Preferences under Ambiguity Without Event-Separability*

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Abstract

We propose and axiomatically characterize a representation of ambiguity sensitive preferences. The distinguishing feature of our axiomatization is that we do not require preferences to be event-wise separable over any domain of acts. Even without any such separability restrictions, we are able to uniquely elicit baseline subjective probabilities for a decision maker. The novel axiom that allows us to do so expresses the idea that, at least in the domain of a certain class of acts, the decision maker exhibits a consistent tradeoff between risk and ambiguity concerns. Under our representation of her preferences, any act is assessed by its subjective expected utility with respect to her baseline probabilities and a residual that captures her assessment of the act's exposure to ambiguity.

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1 Introduction

In a famous thought experiment, Ellsberg (1961) demonstrated that many decision makers' choice behavior may be inconsistent with subjective expected utility maximization. Subjective expected utility maximization requires a decision maker's preferences to be event-wise separable. However, as Ellsberg showed, in many situations of uncertainty, this may be too demanding a requirement for decision makers. Following Schmeidler (1989), a series of decision models have been proposed for such ambiguity-sensitive decision makers that relax event-separability restrictions. A feature common to these models is that although preferences are not required to be event-wise separable over the domain of all acts, each of them identifies a subdomain on which event-separability still holds.

In an important contribution, Machina (2009) has shown that such ambiguity-sensitive models may be susceptible to the same kind of difficulties that the Ellsberg paradox poses for subjective expected utility.[1] In other words, the conflict between event-separability of preferences and ambiguity-sensitive behavior may be of a more serious nature than what these models can accommodate. As Machina writes "the phenomenon of ambiguity aversion is intrinsically one of nonseparable preferences across mutually exclusive events, and the models that exhibit full–or even partial–event-separability cannot capture all aspects of this phenomenon." (Machina 2009, p. 390) Therefore, in modeling ambiguity-sensitive behavior, reliance on event-separability assumptions should be kept to the minimum and, if possible, done away with completely.

[1]Machina (2009) makes this point by proposing an Ellsberg-style paradox for the Choquet expected utility model (Schmeidler (1989)). Baillon, Haridon, and Placido (2011) have shown that the conflict between ambiguity-sensitive behavior and event-separability of preferences that Machina's example highlights causes difficulties for other models as well, e.g., maxmin expected utility (Gilboa and Schmeidler (1989)), variational preferences (Maccheroni, Marinacci, and Rustichini (2006)), α-maxmin (Ghirardato, Maccheroni, and Marinacci (2004)) and the smooth ambiguity model (Klibanoff, Marinacci, and Mukerji (2005)).

In this paper, we attempt to implement the "Machina program" by proposing and axiomatically characterizing a representation of ambiguity-sensitive preferences that does away with event-separability restrictions completely. Instead, our focus is to identify whether the decision maker (DM) that we model exhibits a consistent tradeoff between risk and ambiguity concerns, at least over some class of acts. We propose such a class of acts and show that if her choices do reveal a consistent ambiguity-risk tradeoff over this class, then it is possible to uniquely elicit baseline subjective probabilities for her. Further, we show that her preferences permit a representation under which she assesses any act based on a subjective expected utility evaluation with respect to her baseline probabilities and a residual that captures her assessment of the act's exposure to ambiguity. More formally, let Z denote an underlying set of outcomes and $\Delta(Z)$ the set of objective lotteries on Z. Consider an act $f = (f_1, \ldots, f_n) \in (\Delta(Z))^n$, which is a mapping from an underlying set of states, $S = \{1, \ldots, n\}$, to $\Delta(Z)$. The DM's assessment of f under our representation is given by

$$V(f) = \sum_{s \in S} \mu(s)\mathbb{E}u(f_s) + \phi(f),$$

where $\mu : S \to [0, 1]$ specifies the DM's subjective probabilities on the state space, u is a vNM-utility index, with $\mathbb{E}u(p) = \sum_{x \in Z} p(x)u(x)$ denoting the expected utility of any lottery p, and $\phi(f)$ captures how the DM's assessment of f is affected by the ambiguity that she perceives. We call this representation the fully non-separable ambiguity (FNSA) representation.

