Repeated Bargaining: A Mechanism Design Approach

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Abstract

Through a model of repeated bargaining between a buyer and a seller, with changing private information on both sides, this paper addresses questions of efficiency and institutional structures in dynamic mechanism design. A new technical device in the form of a history dependent version of payoff equivalence is established. A new notion of interim budget balance is introduced which allows for the role of an intermediary but with bounded credit lines. We then construct a mechanism, which provides a necessary and sufficient condition for efficiency under interim budget balance. The existence of a future surplus can be used as collateral to sustain efficiency, and its size determines the possibility. The mechanism also offers a simple and realistic implementation. A characterization of efficient implementation under ex post budget balance is also provided. Further, a characterization for the second best is presented, and its equivalent Ramsey pricing formulation is established. A suboptimal, but incentive compatible mechanism for the second best with intuitive properties is presented. When property rights are fluid, that is, the good can be shared, a folk theorem with a simple mechanism is constructed.

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1 Introduction

In a paper that would generate much interest amongst economists and legal scholars, Coase [1960] argued that if transactions costs are low enough and trade a possibility, bargaining will eventually lead to an efficient outcome independent of the initial distribution of property rights. A few decades later, in perhaps an equally influential paper, Myerson and Satterthwaite [1983] showed that under reasonable institutional assumptions, asymmetric information precludes efficient trade. A key missing link in Coase's argument was established as part of the growing acceptance of the role of information in economic transactions.[1]

Myerson and Satterthwaite [1983] , along with many other papers that came before and after, asked important questions of institution design under varying objectives- efficiency, revenue maximization, etc. Public goods provision, procurement, auctions, optimal taxation, bilateral trading, wage contracts are just some applications of the general theory of mechanism design that has thus developed.[2]

Most of these papers, including the two aforementioned, dealt with static or one-time interactions. Arguably, many of these economic transactions are inherently dynamic, where information revealed today can be used for contract design tomorrow. Food subsidies are provided repeatedly. In a fast changing technological landscape, spectrum auctions and buybacks are taking place repeatedly. Taxation is often dynamic and tagged with age, social security being a case in point. Wage contracts and bonuses depend on performance parameters evaluated over time. Online selling can now rely on a huge treasure trove of past buying data.

This paper seeks to provide a theory of such dynamic institutions and contribute towards the burgeoning literature on dynamic mechanism design.[3] When are efficient institutions self enforcing? There are three key words in the preceding statement. By, efficiency we mean first-best or the optimal allocation of resources without any additional frictions or binding constraints. Institution is an environment characterized by a set of rules that internalize underlying frictions. And, self enforcing, refers to the ability of the institution to implement the desired objective under no external subsidies.

To answer this question, we choose the well studied static problem of bargaining under two-sided asymmetric information that concerned Myerson and Satterthwaite [1983]. A seller wants to repeatedly sell a non-durable good to a buyer. Their valuations for the good

[1]Introducing the *Myerson and Satterthwaite Theorem*, as now it is popularly called, Milgrom [2004] writes

> "Doubts about the [Coase's] efficiency axiom are based partly in concern about bargaining with incomplete information. After all, a seller is naturally inclined to exaggerate the cost of his good, and a buyer is inclined to pretend that her value is low. Should we not expect these exaggerations to lead sometimes to missed trading opportunities?"

[2]See Mas-Collel, Whinston and Green [1995], and Jackson [2003] for a thorough overview of the literature.

[3]See Bergemann and Said [2010] and Vohra [2012] for insightful surveys.

are privately known and can change over time. Repetition can blunt the impossibility of efficiency result of Myerson and Satterthwaite [1983]. We formalize the extent and logic of this "blunting".

First, we introduce a new concept of budget balance which we term *interim budget balance*.[4] The motivation is to allow for the role of a financial intermediary or mechanism designer, but one who cannot have access to an unbounded credit line. At any given history, the expected value of current and future cash flows from the buyer and the seller must be non-negative. Interim budget balance can be seen as the mechanism design counterpart to self enforcing constraints from the relational contracting literature[5], but not on the side of the agents, rather the institution itself; and standard bond issuing deficit financing constraints in macro models.[6]

Conceptually, we want to temper the role commitment plays in dynamic mechanism design models. In standard dynamic screening contracts, the effect of incentive constraints mitigates over time, thereby pushing most inefficiency to the early periods[7]; and individual rationality constraints bind only in period 1, putting restrictions just on the ex ante aggregate expected transfers. But with interim budget balance, the incentive constraints can bind "strongly" at any point in the future depending on the extent of budgeting, and individual rationality constraints can bind beyond the first period. The intuition is similar to the static case. It is the simultaneous interaction of incentives with budget balance and participation that can create inefficiency. With interim budget balance, this interaction takes place every period and hence the expected transfers have to be re-calibrated repeatedly, not just in the beginning.

Second, we state and prove a history dependent version of the *payoff equivalence result*. Any incentive compatible mechanism must differ from another that seeks to implement the same allocation only through linear translations of the history dependent expected utilities of the agents. The, undoubtedly insightful, ex ante dynamic versions of the payoff equivalence formula that have been presented so far[8] cannot answer the questions- what impact will changes in transfers in any particular history have on incentives going forward, and how can these incentives be preserved? Our dynamic payoff equivalence formula precisely answers these.

Third, exploiting this payoff equivalence, we ask the question: balancing the budget in the interim sense as described above, when can efficiency be attained under perfect Bayesian and ex post notions of implementability? We offer a necessary and sufficient condition on the fundamentals of the model and an intuitively appealing implementation in the form of the *Collateral Dynamic VCG mechanism.* Each period, both agents set aside

[4]The standard notions in the literature are that of ex ante budget balance- aggregate ex ante expected cash flow to the mechanism designer is non-negative, and ex post budget balance- transfers sum to zero every period for any history.

[5]See Thomas and Worrall [1988], and Levin [2003]

[6]See Ljungqvist and Sargent [2004].

[7]For iid types, in fact, it lasts only one period.

[8]See Pavan, Segal and Toikka [2013], Eso and Szentes [2013] and Skrzypacz and Toikka [2013], and Battaglini and Lamba [2013] for an equivalent characterization with discrete types.

the maximum possible transfers that they can, to be used as collateral, while satisfying incentives and participation through the VCG mechanism. These are then used to balance the budget for all possible types. If this mechanism cannot (weakly) produce a surplus, no other mechanism will. The ability of mechanism to produce efficiency in terms of the fundamentals of the model- in particular, the levels of asymmetric information (captured by persistence) and discounting, is discussed carefully.

Fourth, we also offer a precise characterization of the second best, defined to be the maximization of expected gains from trade. Using a simple AR(1) model, we explore the properties of the no-trade regions. The difference between ex ante and interim notions of budget balance become apparent. The interim budget constraints are likely to bind precisely when the ex ante constraints are easy to satisfy, for example, for low levels of persistence. A general closed form solution to the second best has so far proven to be technically elusive. So, we adapt two different routes to extract more economic intuition out of the model.

A dynamic Ramsey pricing interpretation, along the lines of Bulow and Roberts [1989] shows that trade happens only when the virtual surplus, that is surplus with internalized information rents, is positive. Re-calibration of commitment as described above means that each type has many competing notions of virtual surpluses and the pivotal one is determined by the set of binding interim budget balance constraints. Moreover, in order to get around the problem of binding global incentive constraints, we offer a class of suboptimal contracts with appealing features. Loosely speaking, the no-trade regions are monotonic functions of the history of realizations.

Fifth and final, we consider two variations to the model. We allow the seller to share the property rights to the good with the buyer. In the absence of a financial intermediary, that is under ex post budget balance, and under a decentralized version of the problem in the from of a stochastic game (rather than a dynamic mechanism), by selling part of the good to the buyer at the start of very period before bargaining commences, efficiency can be achieved if discounting is high enough to preserve incentives. Lastly, in the appendix, we introduce a suitable notion of dominant strategy for dynamic mechanisms and show that under the strictest possible institutional architecture of dominant strategy implementation with ex post budget balance, only memory-less posted prices are feasible.

It is important to note that all results we describe can be generalized to dynamic mechanism design problems with N agents and quasilinear utilities. In particular, the history dependent payoff equivalence result, notion of interim budget balance, the Collateral Dynamic VCG mechanism, characterization of the second best, Ramsey pricing formulation, monotonic contracts and the folk theorem with shared property rights, each have their N player counterpart at the cost of a bit more notation.

Related Literature. The bilateral trading problem we study has a rich tradition in the static mechanism design literature- Myerson and Satterthwaite [1983], and Chatterjee and Samuelson [1983] being two of the early papers. Myerson and Satterthwaite [1983] impose all the institutional details on the model and then establish the impossibility of efficiency. On the other hand, Williams [1999] and Krishna and Perry [2000] fix an efficient, incentive compatible and individually rational mechanism and show that it can never satisfy budget balance, thereby again proving the same impossibility result using a different technique. We will exploit both approaches in the dynamic model.

In the dynamic mechanism design literature, Baron and Besanko [1982], Besanko [1985], Battaglini [2005], Pavan, Segal and Toikka [2013], Battaglini and Lamba [2013], and Eso and Szentes [2013] present various versions of the dynamic envelope formula that summarizes the local incentive compatibility constraints.[9] The technique adopted for proving the history dependent payoff equivalence result, viz. the change of variables from transfers to expected utility vectors, parallels Battaglini and Lamba [2013], who do the same for discrete types.

Athey and Segal [2007, 2013] and Bergemann and Valimaki [2010] directly construct dynamic versions of the VCG mechanism, while the former is concerned about budget balance, the latter has an efficient exit condition. They key difference with Athey and Segal [2007, 2013] is that in the dynamic mechanism design problem they do not allow for individual rationality in every period, thereby demanding a very strong form of commitment. When they do allow the agents to walk away, they establish a folk theorem for efficiency in a stochastic game version of their model.

This paper is the most closely related to Athey and Miller [2007] and Skrzypacz and Toikka [2013]. Athey and Miller [2007] study the repeated bilateral trading problem under iid types, ex ante and ex post budget balance, and ex post incentive compatibility. They use a bounded budget account to show approximate efficiency under ex post budget balance. Skrzypacz and Toikka [2013] analyze the same problem with persistent types and multidimensional initial information. They establish a necessary and sufficient condition for efficiency under ex ante budget balance, thereby allowing for unbounded credit lines.[10]

While stressing voluntary participation in each period, we seek to characterize efficiency for an intermediate notion of budget balance, one that allows for the role of an intermediary with a bounded credit line. We also want to be able to impose greater restrictions on the cash flows to the intermediary. The implementation of the Collateral Dynamic VCG mechanism requires distribution of future surplus as collateral every period, in comparison to the one time participation fees in Skrzypacz and Toikka [2013], which may require large amounts of seed capital on the part of the agents, in addition to the unbounded credit line being offered by the mechanism designer.

Skrzypacz and Toikka [2013] provide a characterization of the second best under an infinite horizon Gaussian model. Our characterization for the second best, modulo global incentive constraints, is the most general possible. Unlike the rest of the literature, we seek to get a better understanding of the second best mechanism under the various notions

[9]See also Courti and Li [2000] and Eso and Szentes [2007] for sequential screening models.

[10]Using the balancing trick of Athey and Segal [2007, 2013], this condition also guarantees implementation under ex post budget balance, but then like Athey and Segal, a strong form of commitment is required on part of the agents by allowing individual rationality only in period 1.

of budget balance. To that end we provide a complete characterization of no trade regions for a class of AR(1) models.

The section on property rights is inspired by Cramton, Gibbons and Klemperer [1987], who show that efficiency can be attained in the Myerson and Satterthwaite [1983] setting if one allows for property rights to be shared. And, the section on detail free mechanisms draws on Hagerty and Rogerson [1987], who show that in a static model, under dominant strategies only posted prices are feasible.

Plan. We introduce the model in section 2. This is followed by the institutional architecture in section 3, that is, definitions of incentive compatibility, individual rationality, budget balance, and efficiency. A change of variables is also undertaken which is a key step in proving the payoff equivalence result. In section 4, we present a simple two period model to motivate some of the key forces that apply in the general results in the two sections that follow it. Section 5 constructs the Collateral Dynamic VCG mechanism and establishes its salience in completely characterizing efficiency. Section 6 provides a precise formulation of the second-best mechanism, characterizes an AR(1) model, then discusses a dynamic Ramsey pricing formulation and applies monotonic contracts to deal with binding global incentive compatibly constraints. Section 7 establishes for a folk theorem when property right are fluid. Section 8 discusses avenues for future research. Proofs and other details not in the main text can be found in the appendix.

2 Model

Two agents, each with private information, agree to be in a dynamic bilateral trading relationship for a non-durable good. The buyer (B) has a hidden valuation for the good and the seller (S) is endowed with a technology to produce the good each period at a hidden cost. We assume that buyer's valuation and the seller's cost are random variables[11], denoted v and c, distributed according to priors F and G on $\mathcal{V} = [\underline{v}, \overline{v}]$ and $\mathcal{C} = [\underline{c}, \overline{c}]$, that evolve according to independent Markov processes $F(.|.) : \mathcal{V} \times \mathcal{V} \to [0, 1]$ and $G(.|.) : \mathcal{C} \times \mathcal{C} \to [0, 1]$, respectively.[12] The priors and Markov processes have continuous densities, denoted by f, g, $f(.|.)$ and $g(.|.)$, and full support, and the Markov processes are differentiable in the second argument, that is, the past type.

Each period p_t determines the probability of trade, that is, the production and allocation of the good from the seller to the buyer, $x_{B,t}$ the transfer from the buyer to the mechanism designer and $x_{S,t}$ the transfer to the seller from the mechanism designer. The mechanism designer here can be considered as a financial intermediary, an institution as part of a larger social contract facilitating trade, or a simple transfer scheme in case $x_B = x_S$. The per period payoffs are given by $v_t p_t - x_{B,t}$ and $x_{S,t} - c_t p_t$, for the buyer

[11] These shall be interchangeably referred to as their types.

[12] All the main results can accommodate moving supports. It would simply entail a change of notation to $\mathcal{V}_t$ and $\mathcal{C}_t$, to denote the respective supports in each period.

and seller respectively.[13,14]

We assume that taking the institutional details as given, both the buyer and seller can commit to the mechanism. The institutional details temper the role commitment will play in the model, as we elaborate below. Both the agents know their first period valuation and cost respectively when the contract is signed, and these then stochastically evolve over time. This assumption is crucial for it sows the seeds of asymmetric information in the model with commitment.

The (contractual) relationship lasts lasts for T discrete periods, where $T \leq \infty$. Both the agents discount future payoffs with a common discount factor δ. The static version of this model, $\delta = 0$, is exactly the one studied by Myerson and Satterthwaite [1983], and Chatterjee and Samuelson [1983].

It is easy to show that a form of revelation principle holds and thus we can, without loss of generality, consider direct mechanisms. Every period the agents learn their own types, and then send a report to the mechanism, which in turn, spits out the allocation and transfers rules. Employing the revelation principle, however, demands a moral call on the information the mechanism itself reveals to the agents. In particular, does the buyer observe the seller's announcement and vice-versa? We shall work in an environment where the announcements are publicly observed. There is a close information theoretic relationship between this *public* mechanism and the *blind* one where the announcements are not publicly observed. We refer the reader to Skrzypacz and Toikka [2013] for a discussion on this.

The direct mechanism, say m, is then a collection of history dependent probability and transfer vectors, $m = \langle \mathbf{p}, \mathbf{x} \rangle = \left(p\left(v_t, c_t \left| h^{t-1}\right.\right), x_B\left(v_t, c_t \left| h^{t-1}\right.\right), x_S\left(v_t, c_t \left| h^{t-1}\right.\right)\right)_{t=1}^{T}$, where h^{t-1} and (v_t, c_t) are, respectively, the public history up to period $t-1$ and the types revealed at time t. These can also be succinctly written as $p(h^t)$, etc. In general, h^t is defined recursively as $h^t = \left\{h^{t-1}, (v_t, c_t)\right\}$, with $h^0 = \emptyset$. The set of possible histories at time t is denoted by H^t (for simplicity $H = H^T$).

