New Contest Success Functions

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Abstract

Skaperdas (1996) characterized the contest success function (CSF), which stipulates the winning probabilities of the contestants, using respectively the scale invariance and translation invariance axioms. This paper first characterizes the entire family of CSFs that fulfils a convex mixture of the two axioms. This family contains the Skaperdas CSFs as special cases. Next, we consider two ordinal axioms, scale consistency and translation consistency, and characterize the respective classes of CSFs. While the former consists of the Skaperdas scale invariant and translational invariant CSFs and some new functional forms, the latter contains the Skaperdas translation invariant CSF and some additional CSFs that were not considered in the literature earlier.

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1. Introduction

A contest refers to a non-cooperative game in which two or more participants contend for a prize. Models of contest have been employed extensively to analyze a variety of phenomena like rent seeking (Tullock 1980, Nitzan 1991, Baye et al. 2005, Amegashie, 2006), conflict (Hirshleifer 1991, Skaperdas 1992), polarization (Esteban and Ray 2011), electoral candidacy (Snyder 1989, Skaperdas and Grofman 1995), sporting tournament (Szymanzki 2003), provision of public goods (Kolmar and Wagener 2011) and reward structure in firms (Rosen 1986)[1]. In a contest, agents make irretrievable investments, which depending on the situation; can be money, effort or any other valuable resource.

Essential to the notion of a contest is a contest success function (CSF), which specifies a contestant's probability of winning the contest and obtaining a prize. An increase in each contestant's outlay increases his chances of winning the contest and reduces his opponents' chances. In a highly interesting contribution, Skaperdas (1996) characterized this probability for any contestant as the ratio between the level of effective investment made by the contestant and the sum of effective investments across all the contestants. The effective investment of a contestant can be interpreted as the output determined by his effort, which may be regarded as his input in the contest. It is assumed to be an increasing and positive valued function of effort.

Skaperdas (1996) also developed axiomatic characterizations of two functional forms of CSFs. One of the axioms employed by Skaperdas (1996) is an anonymity principle which demands that a contestant's probability of success depends only on his outlays. Thus, the agents are not distinguished by any characteristic other than their outlays. Clark and Riis (1997) broadened the Skaperdas (1996) framework by allowing the contestants to differ with respect to their contest-related personal characteristics. Rai and Sarin (2009) generalized the characterizations of Skaperdas (1996) to the situation where agents can have investments that are of multiple types in nature. Münster (2009) extended the Skaperdas (1996) and Clark and Riis (1997) characterizations to contests between groups.

The two characterizations of the functional forms advanced by Skaperdas (1996) invoke two alternative axioms of invariance. The first axiom, the scale invariance postulate, demands that an equiproportionate change in the efforts of all the agents will keep the winning probabilities

[1] The literature has been surveyed by Nitzan (1994), Corchon(2007), Konrad (2009) and Skaperdas and Garfinkel (2012). See also Dixit (1987) for a general discussion.

unchanged. In contrast, the second axiom, which is known as the translation invariance postulate, requires invariance of winning probabilities under equal absolute changes in the efforts of all the agents. The underlying effective investment functions turn out to be of power function and logit function type respectively. (A systematic comparison of the properties of these two functional forms is available in Hirshleifer (1989).)

A natural generalization of scale and translation invariance axioms is an intermediate condition , which stipulates that a convex mixture of an equiproportionate change and an equal absolute change in the efforts should keep winning probabilities unchanged. One objective of this paper is to characterize the entire class of CSFs that satisfies this generalized invariance concept. It is explicitly shown that the two functional forms characterized by Skaperdas (1996) become particular cases of the CSF that meets intermediate equivalence.

