R&D Networks and Uncertainty in Oligopoly Models

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Abstract

We consider an oligopoly setting in which firms form pair-wise collaborative links in R&D with other firms, and then compete in an oligopoly. Each collaborative link allows firms involved in to obtain process innovation with some idiosyncratic probability. First, we assume that the process innovation impacts identically the two firms involved in a collaborative link. We provide a condition satisfied by any equilibrium network. Second, we deal with situations where the process innovation associated with a collaborative link between two firms affects these firms in a different way. We highlight two results. The first is called *the tyranny of the weakest*: firms which are the most able to use the process innovation cannot form links while firms which are the least able to use the process innovation are linked together. The second is called *positive assortative matching* with regard to the ability to take advantage of a process innovation.

JEL classification: C70, L13, L20.

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1 Introduction

Research shows that collaborations among firms in innovative activities has become widespread, especially in industries characterized by rapid technological change like the pharmaceutical, chemical and computer industries (see Hagedoorn, 2002; Powell et al., 2005). R&D collaborations improve the ability of firms to innovate. In addition, they provide access to indirect spillovers since they allow for the diffusion of information across firms (see Ahuja, 2000; Powell et al., 2005). The increasing importance of this phenomenon has spurred economic research on the structural features of the network of R&D collaboration, and on their impact on industry performance. Empirical studies have shown that such real-world networks typically have asymmetric architectures. In fact, it is not uncommon to find that there simultaneously exist firms having many collaborations with others having only a few collaborations (see for example Powell et al., 2005).

In this paper, we study the incentives for R&D collaboration between horizontally related firms. We consider a two stage oligopoly setting in which firms first form pair-wise collaborative R&D links with other firms, and then compete in an oligopoly. The collection of pair-wise links defines a collaboration network and induces a distribution of (expected) costs across the firms in the industry. Given these costs, firms compete in the market. By its very nature, the outcome of R&D collaboration is uncertain and depends on the characteristics of the firms (Gomes-Casseres, Hagedoorn and Jaffe, 2006). Our model allows us to incorporate several realistic aspects of R&D collaboration. To the best of our knowledge this is the first instance when these features have been studied in network models of collaborative oligopolies. First, we model the fact that collaborative links do not always lead to a process innovation. In other words, we introduce uncertainty regarding the outcome of collaborative links. Second, we capture the idea that the success probability of a pair-wise R&D collaboration depends on the identity of the firms engaged in. Finally, unlike the earlier literature we account for the fact that a process innovation may not affect the firms involved in a collaborative link in an identical manner.

In our paper, the pair-wise collaborative links involve a commitment of resources on the part of the collaborating firms. One can imagine that these resources are utilized in R&D,

thereby increasing the probability that the firms will hit upon a desirable process innovation. If a process innovation occurs, it results in lower costs of production for the firms involved in the pair-wise collaborative link. Note that each pair-wise collaborative link is associated with a specific process innovation. As already noted an important feature of our model is that a pair-wise collaborative link does not always lead to a successful process innovation. We assume that the probability a pair-wise collaborative link leads to a process innovation depends on the characteristics of the two firms involved in the link. We assume that firms cannot modify these characteristics in the short run. For instance, the success probability may depend on the geographic proximity of firms and it is reasonable to assume that locations are fixed in the short run. We then consider two possible ways in which the outcome of a successful collaboration impacts the two firms. In the first scenario, a successful pair-wise collaborative link allows the two firms involved to improve their production process in the same way. Next, we consider situations where the two firms involved in the link do not benefit in an identical manner from the process innovation.
We use the notion of pair-wise equilibrium network borrowed from Goyal and Joshi (2006) to model stable collaborations. A pair-wise equilibrium network is a network where firms have no unilateral incentive to remove some of their links and where there does not exist a pair of unlinked firms which have an incentive to form a link. Our objective is to examine the set of pair-wise equilibrium networks.

We first analyze the case where the process innovation obtained by two collaborating firms induces the same reduction in the expected marginal cost for these firms. The classic example of a market with homogeneous products under quantity competition forms our starting point. We first show that under zero costs of link formation we obtain the same results as GJ (2003). Moreover, GJ's results do not change when we introduce uncertainty in the process innovation as long the innovation probabilities are not very different. Second, we identify a property that is satisfied by all pair-wise equilibrium networks, providing an easy and quick way to look for these networks. Then, we show that there exist situations where pair-wise equilibrium networks are clearly asymmetric: firms that have formed links are in asymmetric positions in the network. Note that this result does not arise in models where pair-wise collaborative links yield a process innovation with certainty. Next, we show that the innovation success probabilities satisfy the following non-monotonicity property. Consider an equilibrium network. Suppose the innovation success probability between the linked firms goes up (while that between unlinked firms remain unchanged), then there exist situations where linked firms have an incentive to delete some of their existing links. This result puts into perspective the positive impact of public policy aimed at promoting R&D collaboration. We then investigate what happens when the innovation success probability depends on firm characteristics by focussing specifically on one characteristic: firms location. Hence we use geographical proximity as a measure of similarity between firms, and firms that are closer have higher innovation success probabilities. Interestingly, we find that in a pair-wise equilibrium network sometimes two firms that are close may not collaborate, while two firms that are far may in fact collaborate. In the next section of the paper, we go beyond the linear oligopoly and demonstrate how the results described above can be generalized further. Here we provide results about cost-reducing collaboration for both Cournot and Bertrand differentiated oligopolies.

The last part of the paper studies a question that can only be examined in the presence of heterogeneity. We analyze situations where the process innovation associated with a collaborative link between firms i and j affects the production process of i and j differently[1]. Here, we identify two interesting situations that can occur in a pair-wise equilibrium network. In some pair-wise equilibrium networks, firms that are able to use the process innovation the most efficiently may have formed no links while the firms that are the least efficient in their ability to use the process innovation may have formed links. We call this result the *tyranny of the weakest*. Moreover, in some situations, pair-wise equilibrium networks can be partitioned into two groups: one consisting of the most efficient firms with respect to process innovation and another consisting of the least efficient firms with respect to process innovation. Firms form links with all the other firms of their group while having no links with firms that belong to the other group. We call this result *positive assortative matching* in collaborative networks. Not surprisingly we are also able to show that the complete network is not always a pair-wise equilibrium network when the costs of forming link are zero. It follows that the result found by Goyal and Joshi (2003) under small cost is not robust to the introduction of a heterogene-

[1]Note that Cohen and Levinthal (1989, p. 149) highlight the fact that firms can differ in the absorptive capacity that will permit effective exploitation of the venture's knowledge output.

ity assumption on values of the process innovations associated with a pair-wise collaboration. Finally, we provide a condition which allows us to characterize the set of pair-wise equilibrium networks.

Our paper is a contribution to the study of group formation and cooperation in oligopolies. The model of collaborative networks we present is inspired by recent research on R&D networks. Goyal and Moraga-Gonzales (2001), Goyal, Moraga-Gonzales, and Konovalov (2008) analyze the interaction between the effort of firms on collaborative links and the effort of firms on other projects in R&D. Some other papers focus on the spillovers of process innovation instead of the effort which yields spillover. For instance, König et al. (2012), Goyal and Joshi (2003) assume that the effort in collaboration is exogenous. König et al. examine stability and efficiency of R&D networks in a model with network dependent spillovers. Our paper is more directly related to Goyal and Joshi (GJ, 2003), where firms always obtain a process innovation due to their collaborative R&D links. Moreover, firms are homogeneous and a link between firms i and j impacts the cost of these two firms in an identical manner.
Our paper differs from GJ (2003) since we take into account the fact that the success of collaborative R&D links is uncertain and depends on the identity of the firms engaged in it. We also examine what happens when process innovation does not affect the firms involved in a R&D link in the same way. Formally, Goyal and Joshi use three properties to obtain their results for the Cournot oligopoly case: (1) all links lead the same reduction in marginal costs (2) the profit function is convex, that is the incentive of firm i to form an additional link is increasing with the number of links it has already formed, (3) the profit function is sub-modular, that is the incentive of firm i to form an additional link is decreasing in the number of links formed by the other firms. Note that the first property is crucial for using the two other properties. The introduction of uncertainty not only leads to a more realistic model, but also allows us to introduce heterogeneity in the framework through probability of success of collaborative R&D links. This allows us to relax property (1) and therefore alter the formal analysis of the GJ (2003)'s model. Roughly speaking heterogeneity can be seen as a *third force* along with the two forces (convexity and sub modularity) that are present in the GJ (2003) model. Thus the GJ (2003)'s framework is a special case of our framework, where the probability of success of an innovation associated with a collaborative link is 1.

Two additional things are worth keeping in mind about the way we introduce heterogeneity in the paper. First we identify in a precise manner the magnitude of uncertainty that is needed in the model before it can affect results. Second, we do not introduce heterogeneity in the way in which it is most frequently done in the networks literature − by making the cost of link formation heterogeneous. This way, as we explain later, will not affect network formation in a strategic manner since the firms treat link formation costs like a fixed cost. Finally, by introducing heterogeneity we are able to answer a question that cannot be addressed in the original framework − what happens to collaborations when firms have different abilities to handle or benefit from innovative activities.