The strategies of the buyer and the seller can potentially depend on a richer set of histories. For the buyer, the information available before his period t report is given by $h_B^t = \{h_B^{t-1}, v_{t-1}, \hat{v}_t\}$, where v_{t-1} is the announced type in period $t-1$, and $\hat{v}_t$ is the actual type in period t, starting with $h_B^0 = \{\hat{v}_1\}$. The seller's information is analogously defined. Let the set of private histories at time t be denoted by H_B^t and H_S^t, respectively. Thus, for a given mechanism, the strategy for the buyer, $(\sigma_{B,t})_{t=1}^{T}$, is then simply a function that maps private history into an announcement every period, $\sigma_{B,t} : H_B^t \mapsto \mathcal{V}$, and similarly for the seller, $\sigma_{S,t} : H_S^t \mapsto \mathcal{C}$.

[13]The t subscript will not be used when the set of histories make the time dimension obvious.

[14]An equivalent model is one where the seller is endowed with a good every period and needs to decide whether she should sell the it to the buyer or consume it.

3 The Institutional Framework

The edifice of the institutional machinery has three key foundations: information rents, voluntary participation and limits on insurance. In the mechanism design lexicon, these would respectively be associated with incentive compatibility, individual rationality and budget balance constraints.

For a fixed mechanism m and strategies $\sigma = (\sigma_B, \sigma_S)$, the expected utilities on the induced allocation and transfers, after each possible history are defined as follows.

$$U_B^{m,\sigma}(h_B^t) = \mathbb{E}^{m,\sigma}\left[\sum_{\tau=t}^{T} \delta^{\tau-1}\left(v_\tau p_\tau - x_{B,\tau}\right)|h_B^\tau\right] \tag{1}$$

and,

$$U_S^{m,\sigma}(h_S^t) = \mathbb{E}^{m,\sigma}\left[\sum_{\tau=t}^{T} \delta^{\tau-1}\left(x_{S,\tau} - c_\tau p_\tau\right)|h_S^\tau\right] \tag{2}$$

Though, along truthful histories the difference between public and private histories is moot, and thus in much of what follows we shall suppress the same. Let $U_i^m = U_i^{m,\sigma^*}$, for $i = B, S$; where σ^* is the truth-telling strategy.

3.1 A Change of Variables

We propose a change of variables in the structure of the mechanism that will be central in our endeavor to establish a history dependent payoff equivalence result. In order to keep notation simple we suppress the type/variable over which expectation is taken. For example

$$p(v_t|h^{t-1}) = \int_{\mathcal{C}} p\left(v_t, c_t \left|h^{t-1}\right.\right) dG(c_t|c_{t-1}),$$

where c_{t-1} is the $t-1$ period announcement of the seller, known to the buyer, and,

$$p(v_{t+1}|h^{t-1}, v_t) = \int_{\mathcal{C}}\int_{\mathcal{C}} p\left(v_{t+1}, c_{t+1} \left|h^{t-1}, v_t, c_t\right.\right) dG(c_t|c_{t-1}) dG(c_{t+1}|c_t)$$

Expected utility of the buyer can be recursively defined as

$$U_B(v_t, c_t|h^{t-1}) = v_t p(v_t, c_t|h^{t-1}) - x_B(v_t, c_t|h^{t-1}) + \delta \int_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v_t, c_t) dF(v_{t+1}|v_t) \tag{3}$$

and,

$$U_B(v_t|h^{t-1}) = v_t p(v_t|h^{t-1}) - x_B(v_t|h^{t-1}) + \delta \int_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v_t) dF(v_{t+1}|v_t)$$

Utility of the buyer of type v_t from misreporting (once) to be type v'_t, for a fixed type c_t of the seller, can be succinctly written as

$$U_B(v'_t; v_t, c_t|h^{t-1}) = v_t p(v'_t, c_t|h^{t-1}) - x_B(v'_t, c_t|h^{t-1}) + \delta \int_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v'_t, c_t) \cdot dF(v_{t+1}|v_t)$$

$$= U_B(v'_t, c_t|h^{t-1}) + (v_t - v'_t) p(v'_t, c_t|h^{t-1}) + \delta \int_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v'_t, c_t) \cdot \big(dF(v_{t+1}|v_t) - dF(v_{t+1}|v'_t)\big) \tag{4}$$

Similarly,

$$U_B(v'_t; v_t|h^{t-1}) = U_B(v'_t|h^{t-1}) + (v_t - v'_t) p(v'_t|h^{t-1}) + \delta \int_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v'_t) \cdot \big(dF(v_{t+1}|v_t) - dF(v_{t+1}|v'_t)\big)$$

The seller's utility, U_S, can be similarly defined.

It is straightforward to note that a mechanism $m = \langle \mathbf{p}, \mathbf{x} \rangle$, which is a collection of history dependent allocation and transfer vectors, can be equivalently defined to be $m = \langle \mathbf{p}, \mathbf{U} \rangle$, where (fixing the allocation) the duality between transfers and expected utility vectors is completely described by equation (3).

3.2 Incentive Compatibility

We say that mechanism, m, is *Bayesian incentive compatible* if

$$U_B^m(v_1) \geq U_B^{m,(\sigma_B, \sigma_S^*)}(v_1) \quad \text{and} \quad U_S^m(c_1) \geq U_S^{m,(\sigma_B^*, \sigma_S)}(c_1),$$

$\forall\ v_1 \in \mathcal{V}$, $\forall\ c_1 \in \mathcal{C}$, all possible strategies σ_B and σ_S. If in addition, the game induced by the mechanism and the type space admits σ^* as a perfect Bayesian equilibrium, we say that the mechanism in *perfect Bayesian incentive compatible* (PBIC). Finally, the mechanism is periodic *ex post incentive compatible* (EPIC), if for both agents, truth telling is a best response after any truthful history and for any realization of current type of the other agent.[15]

Exploiting the one-deviation principle, incentive compatibility can be defined as follows.

Definition 1. *A mechanism $m = \langle \mathbf{p}, \mathbf{U} \rangle$ satisfies perfect Bayesian incentive compatibility if*

$$U_B(v_t|h^{t-1}) \geq U_B(v'_t; v_t|h^{t-1}) \quad \text{and} \quad U_S(c_t|h^{t-1}) \geq U_S(c'_t; c_t|h^{t-1})$$

$\forall v_t, v'_t \in \mathcal{V}$, $\forall c_t, c'_t \in \mathcal{C}$, $\forall h^{t-1} \in H^{t-1}$, $\forall t$.

A *stronger* equilibrium notion as described above is the that of ex-post incentive compatibility. The mechanism in each period is implemented in ex-post equilibrium (see Chung

[15] To quote Bergemann and Valimaki [2010], "We say that the mechanism is *periodic ex post incentive compatible* if truth-telling is a best response regardless of the history or the current state of the other agents."

and Ely [2006]). Formally,

Definition 2. *A mechanism $m = \langle \mathbf{p}, \mathbf{U} \rangle$ satisfies ex post incentive compatibility if*

$$U_B(v_t, c_t|h^{t-1}) \geq U_B(v_t'; v_t, c_t|h^{t-1}) \quad \textit{and} \quad U_S(v_t, c_t|h^{t-1}) \geq U_S(c_t'; v_t, c_t|h^{t-1})$$

$\forall v_t, v_t' \in \mathcal{V}$, $\forall c_t, c_t' \in \mathcal{C}$, $\forall h^{t-1} \in H^{t-1}$, $\forall t$.

3.3 Individual Rationality

Even though commitment is assumed as part of our institutional architecture, we find it compelling to allow the agents- the buyer and the seller- to walk away at any stage if their utility from continuing in the contract falls below their reservation thresholds, which are normalized to zero. We say that a mechanism is *perfect Bayesian individually rational* (PBIR) if

$$U_B^m(h_B^t) \geq 0 \quad \textit{and} \quad U_S^m(h_S^t) \geq 0,$$

$\forall$ $h_B^t \in H_B^t$ and $h_S^t \in H_S^t$. Similar to incentive compatibility, if in addition the utility is required to be greater than or equal to the reservation value for every realization of current type of the other agent, we say the mechanism is *ex post individually rational* (EPIR). In keeping with our notation, we have:

Definition 3. *A mechanism $m = \langle \mathbf{p}, \mathbf{U} \rangle$ satisfies perfect Bayesian individually rationality if*

$$U_B(v_t|h^{t-1}) \geq 0 \quad \textit{and} \quad U_S(c_t|h^{t-1}) \geq 0$$

$\forall v_t \in \mathcal{V}$, $\forall c_t \in \mathcal{C}$, $\forall h^{t-1} \in H^{t-1}$, $\forall t$.

Definition 4. *A mechanism $m = \langle \mathbf{p}, \mathbf{U} \rangle$ satisfies ex post individually rationality if*

$$U_B(v_t, c_t|h^{t-1}) \geq 0 \quad \textit{and} \quad U_S(v_t, c_t|h^{t-1}) \geq 0$$

$\forall v_t \in \mathcal{V}$, $\forall c_t \in \mathcal{C}$, $\forall h^{t-1} \in H^{t-1}$, $\forall t$.

We say that a mechanism is perfect Bayesian implementable if it is perfect Bayesian incentive compatible and individually rational, and it is ex post implementable if it is ex post incentive compatible and individually rational.

3.4 Budget Balace

In mechanism design with many agents budget balance is seen as the limits on insurance or external subsidies available to them. In addition to the traditional notions of ex ante and ex post budget balance, we introduce an intermediate notion of interim budget balance.

We say that a mechanism is *interim budget balanced* if

$$\mathbb{E}^m \left[\sum_{\tau=t}^{T} \delta^{\tau-t} \left(x_{B,\tau} - x_{S,\tau} \right) \mid h^{t-1} \right] \geq 0$$

$\forall\ h^{t-1} \in H^{t-1}$.[16] The mechanism is *ex ante budget balanced* if interim budget balance holds for the null history. Moreover, we say that the mechanism is *ex post budget balanced* if the entire vector of transfers are equal for any history, $x_B = x_S$.

Next, using equations (1) and (2) we can write the expected budget surplus that a mechanism generates after any history h^{t-1} to be

$$EBS(h^{t-1}) = \mathbb{E}^m \left[\sum_{\tau=t}^{T} \delta^{\tau-t} (v_\tau - c_\tau) p_\tau - U_B(v_t|h^{t-1}) - U_S(c_t|h^{t-1}) \mid h^{t-1} \right] \tag{5}$$

The ex ante budget surplus is denoted simply by $EBS = EBS(h^0)$. We have,

Definition 5. *A mechanism $\langle \mathbf{p}, \mathbf{U} \rangle$ satisfies ex ante budget balance if*

$$EBS \geq 0$$

This is the weakest possible notion of budget balance for this dynamic model. It means that the mechanism designer does not loose money in an expected ex ante sense. A more robust definition of budget balance in our opinion, which still allows for the role of an intermediary is the one where a positive budget surplus is guaranteed after every history.

Definition 6. *A mechanism $\langle \mathbf{p}, \mathbf{U} \rangle$ satisfies interim budget balance if*

$$EBS(h^{t-1}) \geq 0 \quad \forall h^{t-1} \in H^{t-1},\ \forall t$$

This can be motivated in many ways. First, it can be viewed as a participation constraint for the mechanism designer- after any history, just like the the two agents, the mechanism designer must have a an incentive to continue in the relationship. Second, it is a bankruptcy constraint for the intermediary. If the contract reaches a stage the where the intermediary is expected to loose money for sure, he or she should be allowed to shut shop.

Finally, the most standard (and strictest) definition of budget balance from the static literature that can be generalized to dynamic environments states the transfers should exactly equal across all histories for all time periods.

Definition 7. *A mechanism $m = \langle \mathbf{p}, \mathbf{x} \rangle$ satisfies ex post budget balance if*

$$x_B(v_t, c_t|h^{t-1}) - x_S(v_t, c_t|h^{t-1}) = 0,$$

$\forall v_t \in \mathcal{V}$, $\forall c_t \in \mathcal{C}$, $\forall h^{t-1} \in H^{t-1}$, $\forall t$.[17]

[16] The exact definition will employ almost sure notions on the set of histories. It will be obvious and is suppressed for the ease of exposition.

[17] Equivalently, a mechanism $m = \langle \mathbf{p}, \mathbf{U} \rangle$ satisfies ex post budget balance if

$$(v_t - c_t)p(v_t, c_t|h^{t-1}) - \left(U_B(v_t, c_t|h^{t-1}) - \delta \int_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v_t, c_t) dF(v_{t+1}|v_t) \right) -$$

A natural way to motivate this in the dynamic model is the absence of an outside insurance provider or financial intermediary. A contractual relationship is thus more the order of interpretation rather than a mechanism.

Note the hierarchy in budget balance

$$\text{ex post BB} \;\Rightarrow\; \text{interim BB} \;\Rightarrow\; \text{ex ante BB}$$

There is also an equivalence relationship between instantaneous (or stage) notion of expected budget surplus[18] and ex post budget balance. Details are provided in the appendix.

3.5 Objectives

One of the most widely accepted objectives of mechanism design is that of efficiency.[19] We shall invoke the strongest possible version in its ex post form.

Definition 8. *A mechanism $m = \langle \mathbf{p}, \mathbf{U} \rangle$ satisfies efficiency if*

$$p(v_t, c_t|h^{t-1}) = \begin{cases} 1 & \textit{if } v_t > c_t \\ 0 & \textit{otherwise} \end{cases}$$

$\forall v_t \in \mathcal{V}$, $\forall c_t \in \mathcal{C}$, $\forall h^{t-1} \in H^{t-1}$, $\forall t$.

Thus, regardless of history, under a positive instantaneous surplus and only then, efficiency demands trade and always with probability 1.

There are a plethora of situations, captured by the fundamentals of the model, where efficient trade is not possible. Many competing notions of second-best are in the fray, some of which are discussed in section 8. We employ the most standard one of ex ante expected gains from trade.

Definition 9. *A mechanism $m = \langle \mathbf{p}, \mathbf{U} \rangle$ satisfies second-best efficiency if it maximizes the ex ante expected gains from trade, that is,*

$$GFT = \mathbb{E}^m \left[\sum_{t=1}^{T} \delta^{t-1} (v_t - c_t)\, p_t \right]$$

In a price theory formulation of the problem, this would be referred to as the maximization of surplus. This benchmark is especially salient because in the absence of asymmetric information or when the budget balance constraints do not bind, it will implement the efficient allocation.

$$\left(U_S(v_t, c_t|h^{t-1}) - \delta \int_{\mathcal{C}} U_S(c_{t+1}|h^{t-1}, v_t, c_t) dG(c_{t+1}|c_t) \right) = 0$$

$\forall v_t \in \mathcal{V}$, $\forall c_t \in \mathcal{C}$, $\forall h^{t-1} \in H^{t-1}$, $\forall t$.

[18] Defined as $EBS_t(h^{t-1}) = \mathbb{E}^m \left[x_B(v_t, c_t|h^{t-1}) - x_S(v_t, c_t|h^{t-1}) \right]$

[19] See Holmstrom and Myerson [1983] for the various notions of efficiency.

4 A Simple Two Period Model

To fix ideas and motivate the intuition is a simple way we first present a two period model. The mechanism designer wishes to maximize expected gains from trade for the model presented above with $T = 2$. Results for both ex ante BB and interim BB are discussed.[20]

The analysis in this section takes a direct approach a'la Myerson and Satterthwaite [1983]. We collapse the necessary local incentive compatibility conditions into an envelope formula which is then plugged into the expected budget surplus constraint. Note, however, that because the problem is dynamic this exercise is repeated for every possible history.