Both the scale and translation invariance conditions are cardinal postulates. They can be relaxed to more general ordinal postulates. One such postulate that ensures ordinal property of CSFs is the scale consistency axiom, which says that if all the agents are participating in two contests and for some agents the probabilities of winning one contest are less than or equal to that of winning the other, then an equiproportionate change in the efforts of the agents in both contests will not alter the agents' ordering of chances of winning the contests. To understand this, suppose the investements are measured in money units, say euro. Then of two contests, CI and CII, suppose some individuals' chances of winning CI are more than that of CII. Now, if investments are converted into dollars from euro, the inequality between chances of winning CI and CII should not alter. Scale consistency demands this condition. Note that since the sum of probabilities of winning a contest across the agents is one if for some agents the probabilities of winning one contest over another are lower, then there will be at least one agent for whom the reverse inequality for probabilities of winning the contests will hold. CSFs satisfying scale invariance are definitely scale consistent. However, there are CSFs that are scale consistent but not scale invariant.

Likewise, we can have a translation consistency axiom, which specifies that inequality between winning probabilities for two contests should remain invariant under equal absolute changes in all the efforts. Translation consistency implies translation invariance but the converse is not true.

A second objective of the paper is to axiomatize the classes of CSFs that are scale and translation consistent respectively. It is shown explicitly that the Skaperdas (1996) CSFs that verify scale and translation invariance axioms become particular cases respectively of the scale and translation consistent CSFs characterized in this paper. It is explicitly shown that the logit CSF satisfies both scale and translation consistencies.

2. The Formal Framework

Let $N = \{1,2,...,n\}$ be a set of agents participating in a contest and let y_i stand for effort or investment of agent $i \in N$ in the contest. We denote the vector of investments $(y_1, y_2,.., y_n)$ $\in [0,\infty)^n$ by y, where $[0,\infty)^n$ is the $n-$fold Cartesian product of $[0,\infty)$. The success of any contestant is probabilistic. For any $y \in [0,\infty)^n$, each contestant i's probability of winning the contest is denoted by $p^i(y)$. Evidently, $p^i : [0,\infty)^n \to [0,1]$. The non-negative function p is called the Contest Success Function (CSF).

The following axioms for a CSF have been suggested by Skaperdas (1996).

(A1) $\sum_{i=1}^{n} p^i(y) = 1$ and for all $y \in [0,\infty)^n$, if $y_i > 0$ then $p^i(y) > 0$.

(A2 $p^i(y)$ is increasing in y_i and decreasing in y_j for all $j \neq i$.

(A3)For any permutation $\pi: N \to N$, $p^{\pi(i)}(y) = p(y_{\pi_1}, y_{\pi_2}, \dots, y_{\pi_n})$.

(A4)For all $M \subseteq N$ with at least two elements, the probability of success of agent $i \in M$ in a contest among the members of M is $p_m^i(y) = \dfrac{p^i(y)}{\sum_{j \in M} p^i(y)}$.

(A5) $p_m^i(y)$ is independent of the efforts of the players not included in the subset $M \subset N$ or $p_m^i(y)$ can be written as $p_m^i(y_m)$, where $y_m = (y_j; j \in M)$.

(A 5′) $p^i(y) = \dfrac{f(y_i)}{\sum_{j \in N} f(y_j)}$ for all $i \in N$ and $p_m^i(y) = \dfrac{f(y_i)}{\sum_{j \in M} f(y_j)}$ for all $i \in M$ $(\subseteq N)$, where

$f : [0,\infty) \to [0,\infty)$ is positive on $(0,\infty)$ and increasing in its argument, where for at least one

$j \in M$, $p^j(y) > 0$.

(A1) states that the sum of winning probabilities across the participants in a contest is 1 and if some participant's outlay is positive he has a positive chance of winning the contest. (A2) says that a participant's probability of success is increasing in his own effort but decreasing in the efforts of the other participants. According to (A3), the probability of success remains invariant under any reordering of the participant. This is an anonymity condition which demands that any characteristic other than individual outlays is irrelevant to the determination of success probabilities. (A4) is a consistency condition, which says that for any subgroup of participants, the probabilities of success of the members of the subgroup are the conditional probabilities obtained by restricting the original probability distribution to the subgroup. (A5) means that for any subgroup of participants, the success probabilities are independent of the outlays of the participants who are not members of the subgroup. Finally, $(A5')$ provides a particular specification of the winning probabilities using a positive valued increasing function of efforts. We can refer to $f(y_i)$ as the effective investment made by contestant i. Increasingness of f reflects the view that an increase in the actual investment increases effective investment. Skaperdas (1996) demonstrated that (A1)-(A5) hold simultaneously if and only if the CSF is of the form specified in $(A5')$.