The first natural property that the function ϕ under an FNSA representation has is that it takes the value zero for any constant act. To understand its second important property, let (q, p_{-s}) denote an act that gives $q \in \Delta(Z)$ in state s and is identical to $p \in \Delta(Z)$ on all other states. We call two such acts (q, p_{-s}) and (r, p_{-s}) almost identical and complementary (AIC) if $\frac{1}{2}q + \frac{1}{2}r \sim p$. Here complementarity is in the sense of Siniscalchi (2009). Under an FNSA representation, the ambiguity assessments of any two pairs of AIC acts $((q, p_{-s}), (r, p_{-s}))$ and $((\tilde{q}, \tilde{p}_{-s}), (\tilde{r}, \tilde{p}_{-s}))$, with $q \succ p \succ r$ and $\tilde{q} \succ \tilde{p} \succ \tilde{r}$, have the property that

$$\frac{\phi(q, p_{-s}) - \phi(r, p_{-s})}{\mathbb{E}u(q) - \mathbb{E}u(r)} = \frac{\phi(\tilde{q}, \tilde{p}_{-s}) - \phi(\tilde{r}, \tilde{p}_{-s})}{\mathbb{E}u(\tilde{q}) - \mathbb{E}u(\tilde{r})}.$$

In other words, the difference in ambiguity assessments of two AIC acts is proportional to the difference in the expected utility of the lotteries under them in the state in which they vary and the constant of proportionality is state-dependent. That is, (pairs of) AIC acts are precisely the minimal class over which we require the DM to exhibit consistent ambiguity-risk tradeoffs.

In this paper, we provide a behavioral foundation for the FNSA representation. In addition, our elicitation of unique baseline subjective probabilities for the DM allows us to provide an intuitive definition of ambiguity aversion. We are also able to identify those acts which when randomized over are considered by the DM to provide her with a hedging advantage.

As mentioned earlier, our paper draws motivation from Machina (2009) and its key message about the fundamental conflict between ambiguity-sensitive behavior and event-separability restrictions. Viewed from this perspective, our paper shares the same spirit as Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011) and Lehrer and Teper (2013), where too the goal is to relax event-separability restrictions. The structure of the representation that we propose is similar to Siniscalchi's VEU (Siniscalchi 2009) and Grant and Polak's mean-dispersion representations (Grant and Polak 2013). In both these representations, like in ours, the DM's assessment of an act is based on a baseline subjective expected utility evaluation and an adjustment term that captures her assessment of the act's exposure to ambiguity.

The paper is organized as follows. Section 2 provides the setup. Section 3 formally defines the FNSA representation and provides its behavioral foundations. Section 4 characterizes ambiguity attitudes. Proofs are provided in the Appendix.

2 Setup

Primitive Objects: Let $S = \{1, \ldots, n\}$ be a finite set of states. Let Z denote a set of outcomes or prizes and $\Delta(Z)$ the set of simple (objective) lotteries on Z. An act f is a function from S into $\Delta(Z)$. Let $\mathcal{H}$ be the set of all such acts. We will denote generic elements of $\Delta(Z)$ by p, q, etc., and that of $\mathcal{H}$ by f, g, etc. We will engage in the usual abuse of notation by not distinguishing between a lottery and a constant act that gives that lottery in every state; for instance, $p \in \mathcal{H}$ as well as $p \in \Delta(Z)$. We will often denote an act in vector-form, e.g., $f = (f_1, \ldots, f_n)$. Further, (q, p_{-s}) shall denote an act that gives q in state s and is identical to p on all other states. For any $p \in \Delta(Z)$, $p(x)$ will denote the probability that p assigns to the outcome $x \in Z$. We define a convex combination of lotteries in $\Delta(Z)$ in the standard way. For instance, if $p, q \in \Delta(Z)$ and $\alpha \in [0, 1]$, then $\alpha p + (1 - \alpha)q$ denotes an element in $\Delta(Z)$ that gives the outcome $x \in Z$ with probability $\alpha p(x) + (1 - \alpha)q(x)$.

Preferences: We consider a decision maker (DM) who has preferences $\succsim$ on $\mathcal{H}$. Observe that preferences on $\mathcal{H}$ induce preferences on the sets $\Delta(Z)$ and Z.

Almost Identical Complementary Acts: We now introduce our notion of almost identical complementary acts that is central to the analysis.

Definition 2.1. *A pair of acts $(q, p_{-s}), (r, p_{-s}) \in \mathcal{H}$ are almost identical and complementary (AIC) if $\frac{1}{2}q + \frac{1}{2}r \sim p$.*

We think of two acts like (q, p_{-s}) and (r, p_{-s}) as almost identical since they are identical in all but one state. Further, these acts are complementary in the sense of Siniscalchi (2009).[2]

[2] Two acts f and g are complementary if for all $s, s' \in S$, $\frac{1}{2}f_s + \frac{1}{2}g_s \sim \frac{1}{2}f_{s'} + \frac{1}{2}g_{s'}$.

3 Representation and Axioms

We now formally define a fully non-separable ambiguity (FNSA) representation. In the way of notation, for a utility function $u : Z \to \mathbb{R}$ and $p \in \Delta(Z)$, $\mathbb{E}u(p) = \sum_{x \in Z} p(x)u(x)$ shall denote the expected utility of p.