We first provide the flavor of the general argument, and then proceed to solve specific examples to formally lay down the key forces in our understanding of efficient self enforcing institutions. A word on the notation is in order. Since this is a two period model, h will represent the history of announced types in the first period, and h_i its component with respect to agent i, for $i = B, S$. We also provide the proof of the lemma below for a quick refresher on the standard techniques of dynamic mechanism design.[21]

Lemma 1. *For any perfect Bayesian incentive compatible mechanism,*

$$EBS + U_B(\underline{v}) + U_S(\overline{c}) = \int_{\mathcal{V}} \int_{\mathcal{C}} \Bigg[\left\{ \left(v_1 - \frac{1 - F(v_1)}{f(v_1)} \right) - \left(c_1 + \frac{G(c_1)}{g(c_1)} \right) \right\} p(v_1, c_1)$$

$$+\delta \int_{\mathcal{V}} \int_{\mathcal{C}} \left\{ \left(v_2 - \frac{1 - F(v_1)}{f(v_1)} \left[-\frac{\partial F(v_2|v_1)/\partial v_1}{f(v_2|v_1)} \right] \right) - \left(c_2 + \frac{G(c_1)}{g(c_1)} \left[-\frac{\partial G(c_2|c_1)/\partial c_1}{g(c_2|c_1)} \right] \right) \right\}$$

$$\times p(v_2, c_2|v_1, c_1) g(c_2|c_1) f(v_2|v_1) dc_2 dv_2 \Bigg] g(c_1) f(v_1) dc_1 dv_1,$$

and

$$EBS(h) + U_B(\underline{v}|h) + U_S(\overline{c}|h) = \int_{\mathcal{V}} \int_{\mathcal{C}} \left[\left(v_2 - \frac{1 - F(v_2|h_v)}{f(v_2|h_v)} \right) - \left(c_2 + \frac{G(c_2|h_c)}{g(c_2|h_c)} \right) \right]$$

$$\times p(v_2, c_2|h) g(c_2|h_c) f(v_2|h_v) dc_2 dv_2$$

Proof. Incentive compatibility in period 2 for any history h gives,

$$\left(v_2 - v_2' \right) p(v_2|h) \geq U_B(v_2|h) - U_B(v_2'|h) \geq \left(v_2' - v_2 \right) p(v_2'|h)$$

[20] Since the horizon is finite, the latter also precisely characterizes the allocations that can be implemented under ex-post BB. See section 10.1 in the appendix.

[21] For simplicity, in this section, in a slight abuse of notation, we will denote $U_B(\underline{v}) = \inf_{v \in \mathcal{V}} U_B(v)$ and $U_B(\overline{c}) = \inf_{c \in \mathcal{C}} U_S(c)$. This is true if Markov processes satisfy first order stochastic dominance, but not in general. It is, however, not essential for our results.

The envelope formula for period 2 thus follows,

$$U_B(v_2|h) = U_B(\underline{v}|h) + \int_{\underline{v}}^{v_2} p(\tilde{v}_2|h)d\tilde{v}_2 \tag{6}$$

Employing incentive compatibility in period 1 gives

$$(v_1 - v_1')p(v_1) + \delta \int_{\mathcal{V}} U_B(v_2|v_1) \cdot \left[dF(v_2|v_1) - dF(v_2|v_1')\right]$$

$$\geq U_B(v_1) - U_B(v_1') \geq$$

$$(v_1 - v_1')p(v_1') + \delta \int_{\mathcal{V}} U_B(v_2|v_1') \cdot \left[dF(v_2|v_1) - dF(v_2|v_1')\right]$$

Using (6) and integration by parts, gives the dynamic envelope formula,

$$U_B(v_1) = U_B(\underline{v}) + \int_{\underline{v}}^{v_1} \left[p(\tilde{v}_1) + \delta \int_{\mathcal{V}} p(v_2).\left(-\frac{\partial F(v_2|\tilde{v}_1)/\partial \tilde{v}_1}{f(v_2|\tilde{v}_1)}\right) dv_2\right] dF(\tilde{v}_1) \tag{7}$$

Similarly, we get

$$U_S(c_2|h) = U_S(\overline{c}|h) + \int_{c_2}^{\overline{c}} p(\tilde{c}_2|h)d\tilde{c}_2 \tag{8}$$

and,

$$U_S(c_1) = U_S(\overline{c}) + \int_{c_1}^{\overline{c}} \left[p(\tilde{c}_1) + \delta \int_{\mathcal{C}} p(c_2).\left(-\frac{\partial G(c_2|\tilde{c}_1)/\partial \tilde{c}_1}{g(c_2|\tilde{c}_1)}\right) dc_2\right] dG(\tilde{c}_1) \tag{9}$$

Now, in period 2, we can write the expected budget surplus as

$$EBS(h) = \int_{\mathcal{V}} \int_{\mathcal{C}} \left[x_B(v_2, c_2|h) - x_S(v_2, c_2|h)\right] g(c_2|h_c) f(v_2|h_v) dc_2 dv_2$$

$$= \int_{\mathcal{V}} \int_{\mathcal{C}} \left[(v_2 p(v_2, c_2|h) - U_B(v_2, c_2|h)) - (c_2 p(v_2, c_2|h) + U_S(v_2, c_2|h))\right] g(c_2|h_c) f(v_2|h_v) dc_2 dv_2$$

which using equations (6) and (8), and integration by parts can be written as

$$EBS(h) + U_B(\underline{v}|h) + U_S(\overline{c}|h) = \int_{\mathcal{V}} \int_{\mathcal{C}} \left[\left(v_2 - \frac{1 - F(v_2|h_v)}{f(v_2|h_v)}\right) - \left(c_2 + \frac{G(c_2|h_c)}{g(c_2|h_c)}\right)\right]$$

$$\times p(v_2, c_2|h) g(c_2|h_c) f(v_2|h_v) dc_2 dv_2$$

Similarly, using equations (7) and (9) in

$$EBS = \int\limits_{\mathcal{V}} \int\limits_{\mathcal{C}} \Big[(v_1 p(v_1, c_1) - U_B(v_1, c_1)) - (c_1 p(v_1, c_1) + U_S(v_1, c_1))$$

$$+\delta \int\limits_{\mathcal{V}} \int\limits_{\mathcal{C}} [v_2 p(v_2, c_2 | v_1, c_1) - c_2 p(v_2, c_2 | v_1, c_1)]\, g(c_2|h_c) f(v_2|h_v) dc_2 dv_2 \Big] g(c_1) f(v_1) dc_1 dv_1$$

and integrating by parts, we get the desired equality. □

It is important to note that the above result can be stated and proven verbatim for ex post incentive compatibility. The key to the indifference is that the expected budget surplus takes expectation over all current and future types of both agents.

Lemma 1 provides some indication of the general results to follow. It exactly characterizes the expected budget surplus for every possible history in terms of the allocation and fundamentals of the model. Characterization under interim BB follows immediately.

Let EBS^{**} and $EBS^{**}(h)$ refer specifically to the incentive compatible and individually rational mechanisms where

$$U_B(\underline{v}) = U_S(\overline{c}) = U_B(\underline{v}|h) = U_S(\overline{c}|h) = 0$$

for all $v \in \mathcal{V}$, $c \in \mathcal{C}$, and $h \in \mathcal{V} \times \mathcal{C}$.

Corollary 1. *A perfect Bayesian (or ex post) incentive compatible and individually rational mechanism can be implemented under interim BB if and only if $EBS^{**} \geq 0$ and $EBS^{**}(h) \geq 0$ for all $h \in \mathcal{V} \times \mathcal{C}$.*

Proof. Sufficiency is obvious. For necessity, note that EBS^{**} and $EBS^{**}(h)$ are the highest expected budget surpluses that can be generated for their respective histories while satisfying IC and IR. If they are not non-negative no other IC and IR mechanism can ensure them to be. □

Removing transfers, the second best mechanism can then be explicitly formulated by the following result.

Corollary 2. *A perfect Bayesian incentive compatible and individually rational allocation maximizes expected gains from trade under interim BB if only if it solves*

$$\max_{\mathbf{p}} \; GFT$$

subject to

$$EBS^{**} \geq 0, \quad EBS^{**}(h) \geq 0 \;\forall h, \;\textit{and}$$

$$\mathbf{p} \textit{ is PBIC}$$

Implementability of $\mathbf{p}$ is essentially a requirement that global incentive constraints be satisfied.[22] We will have more to say on that in section 6. Corollary 2 is a two period version of the general Proposition 3 that will follow in section 6.

A substantial appeal of the static model with linear preferences and continuous types is that the solution is always bang-bang, the probability of trade is always 0 or 1. Since the objective and all constraints are linear in the allocation, the same insight goes through in the dynamic model too.

Perhaps the one of the most studied simple expositions of the competing economic forces of information, commitment and budget balance in mechanism design is the uniform bilateral trading problem- static version of our model with a uniform prior.[23] In order to clearly bring out the prominent economic forces in a simple fashion, we work in a natural extension of this model to the dynamic environment, and present the optimal mechanisms under the extreme ends of the information space. In the next three subsections the types of the buyer and the seller are assumed to be uniform on [0,1] in both periods.[24]

4.1 Static Benchmark

Following Myerson and Satterthwaite [1983], let us first consider the case where $\delta = 0$. Then, writing down the problem in corollary 2 as a Lagrangian[25], it is easy to show that trade happens, that is, $p(v_1, c_1) = 1$, if and only if $v_1 > c_1 + M$, where M solves the binding EBS constraint

$$\int_0^1 \int_0^1 \mathbf{1}_{\{v_1 > c_1 + M\}} \left[2v_1 - 1 - 2c_1\right] dc_1 dv_1 = 0$$

$$\text{i.e.,} \quad \frac{1}{6}(4M - 1)(1 - M)^2 = 0$$

Thus, trade happens if and only if $v_1 > c_1 + \frac{1}{4}$. Since efficiency demands trade for every $v_1 \geq c_1$; $M = \frac{1}{4}$ precisely characterizes the no trade region and the degree of inefficiency. Since the allocation is monotonic, it is implementable. Figure 1a captures the no-trade region pictorially. The solid diagonal represents the locus $v_1 = c_1$. Efficient allocation requests trade above the solid diagonal. The area above the dotted line represents the actual trade region.

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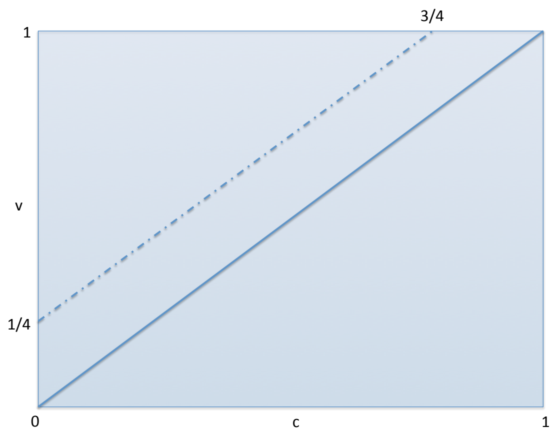
[22] In the static model it is replaced by the familiar monotonicity condition on the allocation.

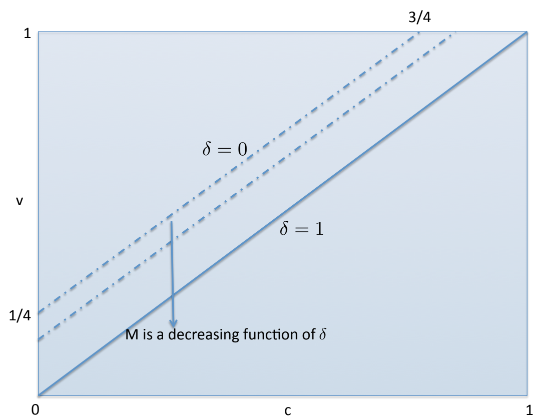
[23] See Myerson [1985] and Gibbons [1992].

[24] We use the iid model to motivate ideas because it offers a simple and complete characterization. Moreover, as we will see in the AR(1) model in section 6, even a two period persistent types model with interim BB is very hard to characterize in general.

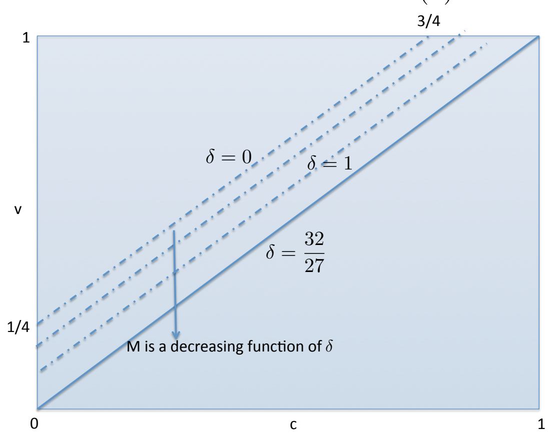
[25] $\int_0^1 \int_0^1 \left[(v_1 - c_1) + \lambda\left(2v_1 - 1 - 2c_1\right)\right] p(v_1, c_1) dc_1 dv_1$, where λ is the multiplier on EBS. Details are in the appendix.



(a) Static



(b) IID with ex ante BB



(c) IID with interim BB

Figure 1: No-trade regions

4.2 IID case

Now, suppose the types of both agents are distributed uniformly on [0,1] in both periods.[26] We first consider implementation under ex ante BB. In the second period distortions in the EBS constraint are 0, and trade is always efficient, generating a maximum possible surplus of $\frac{1}{6}$ irrespective of the history in period 1. Thus, the no trade region in period 1 solves

$$\frac{1}{6}(4M-1)(1-M)^2 + \delta\frac{1}{6} = 0 \tag{10}$$

It is clear that M is a decreasing function of δ. In fact, at $\delta = 0$, $M = 1/4$, replicating the static model, and $M = 0$ for $\delta = 1$, implying the implementability of the efficient allocation.

In the T period version of this problem, trade is always efficient starting period 2. Using a recursive mechanism, Athey and Miller [2007] show that in fact for any distribution, when $T = \infty$, trade in first period will be efficient for any $\delta \geq \frac{1}{2}$.

Next, consider implementation under interim BB. Now, it is easy to show that $EBS^{**}(h)$ will always bind, and the trade in the second period replicates the static model. Thus, $p(v_2, c_2|h) = 1$ if and only if $v_2 > c_2 + \frac{1}{4}$ for all h. In the first period, the no trade region M solves

$$\frac{1}{6}(4M-1)(1-M)^2 + \delta\frac{9}{64} = 0 \tag{11}$$

Again, M is a decreasing function of δ. However, this time even $\delta = 1$ cannot guarantee efficient trade in period 1. Interim budget balance forces the agents to internalize incentives for period 2 while deciding on the optimal mechanism. The contract is no longer efficient in period 2, and hence a smaller *collateral* is available for trade in period 1.

Nevertheless, it is interesting to consider δ as a proxy for the surplus available in the future in a general T period model. Following the said motivation, it is easy to see that trade will be efficient, and the expected budget surplus constraint will not bind in period 1 for $\delta \geq \frac{32}{27}$.

4.3 Perfect Persistence

When types are perfectly persistent, there are no second period expected budget surplus constraints. So, all notions of budget balance are equivalent. The problem reduces to the maximization of gains from trade under ex ante BB and constant types, thus giving us a repetition of the static optimum. In both periods trade happens if and only if $v > c + \frac{1}{4}$.

4.4 Discussion

We know that efficient trade is impossible in the static model. What is the driving force in the dynamic model that presents us with the possibility of efficiency? A casual glance at equations (10) and (11) gives us an indication. Surplus generated in the second period is a function of the levels of asymmetric information through persistence and the limits

[26]Formal proofs are presented in the appendix.

on insurance imposed by varying notions of budget balance. These provide key building blocks for understanding the possibility of achieving efficiency in more general models.

Under iid types and ex ante BB, maximum possible surplus is generated in period 2, which can distributed across types and time, significantly reducing the no-trade region in period 1. However, under a stricter notion of budget balance, there are limits on the depth of the credit line facilitating trade. It reduces the total future surplus, thereby mitigating the advantage dynamics present even for the case with minimal informational constraints.

At the other extreme, when the informational constraints are the most severe, in the form of constant types, it blunts all possible advantages that dynamics present by making all those histories that would relax information constraints to generate future surplus be zero in probability.

The important question we seek to answer in the next section is the following. Under voluntary participation and reasonable limits on budgeting across time; and as a function of information asymmetry and discounting, when can the agents create enough surplus so that an institution guaranteeing efficient trade can be created?

