| > |N| - |S(z) \cup T(z)| > |N| - |S(z^{'}) \cup T(z^{'})|$.
  As $(S(z), T(z)) \in \mathscr{V}$ and $z(i) \in [-1, -0.5)$, so it follows that either $(S(z), T(z)) \in \mathscr{V}_1$ or $(S(z), T(z)) \in \mathscr{V}_3$. Suppose $(S(z), T(z)) \in \mathscr{V}_3$. Then if agent i was the only agent with dip in $[-1, -0.5)$, the it follows that $(S(z^{'}), T(z^{'})) \in \mathscr{V}_4$ otherwise we get $(S(z^{'}), T(z^{'})) \in \mathscr{V}_3$. Now assume $(S(z), T(z)) \in \mathscr{V}_1$. Then if agent i was the only agent with dip in $[-1, -0.5)$, the it follows that $(S(z^{'}), T(z^{'})) \in \mathscr{V}_2$ otherwise we get $(S(z^{'}), T(z^{'})) \in \mathscr{V}_1$. Combining we get $(S(z^{'}), T(z^{'})) \in \mathscr{V}$ which violates our assumption that $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$. So we have $g^{\mathscr{V}}(z) = 01$ or $g^{\mathscr{V}}(z) = 00$ and we can say $g^{\mathscr{V}}(z) R_{z(i)} g^{\mathscr{V}}(z^{'})$.
- $z(i) \in (-0.5, 0]$
  Let us define
  (a) $X := \{i \in N \colon z(i) \in [-1, -0.5)\}$.
  (b) $Y := \{i \in N \colon z(i) \in (0.5, 1]\}$.

  In this case if $(S(z), T(z)) \in \mathscr{V}$ then $g^{\mathscr{V}}(z) = -11$. Otherwise $g^{\mathscr{V}}(z) = 00$ if $X \neq \phi$ and $Y \neq \phi$, or $g^{\mathscr{V}}(z) = -10$ if $X = \phi$ but $Y \neq \phi$, or $g^{\mathscr{V}}(z) = 01$ if $X \neq \phi$ but $Y = \phi$. Now suppose $g^{\mathscr{V}}(z) = 00$ and $X \neq \phi$ and $Y \neq \phi$ and $(S(z), T(z)) \notin \mathscr{V}$. Then it follows that $|S(z) \cup T(z)| \leq |N| - |S(z) \cup T(z)|$. Note that $i \in S(z)$. As we have assumed that $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$, so it follows that $i \in N_A - S(z^{'})$. This implies that
  $|S(z^{'}) \cup T(z^{'})| \leq |S(z) \cup T(z)| \leq |N| - |S(z) \cup T(z)| \leq |N| - |S(z^{'}) \cup T(z^{'})|$.
  So $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and it follows that $g^{\mathscr{V}}(z) R_{z(i)} g^{\mathscr{V}}(z^{'})$. Similar arguments hold when $g^{\mathscr{V}}(z) = -10$ or $g^{\mathscr{V}}(z) = 01$. Now suppose $g^{\mathscr{V}}(z) = -11$. As $i \in S(z)$, so it follows that $g^{\mathscr{V}}(z) R_{z(i)} g^{\mathscr{V}}(z^{'})$.

Combining these cases we can say that $g^{\mathscr{V}}$ satisfies strategy proofness.

Now we show that $g^{\mathscr{V}}(z)$ satisfies country specific Pareto optimality and the far away condition for all $z \in \mathscr{R}$. Consider the following cases.

- Case 1 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \in \mathscr{V}$
  In this case, we have $g^{\mathscr{V}}(z) = (-1, 1)$. So, as the outcome in this case is $(-1, 1)$, we can say that in this case $g^{\mathscr{V}}$ satisfies the far away condition. Now consider the following sub cases.
  - Sub case 1 : $S(z) \neq \phi$
    Then for all agent in $S(z)$, $(-1, 1)$ is the unique best outcome.
  - Sub case 2 : $S(z) = \phi$
    Since $(S(z), T(z)) \in \mathscr{V}$, then we can say $T(z) = N$. Then notice that $(-1, 1)$ is one of the country specific Pareto optimal outcome for this profile.