Definition 3.1. $\succsim$ *has a FNSA representation if there exists*

1. *a non-constant function* $u : Z \to \mathbb{R}$
2. *a probability measure* μ *on* S
3. *a function* $\phi : \mathcal{H} \to \mathbb{R}$ *with the property that*
   (a) *for any* $p \in \Delta(Z)$*,* $\phi(p) = 0$*, and*
   (b) *for any* $s \in S$ *and two pairs of AIC acts* $((q, p_{-s}), (r, p_{-s}))$ *and* $((\tilde{q}, \tilde{p}_{-s}), (\tilde{r}, \tilde{p}_{-s}))$*, with* $q \succ r$ *and* $\tilde{q} \succ \tilde{r}$*,*
$$\frac{\phi(q, p_{-s}) - \phi(r, p_{-s})}{\mathbb{E}u(q) - \mathbb{E}u(r)} = \frac{\phi(\tilde{q}, \tilde{p}_{-s}) - \phi(\tilde{r}, \tilde{p}_{-s})}{\mathbb{E}u(\tilde{q}) - \mathbb{E}u(\tilde{r})},$$

such that the function $V : \mathcal{H} \to \mathbb{R}$ *given by*

$$V(f) = \sum_{s \in S} \mu(s)\mathbb{E}u(f_s) + \phi(f)$$

represents $\succsim$*.*

The FNSA representation deviates from a subjective expected utility (SEU) evaluation in that it incorporates a notion of how the DM's assessment of an act is affected by its exposure to ambiguity. This notion is captured by the function ϕ. As the assessment of constant acts is not influenced by ambiguity, the ϕ-value of any such act is zero. Further, the difference in ambiguity assessments of two AIC acts is proportional to the difference in the expected utility of the lotteries under them in the state in which they vary and the constant of proportionality is state-dependent. That is, for any state $s \in S$, there exists a constant $k(s)$ such that for any pair of AIC acts (q, p_{-s}), (r, p_{-s}), with $q \succ r$,

$$\frac{\phi(q, p_{-s}) - \phi(r, p_{-s})}{\mathbb{E}u(q) - \mathbb{E}u(r)} = k(s)$$

We now introduce a set of axioms that constitute a behavioral foundation for a FNSA representation.

($A1$ – Weak Order) $\succsim$ on $\mathcal{H}$ is transitive and complete.

($A2$ – Archimedean) If $p, q \in \Delta(Z)$, $f \in \mathcal{H}$ are such that $p \succ f \succ q$, then there exists $\overline{\alpha}, \underline{\alpha} \in (0, 1)$ such that

$$\overline{\alpha}p + (1 - \overline{\alpha})q \succ f \succ \underline{\alpha}p + (1 - \underline{\alpha})q$$

($A3$ – Monotonicity) For all $f, g \in \mathcal{H}$, if $f_s \succsim g_s$ for all $s \in S$, then $f \succsim g$.

($A4$ – Non-triviality) There exist $x, y \in Z$ such that $x \succ y$.

($A5$ – Risk Independence) If $p, q, r \in \Delta(Z)$, then for all $\alpha \in (0, 1]$,

$$p \succ q \quad \Rightarrow \quad \alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r$$

Before we present our final axiom note that, following standard terminology, we will refer to a state $s \in S$ as null if $f \sim g$ for all $f, g \in \mathcal{H}$ such that $f_{s'} = g_{s'}$ for all $s \neq s'$.

($A6$ – Consistent Ambiguity-Risk Tradeoff for AIC acts) If $s \in S$ is a non-null state and $((q, p_{-s}), (r, p_{-s}))$, $((\tilde{q}, \tilde{p}_{-s}), (\tilde{r}, \tilde{p}_{-s})) \in \mathcal{H} \times \mathcal{H}$ are pairs of AIC acts with $q' \sim (q, p_{-s})$, $r' \sim (r, p_{-s})$, $\tilde{q}' \sim (\tilde{q}, \tilde{p}_{-s})$ and $\tilde{r}' \sim (\tilde{r}, \tilde{p}_{-s})$, $q', r', \tilde{q}', \tilde{r}' \in \Delta(Z)$, then for all $\alpha \in [0, 1]$,

$$\alpha q' + (1 - \alpha)\tilde{r}' \succsim \alpha r' + (1 - \alpha)\tilde{q}' \quad \Leftrightarrow \quad \alpha q + (1 - \alpha)\tilde{r} \succsim \alpha r + (1 - \alpha)\tilde{q}$$