5 Efficiency: The Collateral Dynamic VCG Mechanism

In order to construct possibility or impossibility results in efficiency with incentive compatible and individually rational mechanisms which satisfy budget balance, two distinct routes are pursued in the static literature. The first one was exploited in the two period model studied in section 4. A second best mechanism was formulated using the envelope formula, and the local incentive compatibility and individual rationality constraints were all summarized in the budget surplus conditions. Efficiency can be attained when these complied budget constraints do not bind, or in other words are non-negative at the efficient allocation. We will generalize this approach in the next section.

The other methodology, reminiscent of Williams [1999] and Krishna and Perry [2000], amongst others, is to start with perhaps the most well known efficient mechanism- VCG (or the Pivot mechanism will do too in this case), and build around it, in an attempt to satisfy budget balance, which the original mechanism does not. In this section, we present a series of general results on efficiency in the dynamic bilateral trade setting, taking this latter route.

To this end, we adapt techniques for static models from Krishna and Perry [2000] to the dynamic framework. Intuitively, why should we expect to achieve efficiency in a dynamic model when an impossibility result holds in the similar static setting? What factors are salient in the ability of the dynamic relationship to overcome the static benchmark? The answer, as was alluded to in section 4, lies in the existence of future surplus, and its size. We show that this surplus, if large enough, can be used as *collateral* to guarantee efficiency in dynamic models. Since efficiency is not attainable in the last period of a finite horizon model under interim BB, we shall assume $T = \infty$ in this section.[27]

[27]Efficiency can be sustained, however, under ex ante BB in a finite horizon model. The condition is

Consider a standard VCG mechanism, where

$$p^{vcg}(v_t, c_t|h^{t-1}) = \begin{cases} 1 & \text{if } v_t > c_t \\ 0 & \text{otherwise} \end{cases}$$

$$x_B^{vcg}(v_t, c_t|h^{t-1}) = \begin{cases} c_t & \text{if } v_t > c_t \\ 0 & \text{otherwise} \end{cases}$$

$$x_S^{vcg}(v_t, c_t|h^{t-1}) = \begin{cases} v_t & \text{if } v_t > c_t \\ 0 & \text{otherwise} \end{cases}$$

Define the associated expected utility vectors to be

$$U_B^{vcg}(v_t, c_t|h^{t-1}) = U_S^{vcg}(v_t, c_t|h^{t-1}) = \mathbb{E}\left[\sum_{\tau=t}^{T} \delta^{\tau-t}(v_\tau - c_\tau)^+ \mid h^{t-1}, (v_t, c_t)\right], \qquad (12)$$

where $x^+ = \max\{0, x\}$, and since the mechanism is fixed, the expectation is taken over the conditional distributions $F(.|.)$ and $G(.|.)$.

Clearly the mechanism always runs a deficit for any notion of budget balance. In what follows, we first use payoff equivalence to pin down the mechanism that guarantees the highest possible expected budget surplus for every history while satisfying individual rationality. Next, we show that this is exactly the mechanism that generates the highest collateral at every history from which the possibility (or impossibility) of efficient trade can be built for interim BB. We start with a history dependent version of payoff equivalence.

Proposition 1. *Payoff equivalence holds after every history. That is, if* $\langle \mathbf{p}, \mathbf{U}\rangle$ *and* $\left\langle \mathbf{p}, \tilde{\mathbf{U}}\right\rangle$ *are two ex post incentive compatible mechanisms that generate utility vectors* $\left(U_B(v_t, c_t|h^{t-1}), U_S(v_t, c_t|h^{t-1})\right)$ *and* $\left(\tilde{U}_B(v_t, c_t|h^{t-1}), \tilde{U}_S(v_t, c_t|h^{t-1})\right)$*, respectively, then, there exists a family of constants* $\left(a_B(c_t|h^{t-1}), a_S(v_t|h^{t-1})\right)$ *such that*

$$U_B(v_t, c_t|h^{t-1}) = \tilde{U}_B(v_t, c_t|h^{t-1}) + a_B(c_t|h^{t-1}), \text{ and}$$

$$U_S(v_t, c_t|h^{t-1}) = \tilde{U}_S(v_t, c_t|h^{t-1}) + a_S(v_t|h^{t-1})$$

Conversely, if $\langle \mathbf{p}, \mathbf{U}\rangle$ *is ex post incentive compatible, and* $\mathbf{U}$ *and* $\tilde{\mathbf{U}}$ *satisfy the above two equations for a finite family of constants* $\left(a_B(c_t|h^{t-1}), a_S(v_t|h^{t-1})\right)$*, then* $\left\langle \mathbf{p}, \tilde{\mathbf{U}}\right\rangle$ *is also ex post incentive compatible.*

This extends the celebrated payoff equivalence result from static mechanism design to dynamic environments. Pavan, Segal and Toikka [2013] and Skrzypacz and Toikka [2013] have presented an ex ante version of this result, showing that for two incentive compatible mechanisms implementing the same allocation, the expected payoff of each initial type of the agent(s) in one differs from the other only by some additive constant. Their results put restriction on the ex ante expected transfers, whereas our's shows a

precisely characterized by Corollary 3, which does not depend on the time horizon.

deeper history dependent connection between transfers that support ex post incentive compatible allocations.

The change of variables from $\langle \mathbf{p}, \mathbf{x} \rangle$ to $\langle \mathbf{p}, \mathbf{U} \rangle$ proves key in establishing this result.[28] If we work in an environment with stage transfers it is hard to keep track of the change in incentives caused by a moving transfers around after any given history. But, moving expected utility vectors endogenously keeps the incentives intact. The bijection from $\mathbf{x}$ to $\mathbf{U}$ through $\mathbf{p}$ precisely determines the associated stage transfers.

Also, note that the result is proven for the stronger equilibrium concept, ex post incentive compatibility. The equivalent version for Bayesian incentive compatibility entails taking the requisite expectations, which will result in the constants simply being $a_B(h^{t-1})$ and $a_S(h^{t-1})$, respectively.

Now, we construct the collateral dynamic VCG mechanism.

Constructing the Collateral Dynamic VGC mechanism

Step 1. Start with the VCG mechanism, $\langle \mathbf{p^{vcg}}, \mathbf{U^{vcg}} \rangle$, as defined above. It is ex post incentive compatible and ex post individually rational.

Step 2. Select the mechanism $\langle \mathbf{p}^*, \mathbf{U}^* \rangle$, where $\mathbf{p}^* = \mathbf{p^{vcg}}$, and $\mathbf{U}^*$ is chosen so that $\inf_{v \in \mathcal{V}} U_B^*(v, c_t | h^{t-1}) = 0 = \inf_{c \in \mathcal{C}} U_S^*(v_t, c | h^{t-1})$ for all v_t, c_t and h^{t-1}. Let $EBS^*(h^{t-1})$ represent the expected budget surplus generated by this mechanism after history h^{t-1}.

Step 3. Show that an ex post incentive compatible and individually rational mechanism guaranteeing efficient trade under interim BB can exist if and only if $\langle \mathbf{p}^*, \mathbf{U}^* \rangle$ runs an expected budget surplus, that is, $EBS^*(h^{t-1}) \geq 0 \ \forall h^{t-1}$, $\forall t$.

Step 4. Using $\langle \mathbf{p}^*, \mathbf{U}^* \rangle$ and equation (3), recover the stage transfers $\mathbf{x}^*$. $\square$

The task, thus, boils down to reformulating the VCG mechanism so that the transfers are normalized to guarantee the "lowest" types[29], the minimum possible expected utility every period, which is zero. This extraction of maximum possible transfers, while ensuring individual rationality, from both the agents generates a *collateral*. In a simple implementation, both agents pay the normalization up front each period, and then simply run a VCG mechanism. We know from Myerson and Satherthwaite [1983] and Krishna and Perry [2000] that the collateral is not sufficient to insure all types in the static model, and hence efficiency cannot be attained under ex post budget balance. In the dynamic model, a higher future (constrained) surplus translates one for one into a higher collateral. When can sufficient collateral be generated in the dynamic model to implement the

[28] This approach is popular in (static) contract theory. See, for example Laffont and Martimort [2001]. The key difference is that change with stage transfers in the dynamic environment must be with the expected utility variables, rather than the stage utility ones.

[29] If we imposed first order stochastic dominance, these would be $\underline{v}$ and $\bar{c}$.

efficient allocation for all types after all histories? As step 3 above requests, the following proposition answers the question precisely.

Proposition 2. *There exists an ex post incentive compatible and ex post individually rational mechanism that implements the efficient allocation under interim BB if and only if* $EBS^*(h^{t-1}) \geq 0 \ \forall h^{t-1}, \forall t$.

Proposition 2 is essentially saying that if the Collateral Dynamic VCG mechanism cannot produce an expected budget surplus in any given history, then no other mechanism can. One might think that while the mechanism may run a deficit in some history[30], it might be possible to transfer resources from some other which runs a surplus. By construction, the Collateral Dynamic VCG mechanism has already taken care of that. If it was possible to make such a transfer in an incentive compatible and individually rational manner, it has been internalized by the mechanism.

Further, the notion of interim budget balance and the Collateral Dynamic VCG mechanism are more flexible than it may seem at a first glance, in at least two ways. First, we have normalized the lower bound of expected surplus in any period to be 0. It can be taken to be any finite negative number. The greater its magnitude, which corresponds to the depth of the credit line on offer, the higher are the chances of sustaining efficiency. Second, we force re-calibration of transfers to take place every period. It can be relaxed so that the interim budget balance constraint is imposed every k periods which too creates greater efficiency without relying on unbounded insurance. In both cases, the Collateral Dynamic VCG mechanism can be accordingly modified.

The condition $EBS^* \geq 0$ is exactly the inequality characterizing efficient trade under ex ante budget balance in Skrzypacz and Toikka [2013]. The equivalence is formally established in the appendix. Thus, their result can be obtained as a consequence of our Proposition 2.[31]

Corollary 3. *There exists an ex post incentive compatible and ex post individually rational mechanism that implements the efficient allocation under ex ante BB if and only if* $EBS^* \geq 0$.

It is imperative to note that the proposition above is *stronger* than most of the results in the static (Bayesian) literature in the sense that it establishes the conditions for ex post implementation. Unsurprising, to the extent that we started with a VCG mechanism. But, perhaps surprisingly, it also precisely characterizes the conditions for Bayesian implementation.

Corollary 4. *There exists a perfect Bayesian incentive compatible and individually rational mechanism that implements the efficient allocation under interim BB if and only if* $EBS^*(h^{t-1}) \geq 0 \ \forall h^{t-1}, \forall t$.

[30] What we actually mean is a set of histories with positive measure.

[31] Like Krishna and Perry [2000], they also consider multidimensional initial types which can be accommodated in our set up too.

The intuition for this is quite simple. The expected budget surplus constraint takes expectation over all current and future types. So, from equation (5), it is evident that for interim budget balance the two notions are equivalent.

An important observation is that if the support is constant and $T = \infty$, it is sufficient to look only at EBS^* and $EBS^*(h^1)$ for all $h^1 \in \mathcal{V} \times \mathcal{C}$, for Markov processes ensure that the expected budget surplus constraints after period 2 have been considered in period 2 itself. Moreover, since the utility vector $\mathbf{U}^*$ starting period 2 is stationary, the associated transfers, using equation (3), are also stationary.

What about implementation under ex post budget balance? Miller [2012] has already shown that ex post implementation under ex post budget balance is impossible even with iid shocks. Thus, it is obviously futile to attempt a characterization under persistence. Though, a similar characterization under perfect Bayesian implementation is an open question. It is straightforward to notice that $EBS^*(h^{t-1}) \geq 0$ in a necessary condition for the same.

In the appendix, we establish another intermediate notion of budget balance that seeks to provide positive expected stage surplus, that is, the expected cash flows to the mechanism designer must be non-negative in expectation every period. Lemma 5 in the appendix shows that perfect Bayesian implementation under this new notion is equivalent to implementation under ex post budget balance.[32] Define

$$EBS_t(h^{t-1}) = EBS(h^{t-1}) - \delta \mathbb{E}^m \left[EBS(h^t) | h^{t-1} \right].$$

Thus, $EBS_t(h^{t-1})$ is the current expected budget surplus after history h^{t-1}, and can be used as a proxy for ex post budget balance.

Using this tool, the Collateral Dynamic VCG mechanism also provides a sufficient condition for efficient implementation under ex post BB. Proof is analogous to Proposition 2, and omitted.

Corollary 5. *If there exists a mechanism that implements the efficient allocation under perfect Bayesian incentive compatibility and individual rationality and ex post budget balance, then $EBS^*(h^{t-1}) \geq 0 \ \forall h^{t-1}$, $\forall t$. In addition, if $EBS_t^*(h^{t-1}) \geq 0 \ \forall h^{t-1}$, $\forall t$, then there exists one such mechanism.*

Exploiting constant support, stationarity can be used to state a tighter sufficiency condition than the one stated above. Let

$$\eta(v, c) = EBS_t^*(v, c) \quad \text{and} \quad \eta^0 = EBS_1^*$$

So, η denotes the expected budget surplus only as a function of types in the last period, and η^0 represents the same for the null history. Then, we have

[32] This a generalization of the result from static mechanism design that ex ante and ex post notions of budget balance are equivalent under Bayesian implementation. See for example Cramton, Gibbons and Klemperer [1987], and Mailath and Samuelson [1990] .

Corollary 6. *If $\eta(v,c) + \delta\mathbb{E}[\eta(\tilde{v},\tilde{c})|(v,c)] \geq 0$ for all (v,c), and $\eta^0 + \delta\mathbb{E}[\eta(\tilde{v},\tilde{c})] \geq 0$, then there exists a mechanism that implements the efficient allocation under perfect Bayesian incentive compatibility and individual rationality and ex post budget balance.*

This result is essentially establishing that even if $EBS_t^*(v,c) < 0$ for some histories, as long as we can borrow from the next period for all those histories for which $EBS_{t+1}^*(\tilde{v},\tilde{c}) \geq 0$, and break even, efficient allocation can still be implemented under static and thus ex post budget balance. In the appendix, we build a heuristic mechanism that shows that if this borrowing from future periods can be done in finite fashion, efficiency under ex post budget balance can be achieved.

Krishna and Perry [2000] emphasize the salience of the generalized VCG mechanism they construct in providing a precise condition for efficient trade. The same emphasis is merited for the Collateral Dynamic VCG mechanism. It unifies some of the existing results, establishes new ones towards a precise characterization of efficiency and offers an intuitive method of implementation.

5.1 Example: The IID Model

For an infinite horizon model with IID types (and constant support), all interim budget balance constraints, including the ex ante one, are the same. Thus, to sustain efficiency we only need to check $EBS^* \geq 0$. Suppose $\mathcal{V} = \mathcal{C}$, and Y is the expected first best gains from trade. Adopting Athey and Miller [2007], and Skrzypacz and Toikka [2013] to our mechanism,

$$EBS^* = (2\delta - 1)\,\mathbb{E}[Y] \geq 0 \Leftrightarrow \delta \geq \frac{1}{2}$$

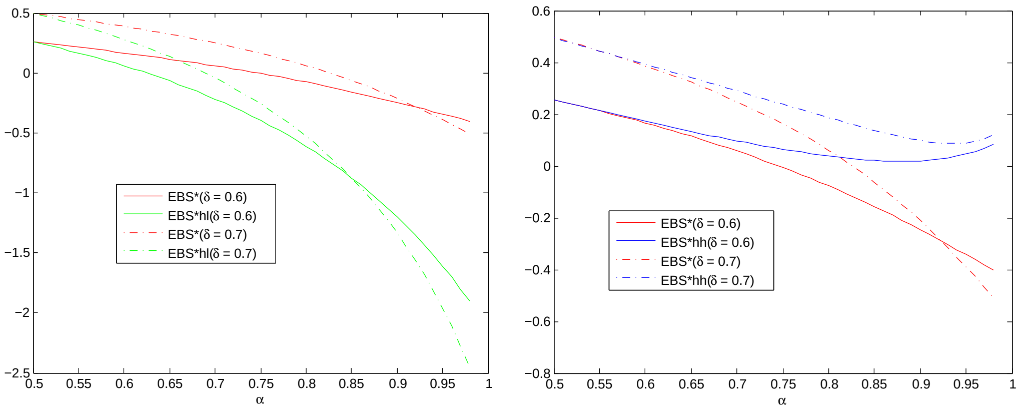
It is a simple and striking result, especially since it is independent of the details of the distribution.[33] But, the unsatisfactory part is the force driving this result: starting at any history, in evaluating the budget constraints, incentive constraints beyond the current ones do not matter.