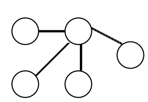
The rest of the paper is organized as follows. In section 2, we present the model setup. In section 3, we provide the results in the textbook example of a market with homogeneous products under quantity competition. In section 4, we propose a generalized framework which allows us to deal with differentiated oligopoly. In section 5, we develop a model where pair-wise collaborative links have some positive probability to create a process innovation, and where the process innovation affects in a different way the efficiency of the production process of the firms involved in the same collaborative link. In section 6, we discuss the differences between our framework and a framework where the costs of forming links are heterogeneous.

2 Model setup

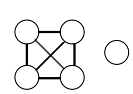
Networks. Consider an industry where $N = \{1, \ldots, i, j, \ldots, n\}$, with $n \geq 3$, denotes the finite set of firms. In the following for each $i \in N$, $N_{-i} = N \setminus \{i\}$. In the game we model, every firm first announces its intended R&D collaborative links: $s_{i,j} = 1$ means that firm i intends to form a collaborative link with firm j, while $s_{i,j} = 0$ means that firm i does not intend to form such a link. Firms only play pure strategies. Thus a strategy of firm i is given by $s_i = \{\{s_{i,j}\}_{j \in N_{-i}}\}$. Let S_i denote the strategy set of firm i. The set $S = \times_{j \in N} S_j$ is the set of strategy profiles of firms and $S_{-i} = \times_{j \in N_{-i}} S_j$ is the joint strategy set of all firms except i; s_{-i} is a typical member of S_{-i}. A link between two firms i and j is formed *if and only if* $s_{i,j} = s_{j,i} = 1$. A strategy profile $s = \{s_1, s_2, ..., s_n\}$ therefore induces a network $g^{[s]}$. For expositional simplicity we shall often omit the dependence of the network on the underlying strategy profile. Thus, the (undirected) *network* or *graph* g is the pair (N, L) consisting of the set of nodes N representing the population of firms, and the set of links $L(g)$ representing R&D collaborations among the firms. The set $L_i(g) \subset L(g)$ is the set of links in which i is involved in g.

We denote by $\mathcal{G}$ the set of *simple networks*[2] whose set of nodes/firms is N. A link $ij \in L(g)$ represents the existence of a R&D collaboration between firms i and j in g. The neighborhood of firm i is the set $g(i) = \{j \in N_{-i} : ij \in L_i(g)\}$. An *isolated firm* i in g is such that $g(i) = \emptyset$. A *walk* $W(i_1, i_k)$ connecting firm i_1 and i_k in g is a sequence of firms $(i_1, i_2, \ldots, i_k)$ such that $i_1 i_2, i_2 i_3, \ldots, i_{k-1} i_k \in L(g)$. A *component* in g is a maximal set of firms such that there exists a walk between any two of them. A *complete component* in g is a maximal set of firms such that there exists a link between any two of them. A *connected network* is a network consisting of only one component without any isolated firms. The *complete network* is a network consisting of only one complete component without any isolated firms. The *empty network* is the network g such that $L(g) = \emptyset$. A network has the $k-$*dominant group architecture* if k firms belong to a complete component and the other $n-k$ firms are isolated. A *star* is a network where there exists a firm, say i, such that $g(i) = N_{-i}$ and for firms $j \in N_{-i}$, we have $g(j) = \{i\}$; i is called the centre of the star.

We define network g_{-i} as the network similar to g except that firm i and all its links are



A star network



A network which has the 4-dominant group architecture

Figure 1: Networks architectures

removed from g. We assume that if $ij \notin L(g)$, then network $g + ij$ is the network obtained

[2]Simple networks are networks without loops (a firm i cannot form a link with itself) or multiple links (firms i and j can establish at most one link between them).

when the link ij is added to g. We assume that if $ij \in L(g)$, then network $g - ij$ is the network obtained when the link ij is removed from g. Finally, we denote by $\bar{g}$ the complement network of g. We have: $ij \in L(\bar{g})$ *if and only if* $ij \notin L(g)$.

Flows/Probabilities of innovation. We assume that a strictly positive value also called *flow* is assigned to each link of the complete network. In our context, flow describes the probability that a R&D collaboration between firms i and j yields a process innovation, and is denoted by $\rho_{i,j} \in (0, 1]$. Although not explicitly modeled, one can imagine that these probabilities $\rho_{i,j}$ depends on the identity and the characteristics of firms i and j. Initially, we will assume that link ij has the same impact on firms i and j, that is $\rho_{i,j} = \rho_{j,i}$. This allows us to define the sequence of all possible probabilities. Note that this sequence is independent of the network g. Let $B = (\rho_{i,j})_{i \in N, j \in N_{-i}}$. When it is useful we note $\rho_{i,j}(B)$. Let $\rho^M(B) = \max\{x : x \in B\}$ and $\rho^m(B) = \min\{x : x \in B\}$. For simplicity we may just write ρ^M and ρ^m when the sequence of probabilities B does not play a role in the analysis.
We associate with each firm $i \in N$ a number $\mathcal{U}_i(g) = \sum_{j \in g(i)} \rho_{i,j}$ representing its "*flow degree*". We provide an interpretation of this flow degree when we present the impact of collaborative links on the marginal cost of firms. For each $(i, j) \in N \times N_{-i}$, we define $\mathcal{U}^m_{i,j}(g) = \min\{\mathcal{U}_i(g), \mathcal{U}_j(g)\}$, and $\mathcal{U}(g) = \sum_{ij \in L(g)} \rho_{i,j}$, and so $\mathcal{U}(g_{-i}) = \mathcal{U}(g) - \mathcal{U}_i(g)$. In the following, we call $\mathcal{U}(g_{-i})$ the total flow degree of firms $j \in N \setminus \{i\}$ in g_{-i}.

Structure of the game. The game played by the firms consists of two stages.

1. In the first stage, firms simultaneously choose the collaborative links they intend to form. These choices induce a network g. Collaborative links allow firms to decrease their expected marginal cost.
2. In the second stage, firms play a simultaneous oligopoly game.

Note that at the end of the first stage, firms are aware of the network g in which they are involved; they play the Oligopoly game given the network g formed in the first stage.

Pair-wise equilibrium network. We assume that in the second stage of the game firms play an oligopoly equilibrium given the network formed in the first stage. For the equilibrium networks, we use the notion of pair-wise equilibrium networks defined by Goyal and Joshi

(2006).

First, we define a Nash equilibrium. Let $\Pi_i^\star(g^{[s]})$ be the oligopoly equilibrium profit of firm i in the second stage given the strategy profile s played by the firms in the first stage. The strategy $s_i \in S_i$ is said to be a best response of firm i to $s_{-i} \in S_{-i}$ if $\Pi_i^\star(g^{[s_i,s_{-i}]}) \geq \Pi_i^\star(g^{[s_i',s_{-i}]})$, for all $s_i' \in S_i$. The set of firm $i's$ best responses to s_{-i} is denoted by $\mathrm{BR}_i(s_{-i})$. A strategy profile $s \in S$ is said to be a Nash equilibrium if $s_i \in \mathrm{BR}_i(s_{-i})$, for all $i \in N$. In the following, to simplify notation we replace $\Pi_i^\star(g^{[s]})$ by $\Pi_i^\star(g)$.

Definition 1 *(Goyal and Joshi, pg. 324, 2006) A network g is a pair-wise equilibrium network if the following conditions hold:*

1. *There is a Nash equilibrium strategy profile which supports g.*
2. *For $g_{i,j} = 0, \Pi_i^\star(g+ij) - \Pi_i^\star(g) > 0 \Rightarrow \Pi_j^\star(g+ij) - \Pi_j^\star(g) < 0$.*

A pair-wise equilibrium network is a refinement of Nash equilibrium: it is a Nash equilibrium where there do not exist two firms with an incentive to form a collaborative link.

3 Cost-reducing collaboration in linear oligopoly

In this section, we consider the textbook linear oligopoly model, and in the following section we generalize our results.

Collaboration links and cost reduction. We assume that a collaborative link requires a fixed investment by each firm, given by $f \geq 0$, and allows to reduce costs of production through process innovations.

We assume that the probability of success of a process innovation depends on the characteristics of firms: all links do not have the same probability of leading to a process innovation. A link between firms i and j leads to a process innovation with probability $\rho_{i,j} \in (0,1]$, and it is independent of the probability of success of an innovation process associated with other links. Moreover, the process innovation targeted by a link is different from the process innovation targeted by another one. To simplify the analysis, we assume that the expected marginal cost function of a firm is affine, that is:

$$c_i(g) = \gamma_0 - \gamma \sum_{j \in g(i)} \rho_{i,j} = \gamma_0 - \gamma \mathcal{U}_i(g), \tag{1}$$

where $\gamma > 0$,[3] $\gamma_0 > \gamma(n-1)$.