  Combining these sub cases, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition.
- Case 2 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_A \subseteq S(z) \cup T(z)$ and $N_B \nsubseteq S(z) \cup T(z)$
  In this case, we have $g^{\mathscr{V}}(z) = (-1, 0)$. Since $(S(z), T(z)) \notin \mathscr{V}$ and $N_B \nsubseteq S(z) \cup T(z)$, we can say that $X := \{i \in N : z(i) \in (0.5, 1]\} \neq \phi$. Note that, $(-1, 0)$ and $(0, 0)$

are the best alternatives for all agents in X, but in this case, $(-1,0)R_{z(i)}(0,0)$ for all $i \in N$. So, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition and the far away condition.

- Case 3 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_B \subseteq S(z) \cup T(z)$ and $N_A \not\subseteq S(z) \cup T(z)$ Similar to case 2.
- Case 4 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_A \not\subseteq S(z) \cup T(z)$ and $N_B \not\subseteq S(z) \cup T(z)$ In this case, we have $g^{\mathscr{V}}(z) = (0,0)$ and $X := \{i \in N : z(i) \in (0.5, 1]\} \neq \phi$ and $Y := \{i \in N : z(i) \in [-1, -0.5)\} \neq \phi$. Note that, $(-1,0)$ and $(0,0)$ are the best alternatives for all agents in X; and $(0,1)$ and $(0,0)$ are the best alternatives for all agents in Y. So $(0,0)$ is the only common alternative that is the best for all agents in $X \cup Y$. Notice that, as $X \neq \phi$ and $Y \neq \phi$, so we can say, $(-1,0)R_{z(i)}(0,0)$ do not hold for all $i \in N$, and $(0,1)R_{z(i)}(0,0)$ do not hold for all $i \in N$. So, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition and the far away condition.

Combining all these cases, we get the required proof.

Now consider the following example. Suppose there are 5 agents in N_A and 2 agents in N_B. Consider the following profile. $z(i) = -0.75, z(j) = -0.25$ for all $j \in N_A - \{i\}$ and $z(k) = 1$ for all $k \in N_B$. Notice that $(S(z), T(z)) \in \mathscr{V}_1 \subset \mathscr{V}$. So $g^{\mathscr{V}}(z) = -11$. Now consider a unilateral deviation $z^{'}$ of agent $i \in N_A$ where $z^{'}(i) = -0.5$. Notice that $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $g^{\mathscr{V}}(z^{'}) = -10$. But $g^{\mathscr{V}}(z)I_{z(i)}g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z)I_{z^{'}(i)}g^{\mathscr{V}}(z^{'})$. This is a violation of the non-corruptibility condition.

4. Example that violates only strategy proofness :
We define subsets of $2^N \times 2^N$ as follows

(a) $\mathscr{N}_1 := \{(U,V) \in 2^N \times 2^N : U = \phi,\ V \subsetneq N\}$

(b) $\mathscr{N} := (2^N \times 2^N) - \mathscr{N}_1$

(c)
$$\mathscr{V}_1 := \left\{ (U,V) \in \mathscr{N} : \begin{array}{l} \exists\, \phi \neq N_1 \subsetneq N_A \text{ and } \phi \neq N_2 \subsetneq N_B \text{ such that } N_1 \subset N_A - (U \cup V), \\ N_2 \subset N_B - (U \cup V) \text{ and } |U \cup V| < min\{|N_A - (U \cup V)|, |N_B - (U \cup V)|\} \end{array} \right\}$$

(d)
$$\mathscr{V}_2 := \left\{ (U,V) \in \mathscr{N} : \begin{array}{l} N_A \subset U \cup V \text{ and } \exists\, \phi \neq N_2 \subsetneq N_B \text{ such that} \\ N_2 \subset N_B - (U \cup V) \text{ and } |U \cup V| < |N| - |U \cup V| \end{array} \right\}$$

(e)
$$\mathscr{V}_3 := \left\{ (U,V) \in \mathscr{N} : \begin{array}{l} N_B \subset U \cup V \text{ and } \exists\, \phi \neq N_1 \subsetneq N_A \text{ such that} \\ N_1 \subset N_A - (U \cup V) \text{ and } |U \cup V| < |N| - |U \cup V| \end{array} \right\}$$

(f) $\mathscr{V}_4 := \{(U,V) \in 2^N \times 2^N : U \cup V = N\}$

(g) $\mathscr{V} := \mathscr{V}_1 \cup \mathscr{V}_2 \cup \mathscr{V}_3 \cup \mathscr{V}_4$
Note that for all $i \neq j, \mathscr{V}_i \cap \mathscr{V}_j = \phi$ where $i, j \in \{1,2,3,4\}$.