What about ex post budget balance? If Y is the expected first best gains from trade, then it is easy to show that for any history h^{t-1},

$$EBS_t^*(h^{t-1}) = (2\delta - 1)\mathbb{E}[Y] - \delta(2\delta - 1)\mathbb{E}[Y] = (1-\delta)(2\delta - 1)\mathbb{E}[Y]$$

So, again efficient trade is possible if and only if $\delta \geq \frac{1}{2}$. However, there is a key difference between the two results. While, the one for interim (and ex ante) notions of budget balance is ex post implementable, for ex post budget balance it is only implementable in perfect Bayesian equilibrium. For iid types, an interesting result between the independence of the critical level of discounting from the underlying distributions here, and the impossibility under ex post implementation established by Miller [2012], is attainable if we consider implementation under perfect Bayesian incentive compatibility, but ex post individual rationality. Proposition 6 in the appendix proves such a result.

[33] This of course is not true for a finite model, as we saw in section 4.



(a) EBS^*, $EBS^*(v_H, c_L)$, $\delta = 0.6$ and 0.7 (b) EBS^*, $EBS^*(v_L, c_H)$, $\delta = 0.6$ and 0.7

Figure 2: EBS for Two Types

5.2 Example: Two type model with Persistence

It is a well known fact that even in the static model with discrete types, efficiency is possible.[34] Instead of pursuing a full characterization, we pin down efficiency for a simple two type model that allows us to study the affects of discounting and persistence in perhaps the simplest possible fashion. In doing so, we rely on the two-type envelope formula for an infinite horizon model, established by Battaglini [2005]. Details are in the appendix.

The buyer and seller types are arranged as $v_H > c_H > v_L > c_L$. Any other arrangement will lead to posted prices being efficient statically and thus dynamically. We assume a uniform prior and simple Markov matrices: $f(v_i|v_i) = \alpha = g(c_i|c_i)$ and $f(v_i|v_j) = 1 - \alpha = g(c_i|c_j)$ for $i \neq j$. Note that efficiency demands $p(c_H, v_L|h^{t-1}) = 0$ for all h^{t-1}, and trade with probability 1 for all other realizations.

Figure 2 shows maximum possible expected budget surplus for an incentive compatible and individually rational mechanism. Values of types are chosen so that efficiency is not feasible in a static model. We plot the EBS^* and $EBS^*(v_H, c_L)$ in figure 2a, EBS^* and $EBS^*(v_L, c_H)$ in figure 2b, against persistence for $\delta = 0.6$ (solid) and $\delta = 0.7$ (dotted).[35] Two observations are immediate. First, the interim budget constraints start from the same point at $\alpha = 0.5$ reiterating that for iid types, ex ante and interim are the same. Second, their value in the positive regions is increasing in δ.[36]

Also, while $EBS^*(v_H, c_L)$ is always below the ex ante budget constraint, $EBS^*(v_L, c_H)$ lies above it. Persistence affects both the probability of an event and the size of distortions or information rents. In $EBS^*(v_H, c_L)$, clearly the size of distortions grows faster than the rate at which the probability of histories with distortions goes to zero, hence it becomes negative. On the other hand $EBS^*(v_L, c_H)$ increases at the tail of highest levels of persistence showing the opposite effect.

[34] See Matsuo [1989].
[35] Note these are closed form solutions.
[36] We plot only two values, but numerical results show that these is generally true for all δ.

5.3 Example: Truncated Normal

We consider now a model with a uniform prior on [0,1] for both types and a Markov evolution governed by a truncated normal.

$$F(v'|v) = \frac{\Phi\left(\frac{v'-v}{\sigma}\right) - \Phi\left(\frac{0-v}{\sigma}\right)}{\Phi\left(\frac{1-v}{\sigma}\right) - \Phi\left(\frac{0-v}{\sigma}\right)},$$

where Φ is the standard normal and σ is the variance. So, the type today determines the mean tomorrow. $G(.|.)$ is similarly defined. Information rents for this problem are not stationary and hence hard to evaluate even numerically for an infinite horizon.[37] So, we cannot exploit the techniques that we did in the two type model above. Instead, we calculate EBS^* for a two period model, and then further calculate $EBS^*(v,c)$ for specific realizations of v, c for a three period model, that is, with two periods to go. This allows us a fair comparison between the ex ante and interim notions of budget balance.

It is important to note that higher values of σ imply "lower" persistence- the bell of the normal curve is flatter. In figure 3 we plot the EBS^*, $EBS^*(0.25, 0.75)$ and $EBS^*(0.75, 0.25)$ against σ for three different values of δ. As expected, EBS^* is an increasing function of σ, and hence a decreasing function of persistence for all values of δ.

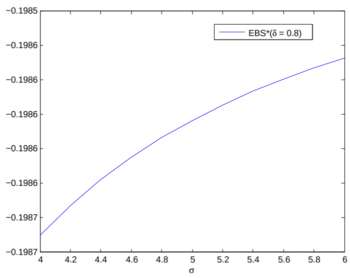
As we saw in the two-type model, interim budget surplus may not always be decreasing in persistence. While $EBS^*(0.25, 0.75)$ is decreasing, $EBS^*(0.75, 0.25)$ (middle graphs) is in fact increasing in persistence, showing that interim budget constraints may bind precisely when the ex ante ones are likely to be satisfied.

Finally, note that the value of each graph is decreasing with δ, which is a bit counterintuitive. It is because, being a two period model, the "surplus" being generated from implementing the efficient allocation in the last period is actually negative, hence an absolute increase in δ decreases the aggregate surplus rather than increasing it. This is however, just a product of the finiteness (two period) of the model.

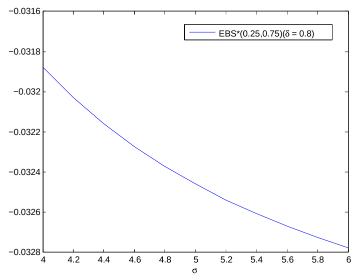
6 Second Best Efficiency

As was expected, though achievable in a large class of environments and institutional structures, efficiency is definitely not the norm. Therefore, an analysis of the second-best is merited. The steps to be followed are pretty much a direct generalization of the two period model studied in section 4. We seek to maximize the expected gains from trade subject to perfect Bayesian incentive compatibility and individual rationality under interim budget balance. First, the dynamic envelope formula gives a characterization of the expected budget surplus sans the transfers.

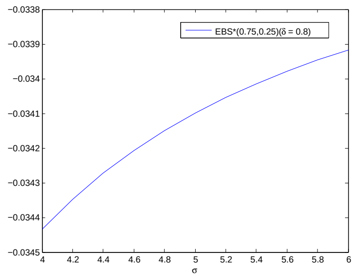
[37] To test the affect of persistence on ex ante budget balance, Skrzypacz and Toikka [2013] use a nice example in the form of a renewal model. Type remains constant with a probability, say p, and is redrawn from the prior with probability $1-p$. It has an easily exploitable stationarity. Unfortunately we cannot use this model, because the history dependent payoff equivalence result does not allow lumpy distributions.



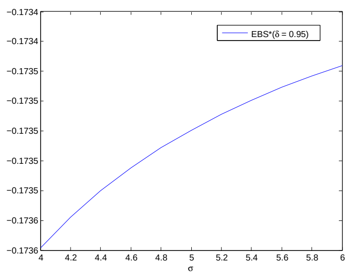
(a) EBS^*, $\delta = 0.8$



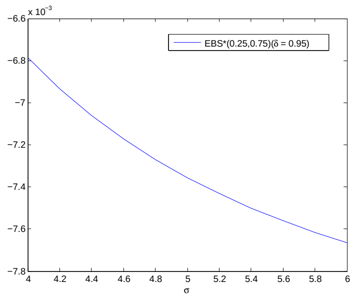
(b) $EBS^*(0.25, 0.75)$, $\delta = 0.8$



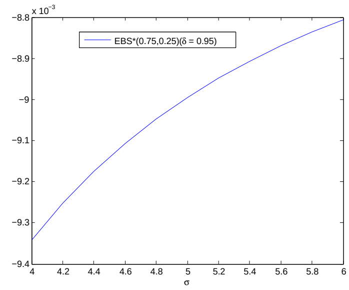
(c) $EBS^*(0.75, 0.25)$, $\delta = 0.8$



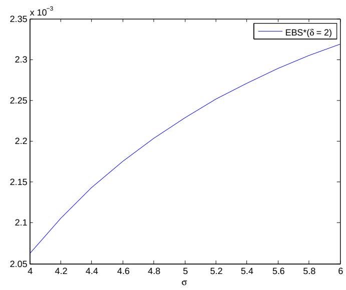
(d) EBS^*, $\delta = 0.95$



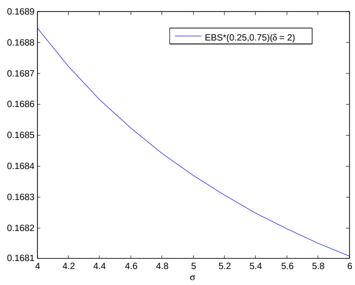
(e) $EBS^*(0.25, 0.75)$, $\delta = 0.95$



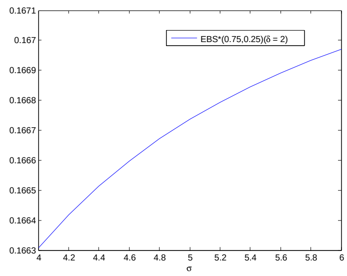
(f) $EBS^*(0.75, 0.25)$, $\delta = 0.95$



(g) EBS^*, $\delta = 2$



(h) $EBS^*(0.25, 0.75)$, $\delta = 2$



(i) $EBS^*(0.75, 0.25)$, $\delta = 2$

Figure 3: EBS for Truncated Normal

Lemma 2. *The expected budget surplus after any history can be expressed only as a function of the allocation, that is, for all h^{t-1} and for all t,*

$$EBS(\mathbf{p}|h^{t-1}) + \inf_{v\in\mathcal{V}} U_B(v|h^{t-1}) + \inf_{c\in\mathcal{C}} U_S(c|h^{t-1}) = \Gamma(\mathbf{p}|h^{t-1}),$$

for some function $\Gamma(.|h^{t-1})$ continuous in the vector of current and future allocations.

Define $EBS^{**}(\mathbf{p}|h^{t-1})$ to be the expected budget surplus that chooses $\inf_{v\in\mathcal{V}} U_B(v, c_t|h^{t-1}) = 0 = \inf_{c\in\mathcal{C}} U_S(v_t, c|h^{t-1})$ above. Then, we have the following characterization of the second best.

Proposition 3. *There exists a mechanism that maximizes the expected gains from trade amongst the class of perfect Bayesian incentive compatible and individually rational mechanisms under interim BB if and only if the allocation solves*

$$\max_{\mathbf{p}} \; GFT$$

subject to

$$EBS^{**}(\mathbf{p}|h^{t-1}) \geq 0 \;\forall h^{t-1},\; \forall t,\; \textit{and}$$

$$\mathbf{p} \textit{ is PBIC}$$

A questions begs to be asked. What if none of the expected budget surplus constraints bind in the aforementioned optimization problem? An immediate implication is in the offing- the efficient allocation then must be implementable. For all h^{t-1}, define $EBS^{**}(h^{t-1})$ to be be $EBS^{**}(\mathbf{p}|h^{t-1})$ evaluated at the efficient allocation.

Corollary 7. *An efficient mechanism is perfect Bayesian (and ex post) implementable under interim BB if and only if*

$$EBS^{**}(h^{t-1}) \geq 0 \;\forall h^{t-1},\; \forall t$$

Proposition 2 and Corollary 7 provide equivalent characterizations of the precise conditions on the fundamentals that guarantee efficient implementation. While the former uses a engineering approach, constructing the Collateral Dynamic VCG mechanism, the latter pursues it through the second best.

In the characterization of the second best above, the dynamic envelope formula allows us to eliminate the transfers, and summarize the IR and local IC constraints through the expected budget surplus conditions. Unlike the static model though, implementability does not necessitate the monotonicity of allocation rule, so global incentive constraints cannot be replaced by simple monotonicity constraints. This makes the problem highly intractable. See, for example, Battaglini and Lamba [2013].

We first describe the second best for a simple AR(1) model and then propose a a class of suboptimal but intuitively appealing incentive compatible contracts.

6.1 Example: AR(1) Model

We consider a two period model. Both agents' type are distributed uniformly on $[0,1]$ in period 1. In period 2, they follow simple AR(1) processes, viz,

$$v_2 = \gamma_B v_1 + (1-\gamma_B)\epsilon_v,$$

$$c_2 = \gamma_S c_1 + (1-\gamma_S)\epsilon_c$$

here $0 \leq \gamma_i \leq 1$, for $i = B, S$ and ϵ_v and ϵ_c are both uniformly distributed on $[0,1]$. Note that types can only increase in value, and values of 0 and 1 for γ_i capture independence and perfect persistence, respectively.

The iid case, $\gamma_B = \gamma_S = 0$, and perfectly persistent case $\gamma_B = \gamma_S = 1$ were completely characterized in section 4. Under interim BB, the $EBS^{**}(h)$ constraints always bind for all h in period 2 in the iid case, whereas for constant types, none do. We show in the appendix that even for the asymmetric extreme cases, $\gamma_B = 0, \gamma_S = 1$, and $\gamma_B = 1, \gamma_S = 0$, $EBS^{**}(h)$ do not bind in the second period, elucidating that perfect persistence of any of agent is enough to transfer the burden of asymmetric information entirely to period 1.

Let $\gamma = (\gamma_B, \gamma_S)$. Under ex ante BB, trade can be characterized as follows.

Lemma 3. *There exists a number $M_{\gamma,\delta}$, $0 \leq M_{\gamma,\delta} < 1$, such that the second best Perfect Bayesian implementable allocation under ex ante BB is given by*

$$p(v_1, c_1) = \begin{cases} 1 & \text{if } v_1 > c_1 + M_{\gamma,\delta} \\ 0 & \text{otherwise} \end{cases}$$

and,

$$p(v_2, c_2 | v_1, c_1) = \begin{cases} 1 & \text{if } v_2 > c_2 + \frac{M_{\gamma,\delta}}{1-M_{\gamma,\delta}}\left[\gamma_B(1-v_1) + \gamma_S c_1\right] \\ 0 & \text{otherwise} \end{cases}$$

where for an interior optimum $M_{\gamma,\delta}$ solves the equation

$$EBS^{**} = 0$$

evaluated at the allocation defined above.

As in standard dynamic screening models, under an AR(1) process, the allocation is monotonic[38] and hence incentive compatible. Note that in the second period, the no trade region is strictly decreasing in v_1 and increasing in c_1. So, a greater chance of trade in period 1 offers an increased probability in period 2.

For implementation under interim BB, we make a further assumption. Let $\gamma_B = \gamma_S = \gamma \leq 1/4$. This is made to ensure that we do not have to consider global incentive compatibility constraints and the problem is solvable in closed form.

[38] In the sense of Definition 11.