When $\gamma = 1$ the flow degree, $\mathcal{U}_i(g)$, can be interpreted as the expected impact of the process innovations from the collaborative links in lowering the marginal costs. Note that we make the following implicit assumption: each successful process innovation in which firm i is involved decreases i's marginal cost by the same amount γ.

A network g induces an expected marginal cost vector for the firms which is given by $c(g) = (c_1(g), c_2(g), \ldots, c_n(g))$.

Demand and expected profit function. To start with, we consider the simplest text-book oligopoly model: the linear Cournot Oligopoly with a homogeneous good. We assume the following linear inverse demand function:

$$p = \alpha - \sum_{i \in N} q_i, \alpha \geq 0,$$

where p is the market price of the good and q_i is the quantity sold by firm i.
Given any network g, the Cournot equilibrium output is:

$$q_i^\star(g) = \frac{\alpha - \gamma_0 + \gamma[(n-1)\mathcal{U}_i(g) - 2\mathcal{U}(g_{-i})]}{n+1}.$$

To ensure that each firm produces a strictly positive quantity, we assume the following condition

$$(C1): \alpha - \gamma_0 > \gamma(n-1)(n-2)\rho^M.$$

The second stage Cournot gross expected profit of firm $i \in N$ is given by:

$$\pi_i^\star(g) = \varphi(\mathcal{U}_i(g), \mathcal{U}(g_{-i})) = \left(\frac{\alpha - \gamma_0 + \gamma[(n-1)\mathcal{U}_i(g) - 2\mathcal{U}(g_{-i})]}{n+1}\right)^2, \tag{2}$$

and the second stage Cournot expected profit of firm $i \in N$ is given by $\Pi_i^\star(g) = \pi_i^\star(g) - |L_i(g)|f$.

[3]Formally, collaborative links reduce unit costs. It follows that any mechanism that can reduce unit costs will lead to the same qualitative results as in our paper.

To simplify the analysis we assume in the following that φ is differentiable.

First, we consider $X \subset L_i(g)$ and provide the expected marginal gross profit obtained by firm i when it maintains the links belonging to X.

$$\zeta_i^-(X,g) = \left(\sum_{ij\in X}\rho_{i,j}\right)(\mathcal{A} + 2\mathcal{B}\mathcal{U}_i(g) - \mathcal{C}\mathcal{U}(g_{-i})) - \mathcal{B}\left(\sum_{ij\in X}\rho_{i,j}\right)^2,$$

where $\mathcal{A} = 2(n-1)\gamma(\alpha-\gamma_0)/(n+1)^2$, $\mathcal{B} = ((n-1)\gamma/(n+1))^2$, $\mathcal{C} = 4\gamma^2(n-1)/(n+1)^2$.[4]

Second, we consider $X \subset L_i(\bar{g})$, a subset of links that firm i *can* form in g, and provide the expected marginal gross profit for i when it forms the links belonging to X:

$$\zeta_i^+(X,g) = \left(\sum_{ij\in X}\rho_{i,j}\right)(\mathcal{A} + 2\mathcal{B}\mathcal{U}_i(g) - \mathcal{C}\mathcal{U}(g_{-i})) + \mathcal{B}\left(\sum_{ij\in X}\rho_{i,j}\right)^2.$$

We set $z^m(g) = \min_{i\in N}\{\zeta_i^-(\{ij\},g) : ij \in L(g)\}$. Roughly speaking $z^m(g)$ is the lowest marginal value of the links formed in g. We also set $z^M(g) = \max_{ij\notin L(g)} \min\{\zeta_i^+(\{ij\},g), \zeta_j^+(\{ij\},g)\}$. Roughly speaking $z^M(g)$ is highest marginal value of the unformed links in g, accounting for the fact that the formation of a link needs the consent of both firms involved in. Moreover, note that $\mathcal{U}(g_{-k}) = \mathcal{U}(g) - \mathcal{U}_k(g)$, so if $\mathcal{U}_i(g) \geq \mathcal{U}_{i'}(g)$, then $\mathcal{U}(g_{-i}) \leq \mathcal{U}(g_{-i'})$.

3.1 Pair-wise Equilibrium Networks and Uncertainty

Before proceeding further, we define a property that allows us to check for pair-wise equilibrium in a quick and easy manner.

(P1) For all $X \subset L_i(g)$, $\zeta_i^-(X,g) > |X|f$. In other words, firm i has an incentive to maintain its links in g.

Lemma 1 *(P1) is satisfied for all firms i in g and $f > z^M(g)$ if and only if g is a pair-wise equilibrium network.*

[4]It is worth noting that if (C1) is satisfied, then $\zeta_i^-(X,g)$ is increasing with $\sum_{ij\in X}\rho_{i,j}$. Indeed, $\zeta_i^-(X,g)$ can be rewritten as follows: $\left(\sum_{ij\in X}\rho_{i,j}\right)(\mathcal{A} + \mathcal{B}(\sum_{ij\in X}\rho_{i,j} + 2\sum_{ij\in L_i(g)\setminus X}\rho_{i,j}) - \mathcal{C}\mathcal{U}(g_{-i}))$.

Proof Suppose (P1) is satisfied. Then, no firm has an incentive to remove some of their links. Let i and j be such that $ij \notin L(g)$. Since $f > z^M$, either $\zeta_i^+(\{ij\}, g) < f$ or $\zeta_j^+(\{ij\}, g) < f$. Therefore, no unlinked pair of firms have an incentive to form a link. Now suppose g is a pair-wise equilibrium network. Then no firm has an incentive to remove some of their links, and (P1) is satisfied. Moreover, no pair of firms has an incentive to add a link, and $f > z^M(g)$ is satisfied. The result follows. □

We now examine what happens to network formation when links formation is costless.

Proposition 1 *Suppose $f = 0$. Then a pair-wise equilibrium network is the complete network.*

Proof Suppose $f = 0$. The minimal expected marginal profit that a firm i can obtain from a link with firm j is $\Xi = \rho^m(\mathcal{A} - (n-1)(n-2)\rho^M\mathcal{C})$; $\Xi > 0$ since condition (C1) is satisfied giving us the complete network. □

By a continuity argument, if the costs of forming links are sufficiently small, then the complete network is always the pair-wise equilibrium network. This result is similar to the result found by GJ (2003) where the process innovation between two collaborative firms is always successful. In other words, GJ (2003)'s result is robust to the introduction of uncertainty regarding the outcome of pair-wise collaborative links concerning process innovation. The mechanism behind this result is the same as in GJ where $\rho_{i,j} = 1$ for all $i, j \in N$: each additional link formed by firm i allows it to increase its quantity, and so its profit.

We now provide a necessary condition for a pair-wise equilibrium network by comparing a link that exists with a link that does not.

Proposition 2 *Let g be a pair-wise equilibrium network and suppose $ij \in L(g)$ and $i'j' \notin L(g)$. Then, $\mathcal{U}^m_{i,j}(g) > \mathcal{U}^m_{i',j'}(g)$ or $\rho_{i,j} > \rho_{i',j'}$.*

Proof To introduce a contradiction suppose a pair-wise equilibrium network g *does not satisfy* the necessary condition. Wlog, we assume that $\mathcal{U}_i(g) \leq \mathcal{U}_j(g)$ and $\mathcal{U}_{i'}(g) \leq \mathcal{U}_{j'}(g)$. Since $\mathcal{U}_i(g) \leq \mathcal{U}_{i'}(g)$, we have $\mathcal{U}(g_{-i}) \geq \mathcal{U}(g_{-i'})$. Set $\rho_{i',j'} = \rho_{i,j} + \delta$, $\delta \geq 0$. We note that $\zeta_{i'}^+(\{i'j'\}, g) - \zeta_i^-(\{ij\}, g)$ is equal to

$$\left(\frac{(n-1)\gamma}{(n+1)^2}\right)\rho_{i,j}[2\gamma(n-1)(\mathcal{U}_{i'}(g) - \mathcal{U}_i(g) + \rho_{i,j}) - 4(\mathcal{U}(g_{-i'}) - \mathcal{U}(g_{-i}))] + \delta T,$$

where $T > 0$ by (C1). Since $\rho_{i,j} > 0$, $\mathcal{U}_{i'}(g) \geq \mathcal{U}_i(g)$, and $\mathcal{U}(g_{-i'}) \leq \mathcal{U}(g_{-i})$, $\zeta^+_{i'}(\{i'j'\}, g) - \zeta^-_i(\{ij\}, g) > 0$. It follows that $\zeta^+_{i'}(\{i'j'\}, g) > \zeta^-_i(\{ij\}, g) \geq f$. The last inequality comes from the fact that firm i has an incentive to maintain the link ij in g. Since $\zeta^+_{j'}(\{i'j'\}, g) > \zeta^+_{i'}(\{i'j'\}, g)$, the result follows.