Axioms (A1)-(A5) are standard. The novel axiom in this paper is that of consistent ambiguity-risk tradeoff for AIC acts (the CART axiom, for short). The CART axiom implies that for any pair of AIC acts, the difference in the DM's assessment of the two acts is proportional to the difference in the expected utility of the two lotteries under them in the state in which they vary. This is of course true for any SEU maximizer with the constant of proportionality being the probability of that particular state. For someone who is not an SEU maximizer and is sensitive to ambiguity, this axiom suggests that her assessment of the difference in the ambiguity associated with two AIC acts is also of an order proportional to the difference in the expected utility of the two lotteries under them in the state in which they vary. In other words, the axiom guarantees that at least in the domain of AIC acts, the DM exhibits a consistent tradeoff between risk and ambiguity concerns. As we will show below, this consistency is what is critical in eliciting unique subjective probabilities for the DM.

We can now state the main result of our paper.

Theorem 3.1. *$\succsim$ satisfies (A1) - (A6) if and only if it has a FNSA representation. If (u, μ, ϕ) and (u', μ', ϕ') are both FNSA representations of $\succsim$, then $\mu = \mu'$ and there exist constants $a > 0$ and b such that $u = au' + b$ and $\phi = a\phi'$.*

The second statement of the theorem establishes that the DM's subjective probabilities are unique and that her utility function is unique up to a positive affine transformation. Further, the function ϕ is unique up to a positive transformation.

4 Attitude towards Ambiguity

We now provide a definition that summarizes the DM's attitude towards ambiguity. To that end, let (u, μ, ϕ) be a FNSA representation of $\succsim$. Further, for any $f = (f_1, \ldots, f_n) \in \mathcal{H}$, let $\bar{p}_f := \sum_{s \in S} \mu(s) f_s \in \Delta(Z)$.

Definition 4.1. *A DM whose preferences have a FNSA representation is ambiguity-averse (respectively, ambiguity-loving) if for all $f \in \mathcal{H}$, $\overline{p}_f \succsim f$ (respectively, $f \succsim \overline{p}_f$).*

The definition introduces a notion of ambiguity aversion that mirrors the notion of risk aversion for decisions under risk. Whereas there, when considering monetary gambles, a DM is considered to be risk averse when she prefers the expected value of any gamble to the gamble itself, here we think of the DM as ambiguity averse, if she prefers the objective lottery generated by any act (w.r.t. her probabilistic assessment) to the act itself. The DM's ambiguity attitudes are neatly characterized by the ϕ-function as the following proposition shows.

Proposition 4.1. *An individual with a FNSA representation (u, μ, ϕ) is ambiguity-averse (respectively, ambiguity-loving) if and only if $\phi(.)$ is non-positive (respectively, non-negative).*

We next show that we can elicit a revealed ambiguity relation for a DM with a FNSA representation. To that end, note that for any $f \in \mathcal{H}$, there exists $p_f \in \Delta(Z)$ such that $f \sim p_f$.

Definition 4.2. *We define the revealed ambiguity relation $\succsim^A$ on $\mathcal{H}$ as follows: for any $f, f' \in \mathcal{H}$: $f \succsim^A f'$ if $.5p_f + .5\overline{p}_{f'} \succsim .5p_{f'} + .5\overline{p}_f$*

Observe that the preference $.5p_f + .5\overline{p}_{f'} \succsim .5p_{f'} + .5\overline{p}_f$ represents a tradeoff condition. Given that the DM behaves like a vNM expected utility maximizer on $\Delta(Z)$, this preference tells us that she considers the difference between p_f and $p_{f'}$ to be at least as great as that between $\overline{p}_f$ and $\overline{p}_{f'}$. Further, the difference between p_f and $p_{f'}$ is the same as that between f and f', and $\overline{p}_f$ and $\overline{p}_{f'}$ are, respectively, equivalent to the assessments of these acts based on risk considerations alone. Therefore, the preference $.5p_f + .5\overline{p}_{f'} \succsim .5p_{f'} + .5\overline{p}_f$ reveals that the difference between f and f' exceeds what is warranted by the difference in their risk assessments. This excess must, accordingly, be accounted for by the fact that f has an ambiguity advantage over f' for the DM. For an ambiguity-averse DM, $f \succsim^A f'$, therefore, means that f' is at least as ambiguous as f. On the other hand, for an ambiguity-loving DM, $f \succsim^A f'$ means that f is at least as ambiguous as f'. The function ϕ represents the ambiguity relation $\succsim^A$ as the following proposition clarifies.