In order to identify specific functional forms of CSFs, Skaperdas (1996) imposed the following axioms:

(A6) (Scale invariance): $p^i(y) = p^i(\lambda y)$ for all $\lambda > 0$ and for all $i \in N$.

(A7)(Translation invariance) $p^i(y) = p^i(y + c1^n)$, where 1^n is the $n-$coordinated vector of ones and c is a scalar such that $y_i + c \geq 0$ for all $i \in N$.

The scale invariance axiom (A6) is a homogeneity condition, which says that proportional changes in the efforts of all the contestants do not change the winning probabilities. In contrast, (A7) is a translation invariance axiom, which demands that winning probabilities remain unchanged when all the efforts are augmented or diminished by the same absolute quantity.

It has been shown in Skaperdas (1996) that a continuous CSF satisfies (A1) – (A5) and (A6) if and only if it is of the power function type, that is, of the form $p^i(y) = \frac{y_i^\delta}{\sum_{j \in N} y_j^\delta}$, where $\delta > 0$ is a constant. Continuity of a CSF ensures that minor observational errors on investment do not change winning probabilities abruptly. The particular case $\delta = 1$ was considered by Esteban and Ray (2011) in a behavioural model of conflict that provides a link between conflict, inequality and polarization. On the other hand, as Skaperdas (1996) established, the logit function, that is, $p^i(y) = \frac{e^{\theta y_i}}{\sum_{j \in N} e^{\theta y_j}}$ is the only continuous CSF that satisfies (A1) – (A5) and (A7), where $\theta > 0$ is a constant. It is easy to verify that the only CSF that satisfies (A6) and (A7) is the constant function $p^i(y) = \frac{1}{n}$. But constancy of a CSF is ruled out by the assumption that $p^i(y)$ is increasing in y_i and decreasing in y_j for all $j \neq i$.

However, (A6) and (A7) turn out to be polar cases of the following intermediate invariance postulate:

(A8) $p^i\big(y + c(\mu y + (1-\mu)1^n)\big) = p^i(y),$

where μ, $0 \le \mu \le 1$, is a parameter which reflects a contestant's view on winning probability equivalence, c is a scalar such that $x + c(\mu x + (1-\mu)1^n \in [0,\infty)^n$ and 1^n , the $n-$coordinated vector of ones, expressed in the unit of measurement of efforts, so that $y = x + c(\mu x + (1-\mu)1^n$ becomes well defined. The scale and translation invariance criteria given by (A6) and (A7) emerge as polar cases of the intermediate notion (A8) when μ takes on the values 1 and 0 respectively. As the value of μ increases (decreases) to one (zero) the contestant becomes more concerned about scale (translation) invariance[2].

The following theorem isolates the CSF that satisfies (A8). We first identify the CSF for the parametric range $0 < \mu < 1$. The two extreme cases will be discussed later.

Theorem 1: Assume that the CSF is continuously differentiable in efforts. Then it satisfies axioms (A1) – (A5) and (A8) if and only it is of the following form

$$p^i(y) = \frac{[1+\mu(y_i - 1)]^{\frac{\eta}{\mu}}}{\sum_{j\in N}[1+\mu(y_j - 1)]^{\frac{\eta}{\mu}}} \quad , \tag{1}$$

where $\eta > 0$ is a constant and $0 < \mu < 1$.