□

Proposition 2 states that it is not possible for a pair of unlinked firms i', j' to simultaneously have (i) a cost competitive advantage, and (ii) an innovation probability advantage over linked firms.

For cost-reducing collaborations in linear oligopoly models we can also provide a sufficient condition for a pair-wise equilibrium network. This condition allows us to easily establish some additional results. Let $L_i(g) = \{ij_1, \ldots, ij_m\}$ with $\rho_{i,j_k} < \rho_{i,j_{k+1}}$ for all $k \in \{1, \ldots, m-1\}$. Let g^{ij_k} be the network identical to g except that links $ij_1, \ldots, ij_k$ are removed. By convention $g^{ij_0} = g$.

(P2) $\zeta^-_i(\{ij_{k+1}\}, g^{ij_k}) > \zeta^-_i(\{ij_{k+2}\}, g^{ij_{k+1}})$ for all $k \in \{0, \ldots, m-2\}$.

In (P2), we rank the collaborative links formed by firm i according to the success probabilities. Then, we assume that the marginal profit is more affected by removing the k^{th} link than removing the $(k+1)^{\text{th}}$ link. In the appendix we provide some additional explanations about (P2).

Proposition 3 *Suppose that (P2) is satisfied, and* $\Pi^\star_i(g) \geq ((\alpha - \gamma_0 - 2\gamma\mathcal{U}(g_{-i}))/(n+1))^2$ *for all firms i in g, and $z^M(g) < f$. Then, g is a pair-wise equilibrium network.*

Proof Consider firm i which has formed links $ij_1, \ldots, ij_m$. By (P2) firm i has an incentive to maintain links $ij_1, \ldots, ij_m$, or remove all its links. To introduce a contradiction suppose that firm i has an incentive to remove only some of its links. Then, firm i will remove the "least valuable" links. For instance, i removes links $ij_1, \ldots ij_{\ell-1}$ and maintain links $ij_\ell, \ldots ij_m$. Then, $\sum_{k=\ell+1}^{m} \zeta^-_i(\{ij_k\},\, g^{ij_{k-1}}) > (m-\ell)f$. Since $\zeta^-_i(\{ij_{\ell-1}\}, g^{ij_{\ell-2}}) > \max_{k\in\{\ell,\ldots m\}}\{\zeta^-_i(\{ij_{k+1}\}, g^{ij_k})\} > f$ firm i has an incentive to maintain link $ij_{\ell-1}$, a contradiction. Since for all $i \in N$, $\Pi^\star_i(g) \geq ((\alpha - \gamma_0 - 2\gamma\mathcal{U}(g_{-i}))/\ (n+1))^2$, no firm has a strict incentive to remove all its links. It follows that no firm has an incentive to remove links. Finally, since $f > z^M(g)$, no pair of firms has an incentive to add a link, since this link decreases the profit of one of these firms. □

Following Proposition 3, we observe that in pair-wise equilibrium networks the most *valuable* links are not always formed by firms. For instance, suppose that $\alpha - \gamma_0 = 4$, $\gamma = 1$. For network g drawn in Figure 2, we have $z^m(g) = 0.21$ while $z^M(g) = 0.12$. If $f = 0.2$, then network g is a candidate for being a pair-wise equilibrium network even if $\rho_{1,5} = 0.25 \geq \rho_{i,j}$, with $i \in N$ and $j \in N_{-i}$. Indeed, the assumptions of Proposition 3 are satisfied. In particular (P2) is satisfied since all the success probabilities associated with the links formed in g are identical.

Network g Network $\bar{g}$

Figure 2: A pair-wise equilibrium network without the most valuable link

Recall that when collaborative links are always successful, GJ (2003) find that the k-dominant group architecture is the only equilibrium network, where connected firms are in a symmetric position. We now highlight the fact that in our model, because of heterogeneous innovation success probabilities of R&D links, it is possible to obtain pair-wise equilibrium networks where firms that have formed collaborative R&D links are in asymmetric positions. More precisely, in the next corollary, we show that if there exists a firm i whose collaborative links lead to an innovation with a high probability, while collaborative links between the other firms have a low innovation success probability, then the pair-wise equilibrium network is a star where i is the centre.

Corollary 1 *Suppose* $\rho_{i,j} = \rho$ *for all* $j \in N_{-i}$ *and* $\rho_{i',j'} \leq \rho'$ *for all* $i' \in N_{-i}$ *and* $j' \in N_{-i}\backslash\{i'\}$. *If (i)* $\rho(\mathcal{A} + \mathcal{B}\rho - \mathcal{C}(\rho(n-2) + \rho'(n-2)(n-3)/2)) > f$ *and (ii)* $\rho'(\mathcal{A} + 2\mathcal{B}(\rho + (n-2)\rho') - (n-2)\mathcal{C}\rho + (n-2)\mathcal{B}\rho') < f$, *then the unique pair-wise equilibrium network is a star where* i *is the centre.*

Proof Let g be a star where i is the centre. We assume that $\rho_{i,j} = \rho$ for all $j \in N_{-i}$. Therefore, (P2) is satisfied for firm i. Using the same arguments as in the proof of Proposition 3, firm i should preserve all its links or remove all its links. Firm i has no incentive to remove its links since condition (i) implies that $\mathcal{R} = \rho(\mathcal{A} + (n-1)\mathcal{B}\rho) > f$, where $(n-1)\mathcal{R}$ is the difference between the gross profit of firm i in g and its gross profit in the empty network. Conditions (i) and (ii) imply that no firm $j \in N_{-i}$ has an incentive to modify its strategy in the star network. Moreover, (i) implies that firms i and $j \in N_{-i}$ and firm i always have an incentive to form a link together in a pair-wise equilibrium network, and (ii) implies that firms $j, j' \in N_{-i}$ never have an incentive to form a link together in the network where there exist links between firm i and each firm $j \in N_{-i}$. The result follows. $\square$

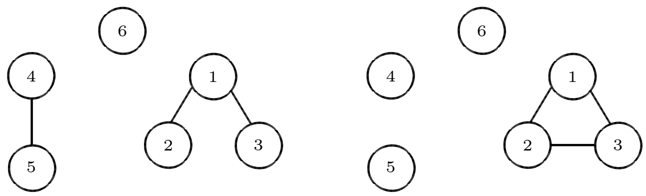
In the previous corollary, we did not state the result for $\rho_{i,j} \geq \rho$ for all $j \in N_{-i}$. We now explain the reason for this statement. If the success probabilities of R&D links in a pair-wise equilibrium network increase, then there exist situations such that some of these links will be removed in pair-wise equilibrium networks.

Example 1 Suppose $N = \{1, \ldots, 6\}$, $\alpha - \gamma_0 = 490$, $\gamma = 1$, $\rho = 0.9$, $\rho' = 0.8918244$, $\kappa = 1.000111$, and $f = 89.752038$. The sequence B is given by $\rho_{i,j} = \rho$ for $ij \in \{12, 13, 45\}$, $\rho_{i,j} = \rho'$ for $ij = 23$, and $\rho_{i,j} = \eta$, η arbitrarily low, for all other links. Finally, we define the sequence B' as follows $\rho_{i,j} = \kappa\rho$ for $ij \in \{12, 13, 45\}$, $\rho_{i,j} = \rho'$ for $ij = 23$, and $\rho_{i,j} = \eta$ for all other links. Network g in Figure 3 is the unique pair-wise equilibrium network when the sequence of probabilities is given by B, while network g' in Figure 3 is the unique pair-wise equilibrium network when the sequence of probabilities is given by B'.[5]

We now explain the intuition behind Example 1. The link between firms 2 and 3 is not formed in g but is formed in g', since ζ_i^+ is increasing in $\mathcal{U}_i(g)$ and the links 12 and 13 are more valuable in g' than in g. Given that the link 23 is formed in g', firms 4 and 5 have an incentive to remove their link in g' since ζ_i^- is decreasing in $\mathcal{U}(g_{-i})$.

Example 1 illustrates a non-monotonic property. In the following proposition, we state this formally. For each sequence of probabilities on collaborative links, B, we denote by $\mathcal{G}^\star(B)$ the set of pair-wise equilibrium networks associated with this sequence. We now consider a net-

[5] Additional details about this example can be found in the Appendix.



Network g Network g'

Figure 3: Example of non-monotonicity

work $g \in \mathcal{G}^\star(B)$ and associate with the sequence B and the network g a sequence $B^{(\kappa,g)}$ which satisfies: $\rho_{i,j}(B^{(\kappa,g)}) = \kappa\rho_{i,j}(B)$, where $\kappa > 1$, for all $ij \in L(g)$, and $\rho_{i,j}(B^{(\kappa,g)}) = \rho_{i,j}(B)$ for all $ij \notin L(g)$.