Proposition 4.2. *For any $f, f' \in \mathcal{H}$,*

$$f \succsim^A f' \quad \Leftrightarrow \quad \phi(f) \geq \phi(f')$$

Finally, we note that for an ambiguity averse agent, the ambiguity preference relation allows us to identify those acts that have a hedging potential. An act $f \in \mathcal{H}$ has a hedging potential if there exists $f' \in \mathcal{H}$ with $f \succsim^A f'$ such that $\alpha f + (1-\alpha)f' \succ^A f'$ for all $\alpha \in (0,1)$. The act f here has the potential to serve as a hedge because when mixed with an act f' that is more ambiguous, the resulting mixture is less ambiguous than f'.

A Appendix

A.1 FNSA representation of $\succsim$

We first establish the sufficiency of our axioms for the FNSA representation. To that end, we define a function $V : \mathcal{H} \to \mathbb{R}$ that represents $\succsim$.

A Representation of $\succsim$: Observe that the axioms of weak order, Archimedean and risk independence imply that preferences restricted to the set of constant acts satisfy the three vNM axioms. Accordingly, there exists a function $u : Z \to \mathbb{R}$ such that the function $V : \Delta(Z) \to \mathbb{R}$, given by $V(p) = \sum_{x \in Z} p(x)u(x)$, represents $\succsim$ restricted to $\Delta(Z)$. Further, for every act $f \in \mathcal{H}$, monotonicity along with the Archimedean and risk independence axioms imply that there exists $p_f \in \Delta(Z)$ such that $f \sim p_f$. By setting $V(f) := V(p_f)$ for all $f \in \mathcal{H}$ we extend the representation of $\succsim$ from $\Delta(Z)$ to $\mathcal{H}$.

Elicitation of Probabilities: Consider any non-null state $s \in S$ and pairs of AIC acts $((q, p_{-s}), (r, p_{-s}))$, $((\tilde{q}, \tilde{p}_{-s}), (\tilde{r}, \tilde{p}_{-s}))$ with $q \succ p \succ r$ and $\tilde{q} \succ \tilde{p} \succ \tilde{r}$. Note that by non-triviality such lotteries exist. Further, let q', r', $\tilde{q}'$ and $\tilde{r}'$ be such that $q' \sim (q, p_{-s})$, $r' \sim (r, p_{-s})$, $\tilde{q}' \sim (\tilde{q}, \tilde{p}_{-s})$ and $\tilde{r}' \sim (\tilde{r}, \tilde{p}_{-s})$. Given that s is non-null, monotonicity implies that $q \succ q' \succ p \succ r' \succ r$. Similarly, $\tilde{q} \succ \tilde{q}' \succ \tilde{p} \succ \tilde{r}' \succ \tilde{r}$. We next show that there exists a unique $\hat{\alpha} \in [0, 1]$ such that $\hat{\alpha} q + (1 - \hat{\alpha})\tilde{r} \sim \hat{\alpha} r + (1 - \hat{\alpha})\tilde{q}$. To do so we define the function $f(\alpha) := V(\alpha q + (1 - \alpha)\tilde{r}) - V(\alpha r + (1 - \alpha)\tilde{q})$. Clearly, f is continuous and strictly increasing in α. Furthermore, as

$$f(\alpha) = \alpha(V(q) - V(r)) - (1 - \alpha)(V(\tilde{q}) - V(\tilde{r}))$$

we have $f(0) < 0$ and $f(1) > 0$. Accordingly, the intermediate value theorem implies that there exists a unique $\hat{\alpha} \in (0, 1)$ such that

$$f(\hat{\alpha}) = V(\hat{\alpha} q + (1 - \hat{\alpha})\tilde{r}) - V(\hat{\alpha} r + (1 - \hat{\alpha})\tilde{q}) = 0$$

and, thus, $\hat{\alpha} q + (1 - \hat{\alpha})\tilde{r} \sim \hat{\alpha} r + (1 - \hat{\alpha})\tilde{q}$. Similarly, for $q' \succ r'$ and $\tilde{q}' \succ \tilde{r}'$, there exists a unique $\check{\alpha} \in (0, 1)$ such that $\check{\alpha} q' + (1 - \check{\alpha})\tilde{r}' \sim \check{\alpha} r' + (1 - \check{\alpha})\tilde{q}'$.

The CART axiom then implies $\hat{\alpha} = \check{\alpha}$. Consequently, we have

$$\hat{\alpha}(V(q) - V(r)) = (1 - \hat{\alpha})(V(\tilde{q}) - V(\tilde{r}))$$
$$\hat{\alpha}(V(q') - V(r')) = (1 - \hat{\alpha})(V(\tilde{q}') - V(\tilde{r}'))$$