Using this $\mathscr{V}$, we define a social choice function as follows

- For any profile $z \in \mathscr{R}$
  (a) $S(z) := \{i \in N : z(i) \in (-0.5, 0.5)\}$.
  (b) $T(z) := \{i \in N : z(i) \in \{-0.5, 0.5\}\}$.
- For every profile $z \in \mathscr{R}$, we may define a social choice function as follows.

$$g^{\mathscr{V}}(z) = \begin{cases} -11 & \text{if } (S(z),T(z)) \in \mathscr{V} \\ -10 & \text{if } (S(z),T(z)) \notin \mathscr{V} \text{ and } N_A \subseteq S(z) \cup T(z) \text{ and } N_B \nsubseteq S(z) \cup T(z) \\ 01 & \text{if } (S(z),T(z)) \notin \mathscr{V} \text{ and } N_B \subseteq S(z) \cup T(z) \text{ and } N_A \nsubseteq S(z) \cup T(z) \\ 00 & \text{otherwise} \end{cases}$$

As the four cases are distinct, so $g^{\mathscr{V}}(z)$ is well defined.

Now we show that $g^{\mathscr{V}}(z)$ satisfies non-corruptibility condition for all $z \in \mathscr{R}$. Let z and $z^{'}$ be a unilateral deviation of an agent $i \in N$. Let $g^{\mathscr{V}}(z) I_{z(i)} g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z) I_{z^{'}(i)} g^{\mathscr{V}}(z^{'})$. To the contrary, suppose $g^{\mathscr{V}}(z) \neq g^{\mathscr{V}}(z^{'})$. So, we may assume that $S(z) \neq S(z^{'})$ or $T(z) \neq T(z^{'})$. Without loss of generality, let $i \in N_A$. As this is a unilateral deviation of agent i, so we can say

- $z(i) \in [-1, -0.5) \Rightarrow z^{'}(i) \in [-0.5, 0]$.
- $z(i) = -0.5 \Rightarrow z^{'}(i) \in [-1, 0] - \{-0.5\}$.
- $z(i) \in (-0.5, 0] \Rightarrow z^{'}(i) \in [-1, -0.5]$.

Since we have assumed that $g^{\mathscr{V}}(z)I_{z(i)}g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z)I_{z'}g^{\mathscr{V}}(z^{'})$, so we get the following cases.