Proposition 4 *Let $\kappa > 1$. There exist B, $g \in \mathcal{G}^\star(B)$, and $B^{(\kappa,g)}$ such that $L(g') \not\subset L(g)$ for all $g' \in \mathcal{G}^\star(B^{(\kappa,g)})$.*

Example 1 has implications for policy. It highlights the fact that there exist situations where the government should not try to improve the probability of success of the innovations associated with collaborative links. Indeed, in this case we obtain $\kappa > 1$. In Example 1, the total expected reduction in marginal costs in g drawn in Figure 3 is 0.27γ, while the total expected reduction in marginal costs in g' drawn in Figure 3 is strictly smaller than 0.27γ.

Till now we have focussed on properties of pair-wise equilibrium networks. We have provided necessary and sufficient conditions for their existence. We have shown that they can lead to networks where firms are in asymmetric positions and the fact the equilibrium networks can be non-monotonic with respect to probabilities. We have also shown that we get the same results as GJ (2003) when links are costless. However, recall that unlike the two forces in GJ (2003)'s model there are three forces at work in our framework. We now investigate how these three forces affect outcomes by examining the relationship between uncertainty, heterogeneity and equilibrium architectures. More precisely, we show that there exists a sufficient condition under which the model with heterogeneous success probabilities in collaborative links yields the same outcome as a model of collaborative R&D with no heterogeneity in probabilities, one special case of which is the model with no uncertainty.

For each B, we set $\bar{\mathcal{U}}^B = \max\{\sum_{j\in N_{-i}} \rho_{i,j} : i \in N\}$. We define the following function for ϵ, with $0 \leq \epsilon \leq \rho^m(B)/2$:

$$z_B(\epsilon) \quad = \quad -2\left(\frac{n-1}{n+1}\right)\left(\frac{(\alpha-\gamma_0)+\gamma(n-1)\bar{\mathcal{U}}^B}{n+1}\right)\epsilon + \gamma\left(\frac{n-1}{n+1}\right)^2 (\rho^m(B))^2.$$

Let $Z_B = \{\epsilon \in [0, \rho^m(B)/2(n-1)] : z_B(\epsilon) \geq 0\}$. By construction, Z_B is bounded since $Z_B \subset [0, \rho^m(B)/2]$. Moreover, we have $z_B(0) > 0$ and by (C1) $z_B(\rho^m(B)/2(n-1)) < 0$. Since $z_B(0) > 0$, Z_B is non-empty. Consequently, Z_B has a least upper bound. Moreover since z_B is continuous, $z_B(0) > 0$ and $z_B(\rho^m(B)/2(n-1)) < 0$, there exists $\epsilon \in (0, \rho^m(B)/2(n-1))$, such that $z_B(\epsilon) = 0$ by the Intermediate Value theorem. Therefore, ϵ, such that $z_B(\epsilon) = 0$, belongs to Z_B. Moreover, z_B is strictly decreasing. It follows that the least upper bound of Z_B belongs to Z_B. We denote by $\bar{\varepsilon}(B)$ the maximal element of Z_B, obviously $z_B(\bar{\varepsilon}(B)) = 0$ and $\bar{\varepsilon}(B) > 0$.

Lemma 2 *Consider a network g such that $ij \notin L(g)$. The gross expected marginal profit obtained by firm i from a link with firm j is at least*

$$\rho_{i,j}\varphi_1(\mathcal{U}_i(g), \mathcal{U}(g_{-i})).^6$$

Proof Let g be such that $ij \notin L(g)$. The gross expected marginal profit that firm i obtains from the link ij is $\varphi(\mathcal{U}_i(g)+\rho_{i,j}, \mathcal{U}(g_{-i})) - \varphi(\mathcal{U}_i(g), \mathcal{U}(g_{-i}))$. By the Mean Value theorem, there is $\omega \in (\mathcal{U}_i(g), \mathcal{U}_i(g) + \rho_{i,j})$ such that

$$\varphi_1(\omega, \mathcal{U}(g_{-i})) = \frac{\varphi(\mathcal{U}_i(g) + \rho_{i,j}, \mathcal{U}(g_{-i})) - \varphi(\mathcal{U}_i(g), \mathcal{U}(g_{-i}))}{\rho_{i,j}}.$$

Since φ is strictly convex in its first argument, we have $\varphi_1(\omega, \mathcal{U}(g_{-i})) > \varphi_1(\mathcal{U}_i(g), \mathcal{U}(g_{-i}))$ and so

$$\rho_{i,j}\left(\frac{\varphi(\mathcal{U}_i(g) + \rho_{i,j}, \mathcal{U}(g_{-i})) - \varphi(\mathcal{U}_i(g), \mathcal{U}(g_{-i}))}{\rho_{i,j}}\right) > \rho_{i,j}\varphi_1(\mathcal{U}_i(g), \mathcal{U}(g_{-i})),$$

and the result follows. □

Proposition 5 establishes that the result obtained in the framework where innovations are always successful (GJ, 2003) is preserved in a framework with low variability in uncertainty

[6] φ_1 is the first derivative of φ wrt to its first argument.

and heterogeneity about the outcome of collaborative links. In particular, if collaborative links have similar probabilities of yielding a process innovation, then a non-empty pair-wise equilibrium network has the $k-$dominant group architecture. Let $\rho^M - \rho^m = \mu$.

Proposition 5 *Suppose $\mu < \bar{\varepsilon}(B)$. Then, a pair-wise equilibrium network has the $k-$dominant group architecture.*

Proof Let g be a pair-wise equilibrium network. Suppose that $\mu < \bar{\varepsilon}(B)$. We show that if $ij \in L(g)$ and $i'j' \in L(g)$, then $ij' \in L(g)$. We have $\zeta_\ell^-(\{\ell k\}, g) = \rho_{\ell,k}(\mathcal{A}+\mathcal{B}(2\mathcal{U}_\ell(g)-\rho_{\ell,k})-\mathcal{C}\mathcal{U}(g_{-\ell})$. Since firm $\ell \in \{i, j, i', j'\}$ has an incentive to maintain its link with firm $k \in \{i, j, i', j'\} \setminus \{\ell\}$ in g, then we have $\zeta_\ell^-(\{\ell k\}, g) \geq f$.
By Lemma 2, if $ij' \notin L(g)$ and firms i and j' form a link together, then firm i obtains a gross expected marginal profit equal to $\rho_{i,j'}(\mathcal{A} + \mathcal{B}(2\mathcal{U}_i(g) + \rho_{i,j'}) - \mathcal{C}\mathcal{U}(g_{-i}))$ which is higher than $[\rho_{i,j'}]\varphi_1(\mathcal{U}_i(g), \mathcal{U}(g_{-i}))$. If $\rho_{i,j'} \geq \rho_{i,j}$, then $[\rho_{i,j'}]\varphi_1(\mathcal{U}_i(g), \mathcal{U}(g_{-i})) > \zeta_i^-(\{ij\}, g)$, and firm i has an incentive to form a link with firm j'. Suppose now that $\rho_{i,j'} < \rho_{i,j}$. We know that $|\rho_{i,j} - \rho_{i,j'}| < \bar{\varepsilon}(B)$. We set $\rho_{i,j'} = \rho_{i,j} - \epsilon$, with $\epsilon \in (0, \bar{\varepsilon}(B))$. Let $H = [\rho_{i,j'}]\varphi_1(\mathcal{U}_i(g), \mathcal{U}(g_{-i})) - \zeta_i^-(\{ij\}, g)$. We have: $H = \mathcal{B}(\rho_{i,j})^2 - \epsilon(\mathcal{A} + 2\mathcal{B}\mathcal{U}_i(g) - \mathcal{C}\mathcal{U}(g_{-i}))$. It follows that

$$\begin{aligned} H/\gamma &\geq -2\left(\frac{n-1}{n+1}\right)\left(\frac{(\alpha-\gamma_0)+\gamma[(n-1)\bar{\mathcal{U}}^B]}{n+1}\right)\epsilon + \gamma\left(\frac{n-1}{n+1}\right)^2(\rho^m(B))^2 \\ &= z_B(\varepsilon) > z_B(\bar{\varepsilon}(B)) = 0. \end{aligned}$$

The last inequality comes from the fact that z_B is strictly decreasing. It follows that firm i has an incentive to form the collaborative link with firm j'. We use similar arguments to establish that firm j' has an incentive to form a collaborative link with firm i. □

The above proposition implies a result shown in GJ (2003).

Corollary 2 *(Proposition 4.1, pg.74, GJ, 2003) If for all $\rho \in B$, we have $\rho = 1$, then a non-empty pair-wise equilibrium network has the $k-$dominant group architecture.*

Thus, the result obtained in the framework where innovations are always successful is preserved when we introduce uncertainty in the outcome of collaborative links. One way to interpret this result is that when probability of success of innovations does not depend on the identity or the characteristics of the firms, *i.e.*, the success probabilities are very similar, the presence or absence of uncertainty does not matter; the "third force", heterogeneity between firms, does not modify the effect of the two other "forces" (strict convexity and sub-modularity). However, when the probabilities of success of process innovations depend on the identity or the characteristics of the firms and these probabilities are very different, this result no longer holds.