Lemma 4. *Suppose* $\gamma_B = \gamma_S = \gamma \leq 1/4$. *There exists a family of numbers numbers* $M_{\gamma,\delta}$ *and* $(M_{\gamma,\delta}(v_1,c_1))_{(v_1,c_1)\in\mathcal{V}\times\mathcal{C}}$ *such that the second best perfect Bayesian implementable allocation under interim BB is given by*

$$p(v_1,c_1) = \begin{cases} 1 & \textit{if } v_1 > c_1 + M_{\gamma,\delta} \\ 0 & \textit{otherwise} \end{cases}$$

and,

$$p(v_2,c_2|v_1,c_1) = \begin{cases} 1 & \textit{if } v_2 > c_2 + M_{\gamma,\delta}(v_1,c_1) \\ 0 & \textit{otherwise} \end{cases}$$

where for an interior optimum $M_{\gamma,\delta}$ *solves the equation*

$$EBS^{**} = 0$$

evaluated at the allocation defined above. Moreover, there exists a number $m_{\gamma,\delta}$, $-1 \leq m_{\gamma,\delta} \leq 1$, *such that*

$$M_{\gamma,\delta}(v_1,c_1) = \begin{cases} \frac{1}{4}\left[\gamma\left(v_1 - c_1\right) + (1-\gamma)\right] & \textit{if } v_1 > c_1 + m_{\gamma,\delta} \\ \\ \frac{M_{\gamma,\delta}}{1-M_{\gamma,\delta}}\left[\gamma - \gamma(v_1 - c_1)\right] & \textit{otherwise} \end{cases}$$

When do the interim budget constraints bind at the optimum and what impact does it have on the no-trade regions? As we saw in the simple examples in section 4 and the examples on efficiency in section 5, interim budget balance constraints are likely to bind when the ex ante constraint is easy to satisfy. The number $m_{\gamma,\delta}$ determines the split in period 2 between the set of histories for which the interim budget constrains bind at the optimum. Unlike, the result with ex ante BB, the no trade regions are no longer an increasing function on the no trade regions in the first period. In fact the allocation does not satisfy monotonicity, but is still incentive compatible.

6.2 A Dynamic Ramsey Pricing Formulation

Maximization of social surplus in the absence of asymmetric information simply entails the implementation of the efficient allocation. The inability of the optimal second best mechanism to be able to do the same in the presence of asymmetric information is due to information rents or binding virtual surplus constraints. The static version of the second best problem can be stated as[39]

$$\max_{\mathbf{p}} \int_V \int_C S(v,c)p(v,c)f(v)g(c)dcdv \tag{13}$$

[39] Under the standard monotone hazard rate conditions, so that the monotonicity constraints can be ignored.

subject to

$$\int_V \int_C VS(v,c)p(v,c)f(v)g(c)dcdv \geq 0, \tag{14}$$

where $S(v,c) = v - c$ denotes the surplus associated with types v and c of the two agents, and $VS(v,c) = \left[v - \frac{1-F(v)}{f(v)}\right] - \left[c + \frac{G(c)}{g(c)}\right]$ the corresponding virtual surplus. The virtual surplus constraint is but of course the summarized expected budget surplus constraint in the mechanism design problem described in proposition 3. In a potent price theoretic interpretation of the problem, Bulow and Roberts [1989] split the virtual surplus into marginal revenue, $MR = \left[v - \frac{1-F(v)}{f(v)}\right]$, and marginal cost, $MC = \left[c + \frac{G(c)}{g(c)}\right]$. Thus, the bargaining problem of Myerson and Satterthwaite [1983] is interpretable as a simple Ramsey pricing problem. To quote Bulow and Roberts [1989], " specify first that trade will occur whenever $MR > MC$. For cases in which $MR < MC$, assign a priority based on the ratio of $(v - c)/(MC - MR)$ ("efficiency" gained to "profits" lost). Allow trade in as many cases as possible using the priority scheme up to the point at which (20) [our equation (14)] equals zero."

A direct dynamic generalization of their insight is in order. Suppose for simplicity first that global incentive constraints can be ignored, for example in the iid model or the models described above in lemmas 3 and 4. We consider the two period problem for an easy exposition. The problem of second best under ex ante BB can be stated as follows.

$$\max_{\mathbf{p}} \int_V \int_C \Big[S(v_1,c_1)p(v_1,c_1) \tag{15}$$

$$+\delta \int_V \int_C [S(v_2,c_2)p(v_2,c_2|v_1,c_1)f(v_2|v_1)g(c_2|c_1)dc_2dv_2\Big] f(v_1)g(c_1)dc_1dv_1$$

subject to

$$\int_V \int_C \Big[VS(v_1,c_1)p(v_1,c_1)$$

$$\delta \int_V \int_C VS(v_2,c_2|v_1,c_1)p(v_2,c_2|v_1,c_1)f(v_2|v_1)g(c_2|c_1)dc_2dv_2\Big] f(v_1)g(c_1)dc_1dv_1 \geq 0, \tag{16}$$

where the important addition to the static world is the virtual valuation term in the second period: $VS(v_2,c_2|v_1,c_1) = \left[v_1 - \frac{1-F(v_1)}{f(v_1)}\left(-\frac{\partial F(v_2|v_1)/\partial v_1}{f(v_2|v_1)}\right)\right] - \left[c_1 + \frac{G(c_1)}{g(c_1)}\left(-\frac{\partial G(c_2|c_1)/\partial c_1}{g(c_2|c_1)}\right)\right]$.

A similar Ramsey strategy can be adopted. All types (in period 1 and 2) for which $MR > MC$, trade will occur. For the rest, a ranking based on the efficiency-profit ratio and the binding virtual surplus constraint determine (im)possibility of trade. Note that ranking is homogenous for types in both periods, that is, for $VS < 0$, trade is allowed in decreasing order of $S/(-VS)$ across periods till (16) binds.

Ramsey pricing under interim BB is more nuanced. For the second period types, two notions of virtual valuations must be internalized, arising from the history dependent expected budget surplus constraints. We have:

$$\max_{\mathbf{p}} \quad \int_V \int_C \Big[S(v_1, c_1)p(v_1, c_1)$$

$$+\delta \int_V \int_C [S(v_2, c_2)p(v_2, c_2|v_1, c_1)f(v_2|v_1)g(c_2|c_1)dc_2dv_2\Big] f(v_1)g(c_1)dc_1dv_1$$

subject to

$$\int_V \int_C \Big[VS^1(v_1, c_1)p(v_1, c_1)$$

$$+\delta \int_V \int_C VS^1(v_2, c_2|v_1, c_1)p(v_2, c_2|v_1, c_1)f(v_2|v_1)g(c_2|c_1)dc_2dv_2\Big] f(v_1)g(c_1)dc_1dv_1 \geq 0, \tag{17}$$

and for all $(v_1, c_1) \in \mathcal{V} \times C$,

$$\int_V \int_C VS^2(v_2, c_2|v_1, c_1)f(v_2|v_1)g(c_2|c_1)p(v_2, c_2|v_1, c_1)dc_2dv_2 \geq 0, \tag{18}$$

where VS^1 is the same as VS in the problem with ex ante BB, and $VS^2(v_2, c_2|v_1, c_1) = \Big[v_2 - \frac{1-F(v_2|v_1)}{f(v_2|v_1)}\Big] - \Big[c_2 + \frac{G(c_2|c_1)}{g(c_2|c_1)}\Big]$.

In the first period, all types for which $VS^1 \geq 0$ trade for sure. In the second period, all types for which $VS^1(.|.) \geq 0$ and $VS^2(.|.) \geq 0$, trade for sure. The *correct* virtual surplus, that is MR-MC, for the Ramsey ranking contest in period 2 depends on whether constraint (18) binds. If after history (v_1, c_1), it binds, sorting is done based on $S(.|v_1, c_1)/\left(-VS^2(.|v_1, c_1)\right)$. Once we have run through all histories for which (18) binds, these allocations are substituted back in to equation (17). Then, a Ramsey ranking is done for the remaining second period types and all first period types as before.

The number $m_{\gamma,\delta}$ in lemma 4 captures the endogenously determined virtual surplus through binding constraints for the AR(1) model. It shows that for histories (v_1, c_1) such that $v_1 > c_1 + m_{\gamma,\delta}$, constraint (18) binds and the no-trade region is thus characterized by VS^2. On the other hand, for $v_1 \leq c_1 + m_{\gamma,\delta}$, the constraint (18) does not bind, and VS^1 is the *correct* virtual surplus determining trade regions through the binding ex ante budget surplus constraint.

As we have shown before, for the iid model, constraint (18) will always bind and VS^2 is pivotal for all histories. And, at the other extreme, under constant types, none of the (18) constraints bind and all inefficiency is characterized through (17).

For a general T period model, for any history h^{t-1}, exactly t virtual surpluses will be need to be internalized. If $EBS^{**}(h^{t-1})$ binds, VS^t will be the pivotal virtual surplus characterizing trade after history h^{t-1}. If not, then check $EBS^{**}(h^{t-2})$, where $\pi_{t-2}(h^{t-1}) = h^{t-2}$ (where π is the projection mapping), and so on. Courtesy the insights of Bulow and Roberts [1989] and machinery we develop in this paper, we have been able to offer a general dynamic Ramsey pricing interpretation of dynamic mechanism design models.

As a final but important thought, note that we focussed on models in which global incentive compatibility constraints do not bind. The Ramsey story is yet not complete. The crucial bang-bang insight of Myerson [1985], emanating from linearity of preferences is again utilized. Since the binding (global) incentive compatibility constraints are all linear in the allocation, they can eventually be summarized in the form of aggregate virtual surplus equations (17) and (18). The added complexity will be in the structure of VS^1 and VS^2, not in the geometry of the virtual surplus constraints themselves.[40] Thus, the basic framework described above still goes through.

6.3 Monotonic Contracts

No discussion of the second best is complete without a description of how to deal with global incentive compatibility constraints. It is an important puzzle in dynamic mechanism design, one for which we offer a suboptimal solution with intuitive properties.

Informally, the following restriction is imposed on trade. If trade happened for a certain buyer type v and seller type c in period 1, then it will always happen in period 2 for any buyer seller types (v', c'), where $v' \geq v$ and $c' \leq c$, and so on. If the mechanism admits trade for a certain pair of types it will always admit trade for any "higher" types in the future.

Formally, define the following partial order on the set of histories.[41]

Definition 10. *For $h^t, \widehat{h}^t \in H^t$, $h^t \succeq \widehat{h}^t$ if $h^t_{v,j} \geq \widehat{h}^t_{v,j}$ and $h^t_{c,j} \leq \widehat{h}^t_{c,j}$ for all $j \leq t$.*

Then, we can define monotonicity.

Definition 11. *An allocation $\mathbf{p}$ is* monotonic *if $h^{t-1} \succeq \widehat{h}^{t-1} \Rightarrow p(v_t|h^{t-1}) \geq p(v'_t|\widehat{h}^{t-1})$ and $p(c_t|h^{t-1}) \leq p(c'_t|\widehat{h}^{t-1})$ for all $v_t \geq v'_t$ and $c_t \geq c'_t$.*

This, what we believe is a fairly plausible restriction on trade, produces a class of contracts that are completely characterized by the local incentive constraints. They are of course incentive compatible and produce monotonic trade regions in the sense we described above. Moreover, the ex ante expected loss in welfare, in comparison to the optimal second best, completely disappears as the stochastic processes either become perfectly persistent or iid.

The idea of imposing monotonicity on the mechanism has been introduced by Battaglini and Lamba [2013] in the context of a dynamic monopoly problem with changing consumer tastes for a discrete type space. We present the results for a model with multiple agents and a continuous type space.

Define $\mathcal{M}$ as the set of *monotonic mechanisms.*[42] The optimal monotonic mechanism then solves,

$$\max_{\mathbf{p} \in \mathcal{M}} GFT$$

[40] Myerson [1984] communicates the basic idea elegantly.

[41] $h^t_{v,j}$ denotes buyer's report in period $j \leq t$.

[42] Clearly, $\mathcal{M}$ is a closed and convex subset of the set of feasible allocations.

subject to

$$EBS^{**}(\mathbf{p}|h^{t-1}) \geq 0 \ \forall h^{t-1}, \ \forall t$$

Let δ_F be the Markov distribution for the buyer that puts all the probability mass on being the constant type and similarly δ_G for the seller. We write $F(.|.) \to \delta_F$ and $F(.|.) \to F(.)$ for convergence in distribution to constant types and iid distribution respectively. Also, $a \vee b$ denotes a or b, and $a \wedge b$ denote a and b. Then, we have the following result.

Proposition 4. *The optimal monotonic mechanism converges in probability to the second best as* $\{[F(.|.) \to \delta_F] \vee [F(.|.) \to F(.)]\} \wedge \{[G(.|.) \to \delta_G] \vee [G(.|.) \to G(.)]\}$.

The proposition simply states that as types either become perfectly persistent or iid, the optimal monotonic contract converges in probability to the second best. A useful corollary is immediate.

Corollary 8. *The ex ante loss in objective between second best and monotonic contracts converges to zero as* $\{[F(.|.) \to \delta_F] \vee [F(.|.) \to F(.)]\} \wedge \{[G(.|.) \to \delta_G] \vee [G(.|.) \to G(.)]\}$.

Global incentive constraints bind in dynamic models because of a failure of monotonicity. What notion of monotonicity fails can be debated, but its failure is hard to ignore. These intuitively appealing contracts provide a useful alternative.[43]

7 Property Rights

An important assumption embedded in most of the analysis so far is the right to own the object- the seller has the right to produce the good every period. This, perhaps innocuous sounding detail, is actually quite significant. Greater efficiency can be achieved with fluid property rights, that is when both agents can jointly own the object. Shares of a company, auctioning spectrum licensing for different regions, local public goods provision, etc may sometimes be viewed through the lens of joint ownership. Of course, to allow for this, the good in question must be divisible.

We consider the following simple modification to the model. There is no mechanism designer or intermediary who can dynamically break the budget for the agents. Instead of having the opportunity to produce the good for the buyer, the seller is now endowed with the good every period which she values at c_t and the buyer values at v_t.[44] The buyer and seller play an infinitely repeated stochastic game. For period t, let $r_{t,B} = 0$ and $r_{t,S} = 1$ denote the initial share of good, as owned by the buyer and seller respectively, and $r^f_{t,B}$ and $r^f_{t,S}$ be the final share. The stage preferences for the buyer and the seller are given by $r^f_{t,B}v_t - x_t$ and $r^f_{t,S}c_t + x_t$, respectively, where x_t represents the transfer made by the buyer to the seller.

[43] Nuemrical results in Battaglni and Lamba [2013] for the monopoly problem show that monotonic contracts do well in terms of the their distance from the second best for intermediate distributions as well.

[44] As pointed out in section 2, this model is equivalent to the one we have so far discussed, where the seller has the option of producing the good for the buyer.

We ask the following question. When can efficiency be sustained with limited commitment in the form of a decentralized stochastic game (as opposed to a dynamic mechanism), under ex post budget balance but with fluid property rights? To this end, we assume that types are iid. The idea is not to produce the most general possible efficiency result for stochastic games, which in turn, has been done by Athey and Segal [2007, 2013][45], but to show how divisibility of the good and property rights can be exploited to sustain efficiency in a model where we get impossibility in a static setting.

To achieve our objective we exploit the efficiency result by Cramton, Gibbons and Klemperer [1987] for static models. They showed that in the neighborhood of equal ownership, efficiency can be sustained through a simple second price auction implementation. We employ the following mechanism, which we term the Fluid Property Rights (FPR) mechanism.[46]

Constructing the FPR mechanism

Step 1. At the start of every period, the seller hands over half the ownership of the good to the buyer for a fixed cost ν. Therefore, the outside option of no trade for the buyer and seller are now $\frac{1}{2}v_t$ and $\frac{1}{2}c_t$, respectively.

Step 2. If the seller refuses to sell the share, trade breaks down forever.

Step 3. A modified second price auction is then run to determine the ownership of the good. Both players announce bids, b_B and b_S, respectively. The highest bidder gets the good and pays the other half of his/her bid, that is, half of the loosing bid. □

The following result formalizes our claim.

Proposition 5. *There exists a cost ν and $1 > \delta^* > 0$ such that efficiency can be sustained as a perfect Bayesian equilibrium of the stochastic game, with ex post individual rationality, through the FPR mechanism, for all $1 > \delta > \delta^*$.*

Mirroring proposition 6 in the appendix, the result is established for the stronger notion of ex post individual rationality. Also, there is nothing salient about the split to half-half at the first stage of the mechanism. One can propose a a split more in the favor of the seller to guarantee efficiency for lower levels of discounting.