Proof: By Theorem 1 of Skaperdas (1996), axioms (A1) – (A5) are satisfied if and only if the CSF is given by (A$5'$). Consider $(y_1, y_2) \in (0,\infty)^2$ and note that $p^i(y) = \dfrac{f(y_i)}{f(y_1) + f(y_2)}, i = 1,2$. Then by (A8) we get,

$$\frac{f[c(1+\mu)y_1 + c(1-\mu)]}{f(y_1)} = \frac{f[c(1+\mu)y_2 + c(1-\mu)]}{f(y_2)} \quad , \tag{2}$$

where for simplicity it is assumed that $c > 0$. From (2) it follows that $\dfrac{f[c(1+\mu)z + c(1-\mu)]}{f(z)}$ is independent of the effort level z. Differentiating $\dfrac{f[c(1+\mu)z + c(1-\mu)]}{f(z)}$ with respect z we get,

[2] In (A8) if we replace p^i by an inequality index and y by the income distribution in an $n-$person society, then the resulting axiom becomes the Bossert-Pfingsten (1990) intermediate inequality equivalence axiom. See also Chakravarty (2014) for a recent discussion.

$$\frac{d}{dz}\left(\frac{f\left[c(1+\mu)z+c(1-\mu)\right]}{f(z)}\right)=0, \tag{3}$$

which implies that

$$(c\mu+1)f(z)f'\{(c\mu+1)z+c(1-\mu)\}=f'(z)f\{(c\mu+1)z+c(1-\mu)\}, \tag{4}$$

where f' stands for the derivative of f.

Equation (4) holds for all finite $z>0$. Putting $z=1$ on each side of (4) we get

$$(c\mu+1)f(1)f'(c+1)=f'(1)f(c+1), \tag{5}$$

from which it follows that

$$\frac{f'(c+1)}{f(c+1)}=\frac{\eta}{(c\mu+1)}, \tag{6}$$

where $\eta=\frac{f'(1)}{f(1)}>0$ (since f is positive valued and increasing on $(0,\infty)$). Integrating both sides of (6) we get,

$$lnf(c+1)=\frac{\eta}{\mu}ln(c\mu+1)+k, \tag{7}$$

which yields:

$$f(c+1)=(c\mu+1)^{\frac{\eta}{\mu}}e^k. \tag{8}$$

Thus,

$$f(z)=\xi\{\mu(z-1)+1\}^{\frac{\eta}{\mu}} \tag{9}$$

where $\xi=e^k,\eta>0$ are constants. By continuity of f, the solution extends to the case where $z=0$. Substituting this form of f into $p^i(y)=\frac{f(y_i)}{\sum_{j\in N}f(y_j)}$ we get the desired form of the CSF. This establishes the necessity part of the theorem. The sufficiency is easy to verify. Δ

As $\mu\to 0$, $p^i(y)$ in (1) approaches $\frac{e^{\eta y_i}}{\sum_{j\in N}e^{\eta y_j}}$, the Skaperdas(1996) CSF associated with (A7) (given that $\theta=\eta$). (Here for evaluating the limit we use the fact that $\lim_{z\to 0+}(1+z)^{\frac{1}{z}}=e$.) On other hand, for $\mu=1$, $p^i(y)$ given by (1) coincides with the Skaperdas(1996) CSF corresponding to (A6) (given that $\eta=\delta$). Thus, p^i in (1) may be regarded as a generalization of scale and translation invariant CSFs.

Note that the scale invariance condition (A6) can very well be relaxed to the following more general ordinal property.

(A9) Scale Consistency: For $x, y \in [0,\infty)^n$, if for some $i \in N$, $p^i(y) \geq p^i(x)$ holds, then $p^i(\lambda y) \geq p^i(\lambda x)$ for all $\lambda > 0$.

Evidently, scale consistency implies scale invariance, but the converse is not true. Note that satisfaction of $p^i(\lambda y) \geq p^i(\lambda x)$ for all $\lambda > 0$ implies fulfilment of $p^i(y) \geq p^i(x)$. Note also that if $p^i(y) > p^i(x)$ holds, then there is at least one contestant $j \neq i$ such that $p^j(y) < p^j(x)$ holds. The reason for this is that $\sum_{i=1}^{n} p^i(y) = \sum_{i=1}^{n} p^i(y) = 1$. (A9) is an ordinal property in the sense that the inequality remains invariant under any ordinal transformation Ω of p^is. Furthermore, $\Omega(p^i)$s given by $\Omega(p^i(y)) = \frac{\Omega(p^i(y))}{\sum_{j\in N}\Omega(p^j(y))}$, $i \in N$, are probabilities[3].