3.2 Geographical Proximity and Innovation Probability

Till now, we have not imposed any restrictions on the characteristics of firms and on the difference of innovation success probabilities across links. However, in many cases, these probabilities may depend on factors like geographic proximity or cultural similarities (Gomes-Casseres, Hageddorn and Jaffe, 2006). We now illustrate our framework by considering one such factor $-$we assume that geographic location is the main determinant of the success probabilities of process innovations. If two firms are located in the same place, then they belong to the same set (or group), otherwise they belong to two distinct sets. Suppose that there are $m \leq n$ distinct groups of firms. These groups are denoted by $[1], [2], \ldots, [m]$. We assume that if firms i and i' belong to the same group and j and j' belong to another group, then collaborative links between i and j, and between i' and j' have the same probability of yielding a successful process innovation. Formally, if $i, i' \in [i]$ and $j, j' \in [j]$, then $\rho_{i,j} = \rho_{i',j'}$.

We define a ring on which we rank the groups in the natural order. We define the distance between the groups $[x]$ and $[y]$, $d([x],[y])$ as follows $d([x],[y]) = \min\{|x-y|, m-|x-y|\}$. Let ϕ be a function which satisfies:

(A1) If $i_1 \in [i]$ and $j_1 \in [j]$, then $\rho_{i_1,j_1} = \phi(d([i],[j]))$;

(A2) The function ϕ is decreasing.

In the following, let $[\bar{x}] = \{i \in [x] : g(i) \neq \emptyset\}$. In the two following results, we assume that the probabilities associated with the links satisfy assumptions (A1), and (A2). The first proposition establishes some properties that pair-wise equilibrium networks must satisfy when geographic proximity determines innovation success probabilities.

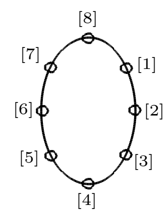


Figure 4: A ring with $m = 8$ groups of firms

Proposition 6 *Let g be a pair-wise equilibrium network and suppose (A1) and (A2) are satisfied.*

1. *If $i_1, i_2 \in [\bar{i}]$, then i_1 and i_2 are linked in g.*
2. *Suppose $\min\{d([j],[i]), d([k],[i])\} \geq d([k],[j])$. If $i_1 \in [i]$ is linked with $j_1 \in [j]$ and i_1 is linked with $k_1 \in [k]$, then j_1 is linked with k_1 in g.*

Proof We prove successively the two parts of the proposition. Let g be a pair-wise equilibrium network.

1. To introduce a contradiction suppose that $i_1, i_2 \in [\bar{i}]$ and $i_1 i_2 \notin L(g)$. Since $i_1, i_2 \in [\bar{i}]$, $g(k) \neq \emptyset$, with $k \in \{i_1, i_2\}$. Therefore, there exist $j_1, j_2 \in N$ such that $i_1 j_1 \in L(g)$ and $i_2 j_2 \in L(g)$. We set $j_1 \in [j_1]$. Since $d([i],[i]) \leq d([i],[j_1])$ and ϕ is decreasing, we have $\rho_{i_1,j_1} = \phi(d([i],[j_1])) \leq \phi(d([i],[i])) = \rho_{i_1,i_2}$. If firm i_1 forms a link with firm i_2, then i_1 obtains an expected marginal profit equal to $\zeta^+_{i_1}(\{i_1, i_2\}, g)$. By straightforward calculations, $\zeta^+_{i_1}(\{i_1, i_2\}, g)) > \zeta^+_{i_1}(\{i_1, j_1\}, g - i_1 j_1) = \zeta^-_{i_1}(\{i_1, j_1\}, g)$. Moreover, $\zeta^-_{i_1}(\{i_1, j_1\}, g) \geq f$ since g is a pair-wise equilibrium network. It follows that firm i_1 has an incentive to form a link with firm i_2. We use similar arguments to show that firm i_2 has an incentive to form a link with firm i_1. Consequently, g is not a pair-wise equilibrium network, a contradiction.
2. We set $\min\{d([j],[i]), d([k],[i])\} \geq d([k],[j])$. Therefore, we have for firms $i_1 \in [i]$, $j_1 \in [j]$ and $k_1 \in [k]$, $\rho_{i_1,j_1} = \phi(d([i],[j])) \leq \phi(d([j],[k])) = \rho_{j_1,k_1}$ since ϕ is decreasing. We assume that firm $i_1 \in [i]$ is linked with firm $j_1 \in [j]$ and i_1 is linked with firm $k_1 \in [k]$, while j_1 is not linked with k_1. If j_1 forms a link with k_1, it obtains an expected marginal profit equal to $\zeta^+_{j_1}(\{j_1, k_1\}, g) \geq \zeta^+_{j_1}(\{j_1, i_1\}, g - i_1 j_1) = \zeta^-_{j_1}(\{j_1, i_1\}, g)$. Moreover, $\zeta^-_{j_1}(\{j_1, i_1\}, g) \geq f$

since g is a pair-wise equilibrium network. It follows that firm j_1 has an incentive to form a link with firm k_1. We use similar arguments to show that k_1 has an incentive to form a link with j_1.

□

It is worth noting that in our model, it is possible to obtain a pair-wise equilibrium network where $i_1 \in [i]$ is linked with $j_1 \in [j]$ but i_1 is not linked with $k_1 \in [k]$, with $k \in \{i, \ldots, j\}$. In other words, geographic proximity between two firms is not the only point to take into account for analyzing R&D networks built by firms. Our result is in line with some empirical findings. For instance, Gomes-Casseres, Hagedoorn and Jaffe (2006) show that success of inter-firm collaborations depends on several factors, with geographical proximity being only one of these factors. Below, we provide an example to illustrate this point.

Example 2 Suppose that $N = \{1, \ldots, 12\}$, with $\{1, 2, 3\} = [1]$, $\{4, 5, 6\} = [2]$, $\{7, 8, 9\} = [3]$, $\{10, 11, 12\} = [4]$. We set $\phi(x) = 1/\sqrt{x + 1}$, $\alpha - \gamma_0 = 34$, $\gamma = 1$, and $f = 3/2$. Then network g drawn in Figure 5 is a pair-wise equilibrium network.

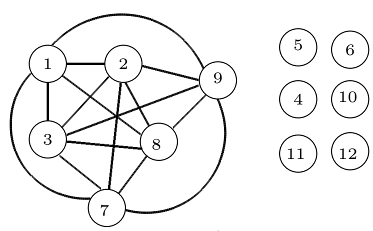


Figure 5: Network g

Example 2 illustrates that in this model the choice of collaborators is not driven only by geographic proximity, but also by innovation uncertainty the firms face. It follows that a collaborative link can appear between two firms which are far from each other, for instance firms 1 and 9, when they have a high flow degree leading to a higher cost advantage, while such a collaborative link may not appear if at least one of the two firms involved in has a low flow degree, for instance between firms 9 and 10.

The next corollary establishes that there exists a condition for geographic proximity to play a crucial role in the pair-wise equilibrium networks. Under this condition all firms which are close enough and have already formed a link are all linked with each other.

Corollary 3 *Suppose that for all* $[i]$, $[j]$ *such that* $d([i],[j]) < m$, $m > 0$, *we have* $\phi(0) - \phi(d([i],[j])) < \bar{\varepsilon}(B)$. *Then, each firm* $i_1 \in [\bar{i}]$ *is linked with each firm* $j_1 \in [\bar{j}]$ *such that* $d([i],[j]) < m$.

4 Results for a Larger Class of Oligopoly Games

In this section, we establish that the characterization result of the previous section (Proposition 2) is true even when the expected profit function satisfies some general properties. Then, we show that models of cost-reducing collaboration in differentiated oligopolies (Cournot and Bertrand) also satisfy these conditions. In other words, we establish that it is possible to obtain results for a general framework where uncertainty regarding the success of innovation is introduced. This result complements the work of Goyal and Joshi (2006).

Let σ be a twice differentiable function. We say that σ is strictly convex if for all y, $\sigma_{11}(x, y) > 0$ for all x. We say that σ is *sub-modular* if for all y, $\sigma_{12}(x, y) \leq 0$ for all x. In the following we assume that the gross expected profit function of firm i is:

$$\pi_i(g) = \sigma(\mathcal{U}_i(g), \mathcal{U}(g_{-i})), \tag{3}$$

where σ is a strictly convex and sub-modular function. In other words the net expected *marginal* profit of firm i satisfies the following conditions:

1. strictly increasing with $\mathcal{U}_i(g)$, that is strictly increasing in the flow degree of firm i; and
2. decreasing with $\mathcal{U}(g_{-i})$, that is decreasing in the total flow degree of firms $j \in N \setminus \{i\}$ in g_{-i}.

Finally, we assume that the expected profit function of firm i is $\Pi_i(g) = \pi_i(g) - |L_i(g)|f$, where $f > 0$.