which implies
$$\frac{V(q)-V(r)}{V(\tilde{q})-V(\tilde{r})}=\frac{1-\hat{\alpha}}{\hat{\alpha}}=\frac{V(q')-V(r')}{V(\tilde{q}')-V(\tilde{r}')}$$
and, thus,
$$\frac{V(q,p_{-s})-V(r,p_{-s})}{V(q)-V(r)}=\frac{V(\tilde{q},\tilde{p}_{-s})-V(\tilde{r},\tilde{p}_{-s})}{V(\tilde{q})-V(\tilde{r})}. \tag{1}$$
Now, for any state $s \in S$ define
$$a(s)=\frac{V(q,p_{-s})-V(r,p_{-s})}{V(q)-V(r)},$$
where $(q,p_{-s}),(r,p_{-s})$ is a pair of AIC acts with $q \succ r$. Equation (1) guarantees that $a(s)$ is consistently defined for any non-null state $s \in S$. On the other hand, for any null state $s \in S$, $a(s)=0$ for any such pair of acts. In turn, this allows us to define state probabilities $\mu(s)=\frac{a(s)}{\sum_{s' \in S} a(s')}$, $s \in S$.

FNSA representation: Consider any $f=(f_1,\ldots,f_n) \in \mathcal{H}$ and let $\overline{p}_f := \sum_{s \in S} \mu(s) f_s \in \Delta(Z)$. Define the function $\phi : \mathcal{H} \to \mathbb{R}$ by
$$\begin{aligned}\phi(f) &= V(f)-V(\overline{p}_f)\\ &= V(f)-\sum_{s\in S}\mu(s)\sum_{x\in Z} f_s(x)u(x)\end{aligned}$$
Accordingly,
$$V(f)=\sum_{s\in S}\mu(s)\sum_{x\in Z} f_s(x)u(x)+\phi(f).$$
represents $\succsim$.

Properties of the FNSA representation: Now, we turn to the properties of the ϕ-function. Clearly, $\phi(p)=0$ for all $p \in \Delta(Z)$. Further, consider two pairs of AIC acts $((q,p_{-s}),(r,p_{-s}))$ and $((\tilde{q},\tilde{p}_{-s}),(\tilde{r},\tilde{p}_{-s}))$ with $q \succ p \succ r$ and $\tilde{q} \succ \tilde{p} \succ \tilde{r}$. The difference in the ϕ-values within the first pair of AIC acts can be expressed as

$$\begin{aligned}\phi(q,p_{-s})-\phi(r,p_{-s}) &= V(q,p_{-s})-V(r,p_{-s})-\mu(s)\left[\mathbb{E}u(q)-\mathbb{E}u(r)\right]\\ &= a(s)[V(q)-V(r)]-\mu(s)[\mathbb{E}u(q)-\mathbb{E}u(r)]\\ &= (a(s)-\mu(s))[\mathbb{E}u(q)-\mathbb{E}u(r)]\end{aligned}$$

Similarly,

$$\phi(\tilde{q},\tilde{p}_{-s})-\phi(\tilde{r},\tilde{p}_{-s}) = (a(s)-\mu(s))[\mathbb{E}u(\tilde{q})-\mathbb{E}u(\tilde{r})]$$

Therefore, the residuals have the property that

$$\frac{\phi(\tilde{q},\tilde{p}_{-s})-\phi(\tilde{r},\tilde{p}_{-s})}{\mathbb{E}u(\tilde{q})-\mathbb{E}u(\tilde{r})}=\frac{\phi(q,p_{-s})-\phi(r,p_{-s})}{\mathbb{E}u(q)-\mathbb{E}u(r)}$$

for any pair of AIC acts $((q,p_{-s}),(r,p_{-s}))$ and $((\tilde{q},\tilde{p}_{-s}),(\tilde{r},\tilde{p}_{-s}))$ with $q\succ r$ and $\tilde{q}\succ\tilde{r}$. This establishes sufficiency of the axioms for the representation.

Necessity of axioms: Necessity of the axioms (A1)-(A5) is straightforward to establish and we omit the details here. We now turn to the proof of necessity of the CART axiom. Suppose (q,p_{-s}), (r,p_{-s}), $(\tilde{q},\tilde{p}_{-s})$, $(\tilde{r},\tilde{p}_{-s})$ and q', r', $\tilde{q}'$, $\tilde{r}'$ are as stated in the axiom and s is a non-null state.