8 Avenues for Future Research

The paper seeks to provide a theory of dynamic institutions. Much work still lies ahead. We give a brief overview of what we think are interesting questions to pursue in the

[45] However they assume discrete types and we can establish the result for discrete and continuous types. Moreover our result holds for the stronger notion of EPIR, while they establish it for PBIR.

[46] The idea of this mechanism appears in footnote 21 of Athey and Miller [2007].

literature.

- The relationship of interim budget balance with self enforcing constraints in the relational contracting literature and the debt financing constraint in macro models is worth exploring.
- Using the Collateral Dynamic VCG Mechanism that we construct and the Balance Budget Account Mechanism for iid types from Athey and Miller [2007], mechanisms for approximate efficiency with persistent types seems like a fruitful direction to push in.
- A precise characterization of the perfect Bayesian implementability under ex post budget balance is an open question. We provide sufficient conditions and a heuristic mechanism in this paper.
- We only looked at the expected gains from trade as a second best mechanism. This is just one point on the Pareto possibility frontier. In particular, the neutral bargaining solution of Myerson [1984] is worth pursuing in the dynamic model.
- Providing theoretical bounds for monotonic mechanisms will make the analysis deeper.
- The appendix, section 9.16, has a section on Dominant Strategy Mechanisms. Through a suitably defined notion of dominant strategies, we show that only memoryless posted prices are feasible. Richer questions can be framed for posted price mechanisms. For example, what is the optimum in the class of posted price mechanisms that can depend on the past history of trade?
- Finally, providing a general theory for discrete types in highly useful too.

9 Appendix

Omitted proofs and other details can be found in this section.

9.1 Relationships between different notions of budget balance

It is easy to see from the definitions 5, 6, and 7 that

$$\text{ex post BB} \Rightarrow \text{interim BB} \Rightarrow \text{ex ante BB}$$

Define,

$$EBS_t(h^{t-1}) = EBS(h^{t-1}) - \mathbb{E}^m \left[EBS(h^t)|h^{t-1}\right] = \mathbb{E}^m \left[x_B(v_t, c_t|h^{t-1}) - x_S(v_t, c_t|h^{t-1})|h^{t-1}\right]$$

to be the current expected budget surplus.

Definition 12. *A mechanism $m = \langle \mathbf{p}, \mathbf{x} \rangle$ satisfies static budget balance if*

$$EBS_t(h^{t-1}) \geq 0 \quad \forall h^{t-1} \in H^{t-1}, \ \forall t$$

Then, we have the following tight characterization of implementation under ex post budget balance.

Lemma 5. *A perfect Bayesian incentive compatible and individually rational mechanism, $m = \langle \mathbf{p}, \mathbf{x} \rangle$, is implementable under static BB if and only if it is implementable under ex post BB.*

Proof. Ex post BB implies static BB is obvious. Suppose $\langle \mathbf{p}, \mathbf{x} \rangle$ satisfies static BB. Fix a history h^{t-1}, and let $\Pi = EBS_t(h^{t-1}) \geq 0$. Define

$$\tilde{x}(v_t, c_t|h^{t-1}) = x_B(v_t|h^{t-1}) - \int\limits_{\mathcal{V}} x_B(v_t, c_t) f(v_t|v_{t-1}) dv_t + x_S(c_t|h^{t-1}) + \alpha\Pi,$$

where $\alpha \in [0, 1]$ is a constant. We have

$$\tilde{x}(v_t|h^{t-1}) = x_B(v_t|h^{t-1}) - (1 - \alpha)\Pi, \text{ and}$$

$$\tilde{x}(c_t|h^{t-1}) = x_S(c_t|h^{t-1}) + \alpha\Pi$$

Repeat this for every possible history. Now, consider the mechanism $\langle \mathbf{p}, \tilde{\mathbf{x}} \rangle$. It is ex post BB by construction. Moreover, using a incentive compatible mechanism we are reducing what the buyer has to pay and increasing what the seller gets. So, the new mechanism must also be incentive compatible and individually rational. □

This is a generalization of a standard result in static mechanism design to the dynamic model.[47] Thus, we get that under perfect Bayesian implementation, static and ex post budget balance are equivalent. Note, however that the same may not be true for ex post implementation. So, for ex post budget balance under voluntary participation, the notions of implementation do not agree.

Moreover for a finite model, it is easy to see through backward induction that static BB is actually equivalent to interim BB. Thus, we have:

$$\text{If } T \text{ is finite, then under } PBIC \text{ and } PBIR, \text{ ex post BB } \Leftrightarrow \text{ static BB } \Leftrightarrow \text{ interim BB}$$

9.2 Details of the Examples presented in Section 4

In each of the first three examples, we maximize expected gains from trade under the respective expected budget surplus constraints. In all the cases below, the allocation will be monotonic in the sense of definition 11, so will ignore the implementability constraint.

[47] See, for example, Cramton, Gibbons and Klemperer [1987], and Mailath and Samuelson [1990].

In the static model,

$$\max_{\mathbf{p}} \int_0^1 \int_0^1 (v_1 - c_1)\, p(v_1, c_1) dc_1 dv_1$$

subject to

$$\int_0^1 \int_0^1 (2v_1 - 1 - 2c_1)\, p(v_1, c_1) dc_1 dv_1 \geq 0$$

The Lagrangian can then we written as

$$\int_0^1 \int_0^1 \left[(v_1 - c_1) + \lambda\,(2v_1 - 1 - 2c_1)\right] p(v_1, c_1) dc_1 dv_1$$

$$= (1 + 2\lambda) \int_0^1 \int_0^1 \left(v_1 - c_1 - \frac{\lambda}{1 + 2\lambda}\right) p(v_1, c_1) dc_1 dv_1$$

Since we know that the efficient allocation is not implementable, we must have $\lambda > 0$. Thus, we must have

$$p(v_1, c_1) = \begin{cases} 1 & \text{if } v_1 > c_1 + M \\ 0 & \text{otherwise} \end{cases}$$

where $M = \frac{\lambda}{1+2\lambda}$. Substituting this allocation rule in the binding constraint, gives $M = \frac{1}{4}$. The two period iid problem under ex ante BB can be written as

$$\max_{\mathbf{p}} \int_0^1 \int_0^1 \left[(v_1 - c_1)\, p(v_1, c_1) + \delta \int_0^1 \int_0^1 (v_2 - c_2)\, p(v_2, c_2 | v_1, c_1) dc_2 dv_2 \right] dc_1 dv_1 \qquad (19)$$

subject to

$$\int_0^1 \int_0^1 \left[(2v_1 - 1 - 2c_1)\, p(v_1, c_1) + \delta \int_0^1 \int_0^1 (v_2 - c_2)\, p(v_2, c_2 | v_1, c_1) dc_2 dv_2 \right] dc_1 dv_1 \geq 0 \quad (20)$$

Setting up the Lagrangian, it is easy to see that the optimal mechanism is efficient in period 2 and M in period 1 solves equation (10).

Finally, in the two period iid problem under interim BB we maximize (19) subject to (20) and

$$\int_0^1 \int_0^1 (2v_2 - 1 - 2c_2)\, p(v_2, c_2 | v_1, c_1) dc_2 dv_2 \geq 0,$$

for all v_1, c_1. Setting up the Lagrangian it is easy to see that allocation in period 2 replicates the static model, and period 1 no-trade region solves equation (11).

9.3 Proof of Proposition 1

We prove the result for the buyer. Analogous arguments apply for the seller.

Necessity. Fix $h^{t-1} \in H^{t-1}$. From equation (4), incentive compatibility implies

$$(v_t - v_t')p(v_t, c_t|h^{t-1}) + \delta \int\limits_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v_t, c_t) \cdot \left[dF(v_{t+1}|v_t) - dF(v_{t+1}|v_t')\right]$$

$$\geq U_B(v_t, c_t|h^{t-1}) - U_B(v_t', c_t|h^{t-1}) \geq$$

$$(v_t - v_t')p(v_t', c_t|h^{t-1}) + \delta \int\limits_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v_t', c_t) \cdot \left[dF(v_{t+1}|v_t) - dF(v_{t+1}|v_t')\right]$$

Using integration by parts, this can be written as

$$(v_t - v_t')p(v_t, c_t|h^{t-1}) + \delta \int\limits_{\mathcal{V}} \frac{\partial U_B(v_{t+1}|h^{t-1}, v_t, c_t)}{\partial v_{t+1}} \cdot \left[F(v_{t+1}|v_t) - F(v_{t+1}|v_t')\right] dv_{t+1}$$

$$\geq U_B(v_t, c_t|h^{t-1}) - U_B(v_t', c_t|h^{t-1}) \geq$$

$$(v_t - v_t')p(v_t', c_t|h^{t-1}) + \delta \int\limits_{\mathcal{V}} \frac{\partial U_B(v_{t+1}|h^{t-1}, v_t', c_t)}{\partial v_{t+1}} \cdot \left[F(v_{t+1}|v_t) - F(v_{t+1}|v_t')\right] dv_{t+1}$$

Since the utility functions and stochastic processes satisfy all standard regularity conditions[48], the usual envelope argument gives us

$$\frac{\partial U_B(v_t, c_t|h^{t-1})}{\partial v_t} = p(v_t, c_t|h^{t-1}) + \delta \int\limits_{\mathcal{V}} \frac{\partial U_B(v_{t+1}|h^{t-1}, v_t', c_t)}{\partial v_{t+1}} \cdot \frac{\partial F(v_{t+1}|v_t)}{\partial v_t} dv_{t+1}$$

A slight modification of the Theorem 1 in Pavan, Segal and Toikka [2013] tells us that this can be done recursively. The modification being an extra variable in conditioning on expectations, viz c_t, since we are using ex post incentive compatibility. Thus[49]

$$U_B(v_t', c_t|h^{t-1}) - U_B(v_t'', c_t|h^{t-1})$$

$$\int\limits_{v_t''}^{v_t'} \left[p(v_t, c_t|h^{t-1}) + \delta \int\limits_{\mathcal{V}} \frac{\partial U_B(v_{t+1}|h^{t-1}, v_t', c_t)}{\partial v_{t+1}} \cdot \frac{\partial F(v_{t+1}|v_t)}{\partial v_t} dv_{t+1} \right] dv_t \qquad (21)$$

The result follows.

Sufficiency. Suppose $\langle \mathbf{p}, \mathbf{U} \rangle$ is ex post incentive compatible. Fix c_t and h^{t-1}. Then,

[48] See sections 2.1 and 3.1 of Pavan, Segal and Toikka [2013].

[49] From any standard integrability theorem, see for example Theorem 5.13 in Royden [1968].

$U_B(v_t, c_t|h^{t-1})$ appears in two kinds of incentive compatibility constraints. First,

$$U_B(v_t, c_t|h^{t-1}) \geq U_B(v_t', c_t|h^{t-1}) + (v_t - v_t')p(v_t', c_t|h^{t-1})$$
$$+\delta \int\limits_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v_t', c_t) \left(f(v_{t+1}|v_t) - f(v_{t+1}|v_t')\right) dv_{t+1}$$

and,

$$U_B(v_t', c_t|h^{t-1}) \geq U_B(v_t, c_t|h^{t-1}) + (v_t' - v_t)p(v_t, c_t|h^{t-1})$$
$$+\delta \int\limits_{\mathcal{V}} U_B(v_{t+1}|h^{t-1}, v_t, c_t) \left(f(v_{t+1}|v_t') - f(v_{t+1}|v_t)\right) dv_{t+1}$$

Clearly addition of any constant $a_B(c_t|h^{t-1})$ to $U(v_t, c_t|h^{t-1})$ for all $v_t \in \mathcal{V}$ does not affect any of these constraints.

Next, fix $(v_{t-1}, c_{t-1}) = h_{t-1}^{t-1}$. Second, we need to consider the constraints,

$$U_B(v_{t-1}', c_{t-1}|h^{t-2}) \geq U_B(v_{t-1}, c_{t-1}|h^{t-2}) + (v_{t-1}' - v_{t-1})p(v_{t-1}, c_{t-1}|h^{t-1})$$
$$+\delta \int\limits_{\mathcal{V}} U_B(v_t|h^{t-1}) \left(f(v_t|v_{t-1}') - f(v_t|v_{t-1})\right) dv_t$$

Thus, addition of the constants $a_B(c_t|h^{t-1})$ to $U(v_t, c_t|h^{t-1})$ for all $v_t \in \mathcal{V}$, and $c_t \in \mathcal{C}$, leads to addition of $a_B(h^{t-1}) = \mathbb{E}\left[a_B(c_t|h^{t-1})\right]$ to $U(v_t|h^{t-1})$ for all $v_t \in \mathcal{V}$ which drops out of the constraint.

Therefore, linear additions of constants as defined in the proposition preserves incentives.

9.4 Proof of Proposition 2

Sufficiency is obvious. If the Collateral Dynamic VCG mechanism satisfies all the necessary properties, then it is one such desired mechanism.

Next, we establish necessity. Most of the substance follows from Proposition 1, that is, payoff equivalence. Suppose there exists a set of histories $\mathcal{H}$ of non-zero measure, such that $EBS^*(h) < 0$ for all $h \in \mathcal{H}$.

Consider any other mechanism $\langle \mathbf{p}, \mathbf{U} \rangle$ that is ex post incentive compatible, individually rational and and implements the efficient allocation under interim budget balance. Then, by construction,

$$U_B(v_t, c_t|h^{t-1}) \geq U_B^*(v_t, c_t|h^{t-1}) \quad \text{and} \quad U_S(v_t, c_t|h^{t-1}) \geq U_S^*(v_t, c_t|h^{t-1})$$

Recollect that expected budget surplus can be written as

$$EBS(h^{t-1}) = \mathbb{E}^m \left[\sum_{\tau=t}^{T} \delta^{\tau-1} \left(v_\tau - c_\tau\right) p_\tau - U_B(v_t|h^{t-1}) - U_S(c_t|h^{t-1}) \mid h^{t-1} \right]$$

Thus, if for any history $h^{t-1} \in \mathcal{H}$, $EBS^{*}(h^{t-1}) < 0$, we must have $EBS(h^{t-1}) < 0$ in $\langle \mathbf{p}, \mathbf{U} \rangle$.

9.5 Equivalence of $EBS^{*} \geq 0$ with Skrzypacz and Toikka [2013]'s worst case expectation condition

Define the aggregate first best gains from trade to be

$$Y = \sum_{t=1}^{T} \delta^{t-1} (v_t - c_t)^{+},$$

where $x^{+} = \max\{0, x\}$. Skrzypacz and Toikka [2013] establish that efficient, incentive compatible and individually rational trade is possible with ex ante budget balance if and only if

$$\inf_{v_1 \in \mathcal{V}} \mathbb{E}[Y|v_1] + \inf_{c_1 \in \mathcal{C}} \mathbb{E}[Y|c_1] \geq \mathbb{E}[Y]$$

where the expectations are taken over F, G, $F(.|.)$ and $G(.|.)$.

From equations (5) and (12), and Step 2 of the mechanism, we get

$$EBS^{*} = \inf_{v_1 \in \mathcal{V}} \mathbb{E}[Y|v] + \inf_{c_1 \in \mathcal{C}} \mathbb{E}[Y|c_1] - \mathbb{E}[Y]$$

9.6 Proof of Corollary 6

If η^0 and $\eta(v, c) \geq 0$, we are done. Suppose, $\eta(v, c) < 0$ for all $(v, c) \in \mathcal{H} \subset \mathcal{V} \times \mathcal{C}$, where $\mathcal{H}$ has positive measure. Fix, $(v, c) \in \mathcal{H}$. Pick types $\tilde{v}, \tilde{c}$ following (v, c) that have positive $EBS^{*}_{t+1}(\tilde{v}, c) > 0$ and increase the expected utility vectors for these histories, while ensuring that associated stage expected budget surplus is never less than zero.

Since $\eta(v, c) + \delta\mathbb{E}[\eta(\tilde{v}, \tilde{c})|(v, c)] \geq 0$, this can be done in a way that the new $\eta'(v, c) \geq 0$ for all $(v, c) \in \mathcal{V} \times \mathcal{C}$. This mechanism is implements the efficient allocation under static budget balance.