Proposition 7 *Suppose that the payoff function is given by (3) where* σ *is strictly increasing in its first argument, strictly convex and sub-modular. Let* g *be a pair-wise equilibrium network, with* $ij \in L(g)$ *and* $i'j' \notin L(g)$. *Then,* $\mathcal{U}^m_{i,j}(g) > \mathcal{U}^m_{i',j'}(g)$ *or* $\rho_{i,j} > \rho_{i',j'}$.

Proof To introduce a contradiction suppose a pair-wise equilibrium network g does not satisfy the condition. Wlog, assume that $\mathcal{U}_i(g) \leq \mathcal{U}_j(g)$ and $\mathcal{U}_{i'}(g) \leq \mathcal{U}_{j'}(g)$. Note that $\mathcal{U}_i(g) \leq \mathcal{U}_{i'}(g)$

implies that $\mathcal{U}(g_{-i'}) \geq \mathcal{U}(g'_{-i}) = \mathcal{U}(g_{-i})$, where $g' = g - ij$ and where the equality comes from the fact links in which i is not involved are the same in g and g'. We have:

$$\begin{aligned}
\pi_{i'}(g + i'j') - \pi_{i'}(g) &= \sigma(\mathcal{U}_{i'}(g) + \rho_{i'j'}, \mathcal{U}(g_{-i'})) - \sigma(\mathcal{U}_{i'}(g), \mathcal{U}(g_{-i'})) \\
&\geq \sigma(\mathcal{U}_{i'}(g) + \rho_{i'j'}, \mathcal{U}(g_{-i})) - \sigma(\mathcal{U}_{i'}(g), \mathcal{U}(g_{-i})) \\
&\geq \sigma(\mathcal{U}_{i'}(g) + \rho_{ij}, \mathcal{U}(g_{-i})) - \sigma(\mathcal{U}_{i'}(g), \mathcal{U}(g_{-i})) \\
&\geq \sigma(\mathcal{U}_{i}(g), \mathcal{U}(g_{-i})) - \sigma(\mathcal{U}_{i}(g) - \rho_{ij}, \mathcal{U}(g_{-i})) \\
&= \pi_i(g) - \pi_i(g - ij)
\end{aligned}$$

The first inequality comes from the fact that $\mathcal{U}(g_{-i'}) \leq \mathcal{U}(g_{-i})$ and σ is sub-modular. The second inequality comes from the fact that $\rho_{i'j'} \geq \rho_{ij}$ and σ is increasing. The third inequality comes from the fact that $\mathcal{U}_{i'}(g) \geq \mathcal{U}_i(g)$, and σ is strictly convex. We now establish the strict inequality between $\pi_{i'}(g + i'j') - \pi_{i'}(g)$ and $\pi_i(g) - \pi_i(g - ij)$. If $\rho_{i,j} < \rho_{i',j'}$, then the second inequality is strict since σ is strictly increasing. If $\mathcal{U}_i(g) < \mathcal{U}_{i'}(g)$, then the third inequality is strict since σ is strictly convex. Since one of these two possibilities must hold under our assumptions, we have $\pi_{i'}(g + i'j') - \pi_{i'}(g) > \pi_i(g) - \pi_i(g - ij)$. Using similar arguments, we establish that $\pi_{j'}(g + i'j') - \pi_{j'}(g) > \pi_i(g) - \pi_i(g - ij)$. Since $\pi_i(g) - \pi_i(g - ij) \geq f$, it follows that firms i' and j' have an incentive to form a link together. Therefore, network g is not a pair-wise equilibrium network, a contradiction. □

We now illustrate how Proposition 7 relates to some specific payoff functions. Let $\theta : (x, y) \mapsto (a_1 + a_2 x - a_3 y)^2$ be a function where $a_1, a_2, a_3 \in \mathbb{R}_+ \setminus \{0\}$ and $a_1 > a_3 y$ for all $y \in \mathbb{R}_+$. We have $\theta_1(x, y) = 2a_2(a_1 + a_2 x - a_3 y) > 0$, $\theta_{11}(x, y) = 2(a_2)^2 > 0$ and $\theta_{12}(x, y) = -2a_2 a_3 < 0$. The function θ is strictly increasing in its first argument, strictly convex and sub-modular; so it satisfies the properties used to establish Proposition 7. We observe that if $a_1 = (\alpha - \gamma_0)/(n + 1)$, $a_2 = \gamma(n - 1)/(n + 1)$, and $a_3 = 2\gamma/(n + 1)$, then θ is equal to the expected gross profit given in equation 2.

In Examples 3 and 4 we consider the cost function given by Equation 1.

Example 3 (*Differentiated Cournot Oligopoly*) *Suppose each firm i faces the following linear inverse demand function: $p_i = \alpha - q_i - \beta \sum_{j \in N_{-j}} q_j$, where p_i is the price of the product sold by firm i, $\alpha > 0$, and $\beta \in (0,1)$.*
The equilibrium expected net profit for firm $i \in N$ is given by $\pi_i^d(g) = \theta(\mathcal{U}_i(g), \mathcal{U}(g_{-i})) = (a_1 + a_2\mathcal{U}_i(g) - a_3\ \mathcal{U}(g_{-i}))^2$, where $a_1 = (\alpha - \gamma_0)/(2 + \beta(n-1)) > 0$, $a_2 = \gamma((n-2)\beta + 2)/((2-\beta)(2+\beta(n-1)) > 0$ and $a_3 = \gamma\beta/((2-\beta)(2+\beta(n-1)) > 0$. Moreover, to ensure that each firm produces a strictly positive quantity in equilibrium, assume that $a_1 > a_3\mathcal{U}(g_{-i})$ for all $\mathcal{U}(g_{-i})$. Then, pair-wise equilibrium networks satisfy the conditions given in Proposition 7.

Example 4 (*Differentiated Bertrand Oligopoly*) *We assume that demand is similar to the one given in Example 3. We let*

$$\lambda = \frac{(1-\beta)(1+(n-1)\beta)}{1+(n-2)\beta}.$$

In the Bertrand equilibrium, the expected net profit for firm $i \in N$ can be written as: $\pi_i^B(\boldsymbol{g}) = \theta_i(\mathcal{U}_i(g), \mathcal{U}(g_{-i})) = \lambda(a_1 + a_2\mathcal{U}_i(g) - a_3\mathcal{U}(g_{-i}))^2$ where

$$\begin{aligned} a_1 &= \frac{(1+(n-2\beta))}{(2+(n-3)\beta)(1+(n-1)\beta)}(\alpha-\gamma_0) > 0, \qquad (4) \\ a_2 &= \gamma\frac{2+(5n-11)\beta+(4n^2-19n+21)\beta^2+((n^2-8n+19)n-14)\beta^3}{(2+(n-3)\beta)(1+(n-1)\beta)(1-\beta)(2+(2n-3)\beta)} > 0, \\ a_3 &= 2\gamma\frac{\beta+(2n-4)\beta^2+(n^2-4n+4)\beta^3}{(2+(n-3)\beta)(1+(n-1)\beta)(1-\beta)(2+(2n-3)\beta)} > 0. \end{aligned}$$

Using arguments similar to those in Example 3, we establish that in the model of cost-reducing collaboration in a differentiated Bertrand oligopoly, pair-wise equilibrium networks satisfy the conditions given in Proposition 7.

We now provide a result in line with Proposition 3. We need to define two notions. First, $z_\sigma^M(g) = \max_{ij \notin L(g)}\{\sigma(\mathcal{U}_{ij}^m(g) + \rho_{ij}, \mathcal{U}(g) - \mathcal{U}_{ij}^m(g)) - \sigma(\mathcal{U}_{ij}^m(g), \mathcal{U}(g) - \mathcal{U}_{ij}^m(g))\}$. Second, we define a property akin to (P2) and useful in this general specification of our framework. Let $L_i(g) = \{ij_1, \ldots, ij_m\}$ with $\rho_{i,j_k} < \rho_{i,j_{k+1}}$ for all $k \in \{1, \ldots, m-1\}$. Let g^{ij_k} be the network identical to g except that links $ij_1, ..., ij_k$ are removed.

(P2') $\sigma(\mathcal{U}_i(g^{ij_k}),\mathcal{U}(g_{-i}))-\sigma(\mathcal{U}_i(g^{ij_{k+1}}),\mathcal{U}(g_{-i})) > \sigma(\mathcal{U}_i(g^{ij_{k+1}}),\mathcal{U}(g_{-i}))-\sigma(\mathcal{U}_i(g^{ij_{k+2}}),\mathcal{U}(g_{-i}))$, for all $k \in \{0,\dots,m-2\}$.

In (P2'), we assume that the marginal profit is more affected by removing the k^{th} link than removing the $(k+1)^{\text{th}}$ link. In the following proposition, we use the same argument as in Proposition 3: no firm has an incentive to remove a link since it has no incentive to remove all its links and (P2') is satisfied; and no pair of unlinked firms i and j can add a link since this link decreases the profit of one of these firms.