- Case 1 : $g^{\mathscr{V}}(z) = (-1, 1)$ and $g^{\mathscr{V}}(z^{'}) = (-1, 0)$
  In this case, we get $(S(z), T(z)) \in \mathscr{V}, (S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $N_A \subseteq S(z^{'}) \cup T(z^{'})$ and $N_B \nsubseteq S(z^{'}) \cup T(z^{'})$. As this is a unilateral deviation of an agent in N_A, so it follows that $N_B \nsubseteq S(z) \cup T(z)$. Since $g^{\mathscr{V}}(z)I_{z(i)}g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z)I_{z'}g^{\mathscr{V}}(z^{'})$, so we must have either $z(i) \in [-1, -0.5)$ and $z^{'}(i) = -0.5$, or $z(i) = -0.5$ and $z^{'}(i) \in [-1, -0.5)$. For the case when $z(i) = -0.5$ and $z^{'}(i) \in [-1, -0.5)$, we have $|S(z) \cup T(z)| > |S(z^{'}) \cup T(z^{'})|$. This contradicts the fact that $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$. Now consider the case when $z(i) \in [-1, -0.5)$ and $z^{'}(i) = -0.5$. As $(S(z), T(z)) \in \mathscr{V}$, so it follows that $|S(z) \cup T(z)| < |N_A - (S(z) \cup T(z))|$. Since $(S(z), T(z)) \in \mathscr{V}$, so we can say $|S(z) \cup T(z)| \geq 1$. So there exists atleast one agent $i \neq j \in N_A$ such that $z(j) \in [-1, -0.5)$. This contradicts the fact that $N_A \subseteq S(z^{'}) \cup T(z^{'})$ because agent j is not the unilaterally deviating agent.
- Case 2 : $g^{\mathscr{V}}(z) = (0, 0)$ and $g^{\mathscr{V}}(z^{'}) = (0, 1)$
  In this case, we get $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $N_B \subseteq S(z^{'}) \cup T(z^{'})$ and $N_A \nsubseteq S(z^{'}) \cup T(z^{'})$. Since z and $z^{'}$ are unilateral deviation of an agent in N_A, so we can say that $N_B \subseteq S(z) \cup T(z)$. Now, if $(S(z), T(z)) \in \mathscr{V}$, then $g^{\mathscr{V}}(z) = (-1, 1)$, a contradiction. Now suppose $(S(z), T(z)) \notin \mathscr{V}$. In this case, if $N_A \subseteq S(z) \cup T(z)$, then as $N_B \subseteq S(z) \cup T(z)$, it follows that $(S(z), T(z)) \in \mathscr{V}_4$ which contradicts $(S(z), T(z)) \notin \mathscr{V}$. So in this case $N_A \nsubseteq S(z) \cup T(z)$. Then $g^{\mathscr{V}}(z) = (0, 1)$, another contradiction.
- Case 3 : $g^{\mathscr{V}}(z) = (0, 0)$ and $g^{\mathscr{V}}(z^{'}) = (-1, 0)$
  In this case, we get $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $N_A \subseteq S(z^{'}) \cup T(z^{'})$ and $N_B \nsubseteq S(z^{'}) \cup T(z^{'})$. Since $g^{\mathscr{V}}(z)I_{z(i)}g^{\mathscr{V}}(z^{'})$ and $g^{\mathscr{V}}(z)I_{z'(i)}g^{\mathscr{V}}(z^{'})$, so we must have either $z(i) \in (-0.5, 0]$ and $z^{'}(i) = -0.5$, or $z(i) = -0.5$ and $z^{'}(i) \in (-0.5, 0]$. Since z and $z^{'}$ are unilateral deviation of an agent in N_A, so we can say that $N_A \subseteq S(z) \cup T(z)$ and $N_B \nsubseteq S(z) \cup T(z)$. Now, if $(S(z), T(z)) \in \mathscr{V}$, then $g^{\mathscr{V}}(z) = (-1, 1)$, a contradiction; and if $(S(z), T(z)) \notin \mathscr{V}$, then $g^{\mathscr{V}}(z) = (-1, 0)$, another contradiction.

Combining these cases we can say that $g^{\mathscr{V}}$ satisfies non-corruptibility.

Now we show that $g^{\mathscr{V}}(z)$ satisfies country specific Pareto optimality and the far away condition for all $z \in \mathscr{R}$. Consider the following cases.

- Case 1 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \in \mathscr{V}$
  In this case, we have $g^{\mathscr{V}}(z) = (-1, 1)$. So, as the outcome in this case is $(-1, 1)$, we can say that in this case $g^{\mathscr{V}}$ satisfies the far away condition. Now consider the following sub cases.
  - Sub case 1 : $S(z) \neq \phi$
    Then for all agent in $S(z), (-1, 1)$ is the unique best outcome.
  - Sub case 2 : $S(z) = \phi$
    Since $(S(z), T(z)) \in \mathscr{V}$, then we can say $T(z) = N$. Then notice that $(-1, 1)$ is one of the country specific Pareto optimal outcome for this profile.

  Combining these sub cases, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition.
- Case 2 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_A \subseteq S(z) \cup T(z)$ and $N_B \nsubseteq S(z) \cup T(z)$
  In this case, we have $g^{\mathscr{V}}(z) = (-1, 0)$. Since $(S(z), T(z)) \notin \mathscr{V}$ and $N_B \nsubseteq S(z) \cup T(z)$,

we can say that $X \coloneqq \{i \in N : z(i) \in (0.5, 1]\} \neq \phi$. Note that, $(-1, 0)$ and $(0, 0)$ are the best alternatives for all agents in X, but in this case, $(-1, 0)R_{z(i)}(0, 0)$ for all $i \in N$. So, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition and the far away condition.