The following theorem identifies the class of CSFs that fulfils (A9).

Theorem 2: Assume that the CSF is continuously differentiable in efforts. Then it satisfies axioms (A1) – (A5) and (A9) if and only it is of the following form

$$p^i(y) = \begin{cases} \dfrac{B^{y_i^{\beta}}}{\sum_{j\in N} B^{y_j^{\beta}}}, \\ \dfrac{y_i^{B}}{\sum_{j\in N} y_j^{B}}, \end{cases} \tag{10}$$

where B is a positive constant and β is a non-zero real number.

Proof: By Theorem 1 of Skaperdas (1996), axioms (A1) – (A5) are satisfied if and only if the CSF is given by (A5′). Consider $(y_1, y_2),(x_1, x_2) \in (0,\infty)^2$. Observe that $p^1(y) = \frac{f(y_1)}{f(y_1)+f(y_2)}$. Then $p^1(y) \geq p^1(x)$ is same as $\frac{f(y_1)}{f(y_1)+f(y_2)} \geq \frac{f(x_1)}{f(x_1)+f(x_2)}$, that is, if and only if $\frac{f(y_2)}{f(y_1)} \leq \frac{f(x_2)}{f(x_1)}$.
Thus, by (A8) we have,

$$\frac{f(y_2)}{f(y_1)} \leq \frac{f(x_2)}{f(x_1)} \text{ if and only if } \frac{f(\lambda y_2)}{f(\lambda y_1)} \leq \frac{f(\lambda x_2)}{f(\lambda x_1)} \text{ for all } \lambda > 0. \tag{11}$$

[3] (A9) becomes Zheng's (2007) unit consistency axiom if we replace p^i by an inequality index, y and x by income distributions in two $n-$person societies and the weak inequality $\geq$ by the strict inequality> in (see also also Chakravarty 2014).

Now, we claim that $\frac{f(\lambda x_2)}{f(\lambda x_1)} = F_\lambda\left(\frac{f(x_2)}{f(x_1)}\right)$ for some non-decreasing function F_λ. To demonstrate this, consider two distinct effort vectors $(y_1', y_2'), (x_1', x_2') \in (0,\infty)^2$. Then we have, $\frac{f(\lambda x_2')}{f(\lambda x_1')} = \frac{f(\lambda y_2')}{f(\lambda y_1')}$ if and only if $\frac{f(x_2')}{f(x_1')} = \frac{f(y_2')}{f(y_1')}$. This implies that $\frac{f(\lambda x_2)}{f(\lambda x_1)}$ is a function of $\frac{f(x_2)}{f(x_1)}$. Non-decreasingness of this function is a consequence of (11)

Define

$$u_\lambda(x_1, x_2) = \frac{f(\lambda x_2)}{f(\lambda x_1)} \tag{12}$$

and

$$q(x_1, x_2) = \frac{f(x_2)}{f(x_1)} \tag{13}$$

Since u_λ and q are functionally related, the Jacobian of u_λ and q with respect to x_1 and x_2 must vanish. More precisely,

$$\begin{vmatrix} \frac{\partial u_\lambda}{\partial x_1} & \frac{\partial u_\lambda}{\partial x_2} \\ \frac{\partial q}{\partial x_1} & \frac{\partial q}{\partial x_2} \end{vmatrix} = 0. \tag{14}$$

This implies that

$$\frac{f(\lambda x_2) f'(\lambda x_1) f'(x_2)}{f(\lambda x_1)} = \frac{f'(x_1) f(x_2) f'(\lambda x_2)}{f(x_1)}. \tag{15}$$

Equation (15) can be rearranged as

$$\frac{f'(\lambda x_1)}{f(\lambda x_1)} \cdot \frac{f'(x_2)}{f(x_2)} = \frac{f'(\lambda x_2)}{f(\lambda x_2)} \cdot \frac{f'(x_1)}{f(x_1)}. \tag{16}$$

Now, (16) holds for all $(x_1, x_2) \in (0,\infty)^2$. Putting $x_1 = z > 0, x_2 = 1$ in (16) and letting $h(z) = \frac{f'(z)}{f(z)}$, we get

$$h(\lambda z) h(1) = h(z) h(\lambda). \tag{17}$$

Given that f is positive valued on $(0,\infty)$ and increasing, h is positive. It is continuous as well. Since (8) holds for all positive z and λ, it is a fundamental Cauchy equation, of which the only continuous solution is given by

$$h(z) = K_1 z^{\alpha} \tag{18}$$

for some $K_1 > 0$ and α is a real number (Aczel, 1966, p. 41, Theorem 3). By continuity of h the solution extends to the case where $z = 0$.