Case 1: $q\succ r$, $\tilde{q}\succ\tilde{r}$
Observe that

$$\begin{aligned}&V(\alpha q'+(1-\alpha)\tilde{r}')-V(\alpha r'+(1-\alpha)\tilde{q}')\\ &=\alpha[V(q')-V(r')]-(1-\alpha)[V(\tilde{q}')-V((\tilde{r}')]\\ &=\alpha[V(q,p_{-s})-V(r,p_{-s})]-(1-\alpha)[V(\tilde{q},\tilde{p}_{-s})-V(\tilde{r},\tilde{p}_{-s})]\\ &=\alpha[\mu(s)(\mathbb{E}u(q)-\mathbb{E}u(r))+\phi(q,p_{-s})-\phi(r,p_{-s})]\\ &\quad-(1-\alpha)[\mu(s)(\mathbb{E}u(\tilde{q})-\mathbb{E}u(\tilde{r}))+\phi(\tilde{q},\tilde{p}_{-s})-\phi(\tilde{r},\tilde{p}_{-s})]\\ &=\alpha(\mathbb{E}u(q)-\mathbb{E}u(r))\left[\mu(s)+\frac{\phi(q,p_{-s})-\phi(r,p_{-s})}{\mathbb{E}u(q)-\mathbb{E}u(r)}\right]\\ &\quad-(1-\alpha)(\mathbb{E}u(\tilde{q})-\mathbb{E}u(\tilde{r}))\left[\mu(s)+\frac{\phi(\tilde{q},\tilde{p}_{-s})-\phi(\tilde{r},\tilde{p}_{-s})}{\mathbb{E}u(\tilde{q})-\mathbb{E}u(\tilde{r})}\right]\end{aligned}$$

We know that

$$\frac{\phi(q, p_{-s}) - \phi(r, p_{-s})}{\mathbb{E}u(q) - \mathbb{E}u(r)} = \frac{\phi(\tilde{q}, \tilde{p}_{-s}) - \phi(\tilde{r}, \tilde{p}_{-s})}{\mathbb{E}u(\tilde{q}) - \mathbb{E}u(\tilde{r})}$$

Further, note that

$$\begin{aligned}
(q, p_{-s}) \succ (r, p_{-s}) &\Rightarrow \mu(s)(\mathbb{E}u(q) - \mathbb{E}u(r)) + \phi(q, p_{-s}) - \phi(r, p_{-s}) > 0 \\
&\Rightarrow (\mathbb{E}u(q) - \mathbb{E}u(r)) \left[\mu(s) + \frac{\phi(q, p_{-s}) - \phi(r, p_{-s})}{\mathbb{E}u(q) - \mathbb{E}u(r)}\right] > 0 \\
&\Rightarrow k := \mu(s) + \frac{\phi(q, p_{-s}) - \phi(r, p_{-s})}{\mathbb{E}u(q) - \mathbb{E}u(r)} > 0,
\end{aligned}$$

since $\mathbb{E}u(q) - \mathbb{E}u(r) > 0$. Hence, it follows that

$$\begin{aligned}
&V(\alpha q' + (1-\alpha)\tilde{r}') - V(\alpha r' + (1-\alpha)\tilde{q}') \\
&= k[\alpha(\mathbb{E}u(q) - \mathbb{E}u(r)) - (1-\alpha)(\mathbb{E}u(\tilde{q}) - \mathbb{E}u(\tilde{r}))] \\
&= k[\alpha\mathbb{E}u(q) + (1-\alpha)\mathbb{E}u(\tilde{r}) - (\alpha\mathbb{E}u(r) + (1-\alpha)\mathbb{E}u(\tilde{q}))] \\
&= k[\alpha V(q) + (1-\alpha)V(\tilde{r}) - (\alpha V(r) + (1-\alpha)V(\tilde{q}))] \\
&= k[V(\alpha q + (1-\alpha)\tilde{r}) - V(\alpha r + (1-\alpha)\tilde{q})]
\end{aligned}$$

Hence, $\alpha q' + (1-\alpha)\tilde{r}' \succsim \alpha r' + (1-\alpha)\tilde{q}'$ if and only if $\alpha q + (1-\alpha)\tilde{r} \succsim \alpha r + (1-\alpha)\tilde{q}$.

<u>Case 2</u>: Next, consider the alternative case where either $r \succsim q$ or $\tilde{r} \succsim \tilde{q}$ or both. Clearly, by monotonicity, $r \succsim q \Rightarrow r' \succsim q'$ and $\tilde{r} \succsim \tilde{q} \Rightarrow \tilde{r}' \succsim \tilde{q}'$. First, observe that if $r \sim q$ and $r' \sim q'$, then $\tilde{r} \sim \tilde{q}$ and $\tilde{r}' \sim \tilde{q}'$. In this case for any $\alpha \in [0, 1]$ we have that

$$\alpha q + (1-\alpha)\tilde{r} \sim \alpha r + (1-\alpha)\tilde{q} \quad \text{and} \quad \alpha q' + (1-\alpha)\tilde{r}' \sim \alpha r' + (1-\alpha)\tilde{q}'$$

and the axiom holds. Next, consider the case $r \succ q$ and $\tilde{r} \sim \tilde{q}$. This implies that $r' \succ q'$ and $\tilde{r}' \sim \tilde{q}'$. In this case for all $\alpha \in (0, 1]$, $\alpha r + (1-\alpha)\tilde{q} \succ \alpha q + (1-\alpha)\tilde{r}$ and $\alpha r' + (1-\alpha)\tilde{q}' \succ \alpha q' + (1-\alpha)\tilde{r}'$. Finally for $\alpha = 0$ we have $\tilde{r} \sim \tilde{q}$ and $\tilde{r}' \sim \tilde{q}'$. So in this case as well, the axioms holds. Finally, for $r \succ q$, $\tilde{r} \succ \tilde{q}$, we are back in the first case and the axiom is thus necessary for the representation.