9.7 A heuristic derivation of efficient implementation under ex post budget balance

Note that

$$EBS_t(h^{t-1}) = EBS(h^{t-1}) - \delta\mathbb{E}\left[EBS(h^{t})|h^{t-1}\right],$$

where

$$EBS(h^{t-1}) = \mathbb{E}^{m}\left[\sum_{\tau=t}^{T} \delta^{\tau-t} (v_\tau - c_\tau) p_\tau - U_B(v_t|h^{t-1}) - U_S(c_t|h^{t-1}) \mid h^{t-1}\right]$$

Start with the Collateral Dynamic VCG mechanism. If $EBS^{*}_1 < 0$, we cannot decrease $U_B(v_1, c_1)$ and $U_S(v_1, c_1)$ for any $(v_1, c_1) \in \mathcal{V} \times \mathcal{C}$ for it will violate individual rationality. But, we can increase increase $U_B(v_2, c_2|v_1, c_1)$ and $U_S(v_2, c_2|v_1, c_1)$. Do it for all the types equally such that the aggregate value equals $\frac{EBS_1^*}{\delta}$. Now, evaluate the new stage expected budget surplus constraints for period 2. For all those histories for which it is negative, increase $U_B(v_3, c_3|h^2)$ and $U_S(v_3, c_3|h^2)$. And, keep doing this exercise. If the mechanism thus constructed is finite, that is, utility vectors required to sustain static budget balance are bounded, then there exists a mechanism that sustains efficiency under static budget balance and hence ex post budget balance. Corollary 6 offers a condition on the primitives when this is indeed possible. Weaker conditions can be similarly stated.

9.8 Equivalence between $EBS^*(h^{t-1})$ and $EBS^{**}(h^{t-1})$

Suppose a mechanism $m = \langle \mathbf{p}, \mathbf{U} \rangle$ implements the efficient allocation. Then, the associated expected budget surplus after any history can be written as

$$EBS(h^{t-1}) = \mathbb{E}^m \left[\sum_{\tau=t}^{T} \delta^{\tau-1} (v_\tau - c_\tau) p_\tau - U_B(v_t|h^{t-1}) - U_S(c_t|h^{t-1}) \mid h^{t-1} \right]$$

Since the Collateral Dynamic VCG mechanism and m both implement the efficient allocation, they differ from each other only through (history dependent) constants in proposition 1. In both the Collateral Dynamic VCG mechanism and the mechanism constructed for the second best optimum the utility of the "lowest type" is normalized to zero. Thus, they must have the same expected utility vectors. Therefore, by the above definition of expected budget surplus, $EBS^*(h^{t-1}) = EBS^{**}(h^{t-1})$.

9.9 Efficiency under ex post IR and perfect Bayesian IC for IID model

Suppose $\mathcal{V} = \mathcal{C}$ and iid types. Then, we can state the following result.

Proposition 6. *There exists a $1 > \delta^* > 0$ such that a perfect Bayesian incentive compatible and ex post individually rational mechanism $m = \langle \mathbf{p}, \mathbf{x} \rangle$ implements the efficient allocation under ex post budget balance for all $\delta > \delta^*$.*

Proof. Fix the efficient allocation. Define

$$\Pi^* = \int_{\mathcal{V}} \int_{\mathcal{C}} (v - c)^+ g(c) f(v) dc dv$$

to be the static first best gains from trade. Next, define transfers to be the following. We don't bother with history, because this is an iid model.

$$x(v, c) = v \int_{\mathcal{C}} \mathbf{1}_{\{v > c'\}} g(c') dc' + c \int_{\mathcal{C}} \mathbf{1}_{\{v' > c\}} f(v') dv' - \int_{\mathcal{V}} \int_{\mathcal{C}} v' \mathbf{1}_{\{v' > c\}} g(c') f(v') dc' dv' + \alpha \Pi^*,$$

for some $\alpha \in (0, 1)$. Then, it is easy to check that the expected stage utilities are given by $u_B = (1 - \alpha)\Pi^*$ and $u_S = \alpha \Pi^*$, respectively. The ex post ones are given by

$$u_B(v, c) = v 1_{\{v > c\}} - x(v, c) \quad \text{and} \quad u_S(v, c) = x(v, c) - c 1_{\{v > c\}}$$

Thus, we need to find a discount factor large enough so that the following two equations are always satisfied for all $(v, c) \in \mathcal{V} \times C$,

$$u_B(v,c) + \frac{\delta}{1+\delta}(1-\alpha)\Pi^* \geq 0 \quad \text{and} \quad u_S(v,c) + \frac{\delta}{1+\delta}\alpha\Pi^* \geq 0$$

It is easy to see for a δ^* large enough this will always be satisfied. In fact, we can choose α, so that $\delta^{**} = \min_\alpha \delta^*(\alpha)$ is the lowest possible. $\square$

Note that we did not need to assume $\mathcal{V} = \mathcal{C}$. As long as there is sufficient overlap to generate gains from trade, we are good. Also, as Miller [2012] shows, this result breaks down if we demand EPIC, so requiring perfect Bayesian incentive compatibility is key to the construction.

9.10 Efficiency in two type model with persistence

We shall exploit second best characterization of efficiency in Corollary 7 and use EBS^{**}. First, let S_{ij} denote the expected surplus after realization of type i for the buyer and type j for the seller in the previous period. Then,

$$S_{HH} = v_H - c_H + \delta\left[\alpha^2 S_{HH} + \alpha(1-\alpha)S_{HL} + \alpha(1-\alpha)S_{LH} + (1-\alpha)^2 S_{LL}\right]$$

$$S_{HL} = v_H - c_L + \delta\left[\alpha(1-\alpha)S_{HH} + \alpha^2 S_{HL} + (1-\alpha)^2 S_{LH} + \alpha(1-\alpha)S_{LL}\right]$$

$$S_{LH} = 0 + \delta\left[\alpha(1-\alpha)S_{HH} + (1-\alpha)^2 S_{HL} + \alpha^2 S_{LH} + \alpha(1-\alpha)S_{LL}\right]$$

$$S_{LL} = v_L - c_L + \delta\left[\alpha^2 S_{HH} + \alpha(1-\alpha)S_{HL} + \alpha(1-\alpha)S_{LH} + (1-\alpha)^2 S_{LL}\right]$$

These can be simultaneously solved in closed form. Further, the ex ante expected surplus is then given by

$$S = \frac{1}{4}\left(S_{HH} + S_{HL} + S_{LH} + S_{LL}\right)$$

Using the envelope characterization in Battaglini [2005], we know that starting at any history, for the buyer distortions last if and only if types are v_L each period and for the seller if types are c_H each period. Let $\Delta v = v_H - v_L$ and $\Delta c = c_H - c_L$. The ex ante distortions (or information rents) are given by

$$R = \frac{1}{2}\sum_{t=1}^{\infty}\delta^{t-1}\alpha^{t-1}\left(\frac{2\alpha-1}{\alpha}\right)^{t-1}[\Delta v + \Delta c] - \frac{1}{4}\sum_{t=1}^{\infty}\delta^{t-1}\alpha^{2(t-1)}\left(\frac{2\alpha-1}{\alpha}\right)^{t-1}[\Delta v + \Delta c]$$

The first term adds the respective distortions and the second one subtracts the events when buyer type is v_L and the seller type is c_H simultaneously because then we know that trade does not happen. R_{HL} and R_{LH} can be similarly calculated. Then, we have

$$EBS^{**} = S - R, EBS^{**}(v_L, c_H) = \frac{S_{LH} - (v_H - v_L)}{\delta} - R_{HL}, EBS^{**}(v_L, c_H) = \frac{S_{LH}}{\delta} - R_{LH}$$

which we plot in Figure 2.

9.11 Proof of Lemma 2

Equation (5), definition of expected budget surplus gives us

$$EBS(h^{t-1}) = \mathbb{E}^m \left[\sum_{\tau=t}^{T} \delta^{\tau-t} (v_\tau - c_\tau) p_\tau - U_B(v_t|h^{t-1}) - U_S(c_t|h^{t-1}) \mid h^{t-1} \right]$$

From equation (21), we get

$$U_B(v_t|h^{t-1}) = \Gamma_B(\mathbf{p}|h^{t-1}) + \inf_{v \in \mathcal{V}} U_B(v|h^{t-1})$$

for some function $\Gamma_B(\mathbf{p}|h^{t-1})$, which from the proof of Proposition 1 is obvious is continuous in $\mathbf{p}$. Similarly,

$$U_S(v_t|h^{t-1}) = \Gamma_S(\mathbf{p}|h^{t-1}) + \inf_{c \in \mathcal{C}} U_S(c|h^{t-1})$$

Substituting these back in $EBS(h^{t-1})$ gives us the result.

9.12 Proof of Proposition 3

Fixing allocation $\mathbf{p}$, from Lemma 2, it is clear that there exists a mechanism that can implement $\mathbf{p}$ under interim BB if and only if $EBS^{**}(\mathbf{p}|h^{t-1}) \geq 0 \ \forall \ h^{t-1}$, $\forall t$. The result thus follows.

9.13 Second Best for the AR(1) Model

For $\gamma_B, \gamma_S < 1$, the objective of the problem is given by

$$\int_0^1 \int_0^1 \Big[(v_1 - c_1) \, p(v_1, c_1) + \delta \frac{1}{1-\gamma_B} \frac{1}{1-\gamma_S}$$

$$\int_{\gamma_B v_1}^{\gamma_B v_1 + (1-\gamma_B)} \int_{\gamma_S c_1}^{\gamma_S c_1 + (1-\gamma_S)} (v_2 - c_2) \, p(v_2, c_2|v_1, c_1) dc_2 dv_2 \Big] dc_1 dv_1 \tag{22}$$

The first period second period expected budget surplus constrain is given by

$$\int_0^1 \int_0^1 \Big[(2v_1 - 1 - 2c_1) \, p(v_1, c_1) + \delta \frac{1}{1-\gamma_B} \frac{1}{1-\gamma_S}$$

$$\int_{\gamma_B v_1}^{\gamma_B v_1 + (1-\gamma_B)} \int_{\gamma_S c_1}^{\gamma_S c_1 + (1-\gamma_S)} (v_2 - \gamma_B(1 - v_1) - c_2 - \gamma_S c_1) \, p(v_2, c_2|v_1, c_1) dc_2 dv_2 \Big] dc_1 dv_1 \geq 0 \tag{23}$$

and, the second period one after history (v_1, c_1) is given by

$$\int\limits_{\gamma_B v_1}^{\gamma_B v_1+(1-\gamma_B)} \int\limits_{\gamma_S c_1}^{\gamma_S c_1+(1-\gamma_S)} \left[(2v_2 - (\gamma_B v_1 + (1-\gamma_B)) - (2c_2 - \gamma_S c_1)\right] dc_2 dv_2 \geq 0 \tag{24}$$

It is easy to see that if either type is constant, the second period constraint is degenerate in the sense that it zero only at corner points of the support of the other type. So, none of these constraints will bind.

Proof of Lemma 3. Here we consider the maximization of the objective, (22), only under (23). In a slight abuse of notation, the Lagrangian is given by

$$(22) + \lambda \cdot (23)$$

Separating coefficient, and denoting

$$M_{\gamma,\delta} = \frac{\lambda}{1+2\lambda}$$

gives us the characterization. □

Proof of Lemma 4. We have $\gamma_B = \gamma_S = \gamma \leq \frac{1}{4}$. The Lagrangian to the problem can be written as

$$(22) + \lambda \cdot (23) + \delta \frac{1}{1-\gamma_B} \frac{1}{1-\gamma_S} \int\limits_0^1 \int\limits_0^1 \lambda(v_1, c_1)(24) dc_1 dv_1$$

Separating coefficients and denoting,

$$M_{\gamma,\delta} = \frac{\lambda}{1+2\lambda}$$

and,

$$M_{\gamma,\delta}(v_1, c_1) = \frac{\lambda}{1+\lambda+2\lambda(v_1,c_1)} \left[\gamma - \gamma(v_1 - c_1)\right] + \frac{\lambda}{1+\lambda+2\lambda(v_1,c_1)} \left[\gamma(v_1 - c_1) + (1-\gamma)\right]$$

Now, fix (v_1, c_1), and let $M = M_{\gamma,\delta}(v_1, c_1)$. Then, a binding (24) is given by

$$\int\limits_{\max\{\gamma c_1 + M, \gamma v_1\}}^{\gamma v_1 + (1-\gamma)} \int\limits_{\gamma c_1}^{\min\{v_2 - M, \gamma c_1 + (1-\gamma)\}} \left[(2v_2 - (\gamma v_1 + (1-\gamma)) - (2c_2 - \gamma c_1)\right] dc_2 dv_2 = 0$$

Both the limits in the brackets fall on the side of the term with M, if $M \geq (v_1 - c_1)$. Suppose it is true. Then, solving it gives a cubic whose non-trivial root is given by

$$M = \frac{1}{4}\left[\gamma(v_1 - c_1) + (1-\gamma)\right]$$

which is greater than or equal to $(v_1 - c_1)$ if and only if $\gamma \leq \frac{1}{4}$.

So, it clear that when (24) binds, M is given by the above value, and when it does not, it is given by the same value as in the ex ante problem. Let $k = v_1 - c_1$ and $z = \gamma k + (1-\gamma)$. Then, the LHS of (24) is given by

$$\frac{1}{6}\left[z - M\right]^2\left[4M - z\right]$$

If no second period constraints bind, define $m_{\gamma,\delta} = 1$, if all second period constraints bind, define $m_{\gamma,\delta} = -1$. For intermediate cases, since the problem is continuous there must exist a z^* such that

$$\frac{M_{\gamma,\delta}}{1 - M_{\gamma,\delta}}\left[1 - z^*\right] = \frac{1}{4}z^*$$

It is clear the $z \leq z^* \Leftrightarrow \frac{M_{\gamma,\delta}}{1-M_{\gamma,\delta}}\left[1 - z\right] \geq \frac{1}{4}z$. Thus, we have that (24) must bind for all $z > z^*$ and not for $z \leq z^*$. Finally,

$$z > z^* \Leftrightarrow v_1 > c_1 + z^*$$

Define $m_{\gamma,\delta} = z^*$.

Finally, it is straightforward to show that the allocations satisfy integral monotonicity as defined in Pavan, Segal and Toikka [2013] and hence is incentive compatible.

□

9.14 Proof of Proposition 4

First, note that monotonicity as defined and local incentive constraints imply that global incentives are satisfied. See Battaglini and Lamba [2013] and Pavan, Segal and Toikka [2013].

We prove that for any given T, the optimal monotonic mechanism converges in probability to the second best. The proof follows the proof of Proposition 8 in Battaglini and Lamba [2013] closely. We show the result for $F(.|.) \to \delta_F$ and $G(.|.) \to \delta_G$. The rest of the cases follow similarly.

Let $\Pi^s(F,G)$, $\Pi^m(F,G)$ and $\Pi^{**}(F,G)$ be the value of the objective from, respectively, the repetition of the optimal static second best, the optimal monotonic mechanism and the optimal second best, when the Markov process are given by F and G respectively. Because the repetition of the optimal static second best is a monotonic mechanism, we must have $\Pi^m(F,G) \in [\Pi^s(F,G), \Pi^{**}(F,G)]$. Moreover, when types are constant, that is, $F(.|.) = \delta_F$, repetition of the static optimum is the optimal second best.

Since the distributions are continuous and $\Pi^m(F,G), \Pi^s(F,G)$ and $\Pi^{**}(F,G)$ are continuous in F and G, by the theorem of the maximum, we must have that for any sequence $(F_n, G_n) \to (\delta_F, \delta_G)$ and $\varepsilon > 0$, there must be an n' such that for $n > n'$ we have $|\Pi^m(F_n,G_n) - \Pi^{**}(F_n,G_n)| \leq |\Pi^s(F_n,G_n) - \Pi^{**}(F_n,G_n)| < \varepsilon$. It is immediate to see that the fact that $\Pi^m(F_n,G_n)$ converges to $\Pi^{**}(F_n,G_n)$ implies that the optimal monotonic mechanism must converge to the mechanism that maximizes second best in probability.