Case I: $\alpha \neq -1$

Then (18) yields:

$$\frac{f'(z)}{f(z)} = K_1 z^{\alpha}. \tag{19}$$

Integrating both sides of (19) we get,

$$\ln(f(z)) = Kz^{\alpha+1} + K', \tag{20}$$

where K and K' are real numbers. Equation (20) is equivalent to

$$f(z) = AB^{z^{\beta}} \quad , \tag{21}$$

where $A = e^{K'} > 0, B = e^{K} > 0$ and $\beta = 1 + \alpha$ is a non-zero real number.

Case II: $\alpha = -1$.

Then (18) becomes:

$$\frac{f'(z)}{f(z)} = K_1 z^{-1} \quad , \tag{22}$$

which, on integration, gives

$$\ln(f(z)) = K_1 \ln(z) + K'. \tag{23}$$

.

That is,

$$f(z) = Az^{B} \tag{24}$$

where $A = e^{K'} > 0$ and B is a real number. Since Since f is increasing, we further require the restriction $B > 0$.

Plugging the forms of f given by (21) and (24) into $p^i(y) = \frac{f(y_i)}{\sum_{j \in N} f(y_j)}$, we get the forms of $p^i(y)$ specified in (10).This completes the necessity part of the proof of the theorem. The sufficiency can be easily verified by checking that CSF given by (10) fulfils (A1)-(A5) and (A9). Δ

The second CSF in (10) is the Skaperdas CSF corresponding to (A6) (assuming that $B=\delta$). However, the first functional form in (10) is new, it has not been characterized earlier in the literature. In this first functional form if we substitute $B=e^{K_1}$ and $\beta=1$, then the resulting CSF coincides with the logit functional form (under the assumption $K_1=\theta$). However, if $\beta\neq 1$ then the underlying CSF is a violator of (A9).

Finally we consider the following ordinal counterpart to (A7):

(A10) Translation Consistency:. For $x,y\in[0,\infty)^n$, if for some $i\in N$, $p^i(y)\geq p^i(x)$ holds, then) $p^i\left(y+c1^n\right)\geq p^i\left(x+c1^n\right)$, where 1^n is the $n-$coordinated vector of ones and c is a scalar such that $y_i+c\geq 0$ for all $i\in N$.

Evidently, (A10) is sufficient but not necessary for (A7). Like (A9), (A10) is also an ordinal property.

In the following theorem we characterize the entire class of CSFs that are translation consistent.

Theorem 3: Assume that the CSF is continuously differentiable in efforts. Then it satisfies axioms (A1) – (A5) and (A10) if and only it is of the following form

$$p^i(y)=\begin{cases}\dfrac{H^{e^{\rho y_i}}}{\sum\limits_{j\in N}H^{e^{\rho y_j}}},\\[2ex] \dfrac{e^{\upsilon y_i}}{\sum\limits_{j\in N}e^{\upsilon y_j}},\end{cases} \tag{25}$$

where H and υ are positive constants and ρ is a non-zero real number.