Uniqueness of the representation: Let (u, ϕ, μ) and $(\tilde{u}, \tilde{\phi}, \tilde{\mu})$ both be FNSA representations of $\succsim$. Further, let $V : \mathcal{H} \to \mathbb{R}$ and $\tilde{V} : \mathcal{H} \to \mathbb{R}$ be the corresponding assessment functions of acts. This implies that both u and $\tilde{u}$ are vNM representations of $\succsim$ restricted to $\Delta(Z)$. Hence, there exists $a > 0$, b such that $\tilde{u} = au + b$. Further, for any $f \in \mathcal{H}$, there exists $p_f \in \Delta(Z)$ such that $p_f \sim f$. Accordingly,

$$V(f) = V(p_f) = \sum_{x \in Z} p_f(x) u(x)$$

and

$$\begin{aligned}
\tilde{V}(f) = \tilde{V}(p_f) &= \sum_{x \in Z} p_f(x) \tilde{u}(x) \\
&= a \sum_{x \in Z} p_f(x) u(x) + b \\
&= aV(f) + b.
\end{aligned}$$

Since u and $\tilde{u}$ are non-constant, there exists q, p, r with $q \succ p \succ r$ and $\alpha', \alpha'' \in (0, 1)$ such that $(q, p_{-s}) \sim \alpha' q + (1 - \alpha') r$ and $(r, p_{-s}) \sim \alpha'' q + (1 - \alpha'') r$. Hence,

$$\begin{aligned}
&V(q, p_{-s}) - V(r, p_{-s}) = (\alpha' - \alpha'')(V(q) - V(r)) \\
\Rightarrow &\frac{\mu(s)(\mathbb{E}u(q) - \mathbb{E}u(r)) + [\phi(q, p_{-s}) - \phi(r, p_{-s})]}{\mathbb{E}u(q) - \mathbb{E}u(r)} = \alpha' - \alpha'' \\
\Rightarrow &\mu(s) + \frac{\phi(q, p_{-s}) - \phi(r, p_{-s})}{\mathbb{E}u(q) - \mathbb{E}u(r)} = \alpha' - \alpha'' \\
\Rightarrow &\mu(s) + \frac{V(q, p_{-s}) - V(\overline{p}_{(q, p_{-s})}) - (V(r, p_{-s}) - V(\overline{p}_{(r, p_{-s})}))}{\mathbb{E}u(q) - \mathbb{E}u(r)} = \alpha' - \alpha'',
\end{aligned}$$

where $\overline{p}_{(q,p_{-s})} = \mu(s)q + (1 - \mu(s))p$ and $\overline{p}_{(r,p_{-s})} = \mu(s)r + (1 - \mu(s))p$. Likewise, we have

$$\tilde{\mu}(s) + \frac{\tilde{V}(q, p_{-s}) - \tilde{V}(\overline{p}_{(q, p_{-s})}) - (\tilde{V}(r, p_{-s}) - \tilde{V}(\overline{p}_{(r, p_{-s})}))}{\mathbb{E}\tilde{u}(q) - \mathbb{E}\tilde{u}(r)} = \alpha' - \alpha''.$$

But, as $\tilde{V}(.) = aV(.) + b$ and $\tilde{u} = au + b$ this implies $\mu(s) = \tilde{\mu}(s)$ for all $s \in S$.

Finally, for any $f \in \mathcal{H}$,

$$\begin{aligned}
\tilde{\phi}(f) &= \tilde{V}(f) - \sum_{s\in S} \tilde{\mu}(s) \sum_{x\in Z} \tilde{f}_s(x)u(x) \\
&= aV(f) + b - a\sum_{s\in S} \mu(s) \sum_{x\in Z} f_s(x)u(x) - b \\
&= a\left(V(f) - \sum_{s\in S} \mu(s) \sum_{x\in Z} f_s(x)u(x)\right) \\
&= a\phi(f).
\end{aligned}$$

A.2 Proofs of Propositions 4.1 and 4.2

The proofs of propositions 4.1 and 4.2 are straightforward and we omit details here.

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