- Case 3 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_B \subseteq S(z) \cup T(z)$ and $N_A \nsubseteq S(z) \cup T(z)$ Similar to case 2.
- Case 4 : $z \in \mathscr{R}$ such that $(S(z), T(z)) \notin \mathscr{V}$ and $N_A \nsubseteq S(z) \cup T(z)$ and $N_B \nsubseteq S(z) \cup T(z)$ In this case, we have $g^{\mathscr{V}}(z) = (0, 0)$ and $X \coloneqq \{i \in N : z(i) \in (0.5, 1]\} \neq \phi$ and $Y \coloneqq \{i \in N : z(i) \in [-1, -0.5)\} \neq \phi$. Note that, $(-1, 0)$ and $(0, 0)$ are the best alternatives for all agents in X; and $(0, 1)$ and $(0, 0)$ are the best alternatives for all agents in Y. So $(0, 0)$ is the only common alternative that is the best for all agents in $X \cup Y$. Notice that, as $X \cup Y \neq \phi$, so we can say, $(-1, 0)R_{z(i)}(0, 0)$ do not hold for all $i \in N$, and $(0, 1)R_{z(i)}(0, 0)$ do not hold for all $i \in N$. So, we can conclude that in this case $g^{\mathscr{V}}$ satisfies the country specific Pareto optimality condition and the far away condition.

Combining all these cases, we get the required proof.

Now consider the following example. Suppose there are 5 agents in N_A and 4 agents in N_B. Consider the following profile. $z = (-0.8, -0.8, -0.75, -0.25, -0.25, 1, 1, 1, 1)$. Notice that $(S(z), T(z)) \in \mathscr{V}_1 \subset \mathscr{V}$. So $g^{\mathscr{V}}(z) = -11$. Now consider a unilateral deviation $z^{'}$ of agent $i \in N_A$ where $z(i) = -0.75$ and $z^{'}(i) = -0.25$. Notice that $(S(z^{'}), T(z^{'})) \notin \mathscr{V}$ and $g^{\mathscr{V}}(z^{'}) = 00$. But $g^{\mathscr{V}}(z^{'})P_{z(i)}g^{\mathscr{V}}(z)$. This is a violation of strategy proofness.

8 Conclusion

In this section, we perturb our model in three aspects keeping everything else fixed.

Firstly, we consider that agents have lexicographic preferences. More precisely an agent i with dip $x(i)$ will weakly prefer (α, β) to (α_1, β_1) under lexicographic preferences iff

- Either $\min\{|\alpha - x(i)|, |\beta - x(i)|\} > \min\{|\alpha_1 - x(i)|, |\beta_1 - x(i)|\}$
- Or $\min\{|\alpha - x(i)|, |\beta - x(i)|\} = \min\{|\alpha_1 - x(i)|, |\beta_1 - x(i)|\}$, $\max\{|\alpha - x(i)|, |\beta - x(i)|\} \geq \max\{|\alpha_1 - x(i)|, |\beta_1 - x(i)|\}$

Notice that, under this preference, agent i will be indifferent between (α, β) to (α_1, β_1) iff $\min\{|\alpha - x(i)|, |\beta - x(i)|\} = \min\{|\alpha_1 - x(i)|, |\beta_1 - x(i)|\}$ and $\max\{|\alpha - x(i)|, |\beta - x(i)|\} = \max\{|\alpha_1 - x(i)|, |\beta_1 - x(i)|\}$. So under this preference, it can be shown that the outcome of a strategy proof, non-corruptible, country specific Pareto optimal social choice function must be one of the corner points of $\mathscr{A}$.

Secondly, we may consider three country model with every thing remaining the same. In this case we have to redefine the far away condition for the middle country. The class of rule will depend on this definition. If we do not define the far away condition for the middle country, then inner solution might be possible.

Thirdly, instead of the fixed interval $[-1, 0]$ and $[0, 1]$; we may assume any arbitrary interval $[a, b]$ and $[b, c]$ as the two countries. Note that in all our proofs, we do not use the relative size of each country. So, our results will hold even in this general setting.