Proof: Take, as in the proof Theorem 2, $(y_1,y_2),(x_1,x_2)\in(0,\infty)^2$. Then $p^1(y)\geq p^1(x)$ is same as $\frac{f(y_2)}{f(y_1)}\leq\frac{f(x_2)}{f(x_1)}$. By (A9),

$$\frac{f(y_2)}{f(y_1)}\leq\frac{f(x_2)}{f(x_1)} \text{ if and only if } \frac{f(y_2+c)}{f(y_1+c)}\leq\frac{f(x_2+c)}{f(x_1+c)} \text{for all } c>0. \tag{26}$$

As in the proof of Theorem 2, one can easily see that there exists a continuous and increasing function G_c such that

$$\frac{f(x_2+c)}{f(x_1+c)}=G_c\left(\frac{f(x_2)}{f(x_1)}\right). \tag{27}$$

Define

$$w_c(x_1,x_2)=\frac{f(x_2+c)}{f(x_1+c)}. \tag{28}$$

Since $w_c$and q are functionally related, the Jacobian of $w_c$and q in (13) with respect to x_1 and x_2 must vanish. This implies that

$$\frac{f'(x_1+c)}{f(x_1+c)}\cdot\frac{f'(x_2)}{f(x_2)}=\frac{f'(x_2+c)}{f(x_2+c)}\cdot\frac{f'(x_1)}{f(x_1)} \tag{29}$$

Equation (29) holds for al $(x_1,x_2)\in(0,\infty)^2$. Putting $x_1=z>0, x_2=\varepsilon>0$ and substituting $\frac{f'(z)}{f(z)}$ by $\psi(z)$, which is positive on $(0,\infty)$, we get

$$\psi(z+c)\psi(\varepsilon)=\psi(z)\psi(c+\varepsilon). \tag{30}$$

Letting $\varepsilon\to 0$ and using continuous differentiability of f , from (30) it follows that

$$\psi(z+c)\psi(0)=\psi(z)\psi(c). \tag{31}$$

From (31) it emerges that $\psi(0)>0$. The equation (31) holds for all positive z and c. The only continuous solution to (31) is given by

$$\psi(z)=\upsilon e^{\rho z} \tag{32}$$

for some positive $\upsilon(=\psi(0))$ and real ρ (see Aczel, 1966, p.84) . By continuity of ψ , the solution extends to the case when $z=0$

From (31) it is evident that

$$\frac{f'(z)}{f(z)}=\upsilon e^{\rho z}. \tag{33}$$

Case I $\rho\neq 0$

Integrating both sides of (33) we get,

$$ln(f(z))=K_3\upsilon e^{\rho z}+K_4, \tag{34}$$

where K_3 and K_4 are real numbers. That is,

$$f(z)=EH^{e^{\rho z}} \tag{35}$$

where $E=e^{K_4}$ and $H=e^{K_3\upsilon}$ are positive constants.

Case II: $\rho=0$

Then (33) becomes:

$$\frac{f'(z)}{f(z)} = \upsilon \ . \qquad (36)$$

Integrating both sides of (36) we get,

$$lnf(z) = \upsilon z + C \qquad (37)$$

for some real number C .

Equation (37) is equivalent to:

$$f(z) = Q\, e^{\upsilon z} \qquad (38)$$

$Q = e^{C} > 0$. For increasingness of f we need the restriction $\upsilon > 0$. Substituting the forms of f given by (35) and (38) in $p^{i}(y) = \frac{f(y_i)}{\sum_{j \in N} f(y_j)}$, the resulting forms of $p^{i}(y)$ become the ones specified in (25) . Hence the necessity part of the theorem is demonstrated. The sufficiency follows easily. Δ

The second CSF in (25) (given that $\theta = \eta$) has been characterized by Skaperdas (1996) using (A7). However, the first CSF in (25) was not suggested in the literature earlier.

3. Conclusions

Axiomatic characterizations of contest success functions enable us to understand them in an intuitively reasonable way in the sense that necessary and sufficient conditions are identified to isolate them uniquely. Skaperdas (1996) characterized the power and logit type forms of success functions. In this paper we have substantially extended the characterizations of Skarpedas (1996) by considering a general axiom and two more axioms, scale and translation consistencies, which are ordinal in nature, a characteristic that has not been explored earlier in the literature. The Skaperdas (1996) functional forms drop out as particular cases of the functional forms axiomatized in the paper. Some new functional forms that have not been suggested earlier in the literature are also analyzed. It is clearly indicated that the Skaperdas (1996) translation invariant CSF is scale consistent although it is not scale invariant.

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