Proposition 8 *Suppose that (P2') and $\sigma(\mathcal{U}_i(g),\mathcal{U}(g_{-i}))-\sigma(0,\mathcal{U}(g_{-i})) \geq f|L_i(g)|$ are satisfied for all firms i in g, and $z^M_\sigma(g) < f$. Then, g is a pair-wise equilibrium network.*

This result allows us to obtain the same kind of results as those obtained in the cost-reducing collaboration under linear oligopoly. Obviously, as we already showed, the class of model where the non-monotonicity result is true is not empty (Proposition 4). Moreover, when the success probabilities are sufficiently similar, the pair-wise equilibrium network is a k-dominant network (Proposition 5).

5 When firms obtain different benefits from the same innovation

Till now the network formation literature has only considered models where innovation is always certain and therefore all firms benefit equally from it. In practice, neither is innovation always certain and nor do all firms always benefit equally from it. In this section, we present an extension of the previous model. Here as before, pair-wise collaborative links have a positive probability of leading to a process innovation, but the value of the process innovation is different for the firms involved in the same collaborative link. In the following, we assume that if the link between firms i and j allows for a process innovation, then firm i reduces its marginal cost by $\gamma\tau'_{i,j} > 0$, while firm j reduces its marginal cost by $\gamma\tau'_{j,i} > 0$, where $\tau'_{i,j}$ and $\tau'_{j,i}$ may be different. Formally the expected marginal cost function is given by:

$$C_i^h(g) = \gamma_0 - \gamma \sum_{j \in g(i)} \rho_{i,j} \tau'_{i,j}, \tag{5}$$

Equation 5 can be rewriten as follows:

$$C_i^h(g) = \gamma_0 - \gamma \sum_{j \in g(i)} \tau_{i,j},$$

where $\tau_{i,j} = \rho_{i,j}\tau'_{i,j}$, so $\tau_{i,j} > 0$.
Let $B' = (\tau_{i,j})_{i \in N, j \in N_{-i}}$, and $\tau^M_{B'} = \max\{x : x \in B'\}$. For simplicity we use τ^M when the set B' does not play a crucial role in the analysis. Analogous to the notion of flow probabilities we define $\mathcal{T}_i(g) = \sum_{j \in g(i)} \tau_{i,j}$, and $\mathcal{T}(g) = \sum_{j \in N} \mathcal{T}_j(g)$.
To simplify, in the following, we consider the linear Cournot oligopoly model with homogeneous goods. Given any network g, the Cournot equilibrium output is:

$$q_i^h(g) = \frac{\alpha - \gamma_0 + \gamma(n\mathcal{T}_i(g) - \sum_{j \in N_{-i}} \mathcal{T}_j)}{n+1}.$$

In order to ensure that each firm produces a strictly positive quantity, we have condition

$$(C1') : \alpha - \gamma_0 > \gamma(n-1)^2 \tau^M.$$

The second stage Cournot expected gross profit of firm $i \in N$ is given by:

$$\pi_i^h(g) = \left(\frac{\alpha - \gamma_0 + \gamma(n\mathcal{T}_i(g) - \sum_{j \in N_{-i}} \mathcal{T}_j(g))}{n+1} \right)^2. \tag{6}$$

The gross marginal expected profit obtained by firm i in g when it maintains the link $ij \in L(g)$ is

$$\xi_i^-(\{i,j\}, g) = (n\tau_{i,j} - \tau_{j,i}) \left(\mathcal{A}' + \mathcal{B}'(2\mathcal{T}_i(g) - \tau_{i,j}) - \mathcal{C}' \left(2 \sum_{j \in N_{-i}} \mathcal{T}_j(g) - \tau_{j,i} \right) \right),$$

where $\mathcal{A}' = 2\gamma(\alpha - \gamma_0)/(n+1)^2$, $\mathcal{B}' = n(\gamma/(n+1))^2$, $\mathcal{C}' = (\gamma/(n+1))^2$. It is worth noting that $\mathcal{B}' = n\mathcal{C}'$ for $n \geq 3$. The gross marginal expected profit obtained by firm i in g when it forms an additional link $ij \notin L(g)$ is:

$$\xi_i^+(\{i,j\},g) = (n\tau_{i,j} - \tau_{j,i}) \left(\mathcal{A}' + \mathcal{B}'(2\mathcal{T}_i(g) + \tau_{i,j}) - \mathcal{C}' \left(2 \sum_{j \in N_{-i}} \mathcal{T}_j(g) + \tau_{j,i} \right) \right).$$

For the following results, we assume that the second stage Cournot expected profit of each firm i is given by $\Pi_i^h(g) = \pi_i^h(g) - |L_i(g)|f$, where $\pi_i^h(g)$ is given by equation 6. To establish the results we define $\sqsubset = \{(i,j) \in N \times N : \tau_{i,j} > n\tau_{j,i}\}$ as a strict order relation over B.[7]

Proposition 9 *Let g be a pair-wise equilibrium network. If $i \sqsubset j$ or $j \sqsubset i$, then $ij \notin L(g)$.*

Proof Suppose $f = 0$ and $n\tau_{i,j} < \tau_{j,i}$. By the condition (C1'), we have

$$\left(\mathcal{A}' + \mathcal{B}'(2\mathcal{T}_i(g) + \tau_{i,j}) - \mathcal{C}' \left(2 \sum_{j \in N_{-i}} \mathcal{T}_j + \tau_{j,i} \right) \right) > 0.$$

Since $n\tau_{i,j} - \tau_{j,i} < 0$, the expected marginal profit obtained by firm i when it forms a link with firm j is negative. Consequently, if $i \sqsubset j$ or $j \sqsubset i$ and $f = 0$, then $ij \notin L(g)$. The expected marginal profit associated with the link ij is decreasing with f. The result follows. □

Corollary 4 *Suppose $f = 0$ and $i \sqsubset j$ or $j \sqsubset i$. Then, a pair-wise equilibrium network is not the complete network.*

Proof The result follows from Proposition 9. □

Corollary 4 shows that when we introduce asymmetry concerning the impact of the process innovation on the firms involved in the same collaborative link, the result obtained under zero costs of forming collaborative links (Proposition 1) is not preserved: a pair-wise equilibrium network is not always the complete network. This is due to the following fact: suppose that a firm i is able to use process innovations in a much more efficient way than a firm j. If firm j forms a link with firm i, this link will greatly increase the competitiveness of one j's competitor while the competitiveness of firm j will only slightly change. As a result, j's profit will decrease, even if the costs of forming the link is null. It follows that j will never consent

[7]The relation $\sqsubset$ is irreflexive, asymmetric and transitive, so it is a strict order relation.

to form the link with i.

The next corollary and example illustrate the fact that there exist situations where firms that are more efficient in the use of the process innovation are the ones which are isolated. Indeed, the other firms will have no incentive to form collaborative links with them since the latter will become very strong competitors during the second stage of the game. Recall that in equilibrium, isolated firms always obtain a smaller expected profit than other firms.

Corollary 5 ***(The tyranny of the weakest).*** *Suppose that $i \not\sqsubset j$ and $j \not\sqsubset i$ for all $i, j \in S$, $S \subset N$, and for all $i' \in N \backslash S$, $j' \in N_{-i'}$, $i' \sqsubset j'$ or $j' \sqsubset i'$. If $f = 0$, then a pair-wise equilibrium network has the $|S|-$dominant group architecture, where only firms in S have formed links.*

Example 5 Suppose $N = \{1, \ldots, 6\}$, $\alpha - \gamma_0 = 5$, $\gamma = 1$, and $f = 0$. We assume that if i belongs to $\{1, \ldots, 4\}$, then $\tau_{i,j} = 0.01$ for all firms $j \in N_{-i}$. Moreover, $\tau_{5,k} = 0.1$ for all firms $k \in N \setminus \{5\}$, and $\tau_{6,k} = 0.7$ for all firms $k \in N \setminus \{6\}$. Then, network g drawn in Figure 6 is the unique pair-wise network. In this network, firm $i \in \{1, \ldots, 4\}$ obtains an expected profit equal to 0.53 while firms 5 and 6 obtain an expected profit equal to 0.49.

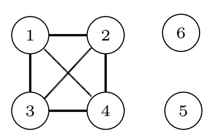


Figure 6: Tyranny of the weakest

Next we illustrate the fact that when there is heterogeneity in the value of the process innovation, pair-wise equilibrium networks may partition firms. More precisely, in a situation where some firms obtain high expected value from collaborative links, while others obtain low expected value, the pair-wise equilibrium network contains two components: the first one consisting of firms that obtain high expected value from process innovations, and the second one consisting of the firms obtaining low expected value from process innovations. We illustrate this situation in the next example.

Example 6 (*Positive Assortative Matching*). Suppose $N = \{1, \ldots, 6\}$, $\alpha - \gamma_0 = 5$, $\gamma = 1$, and $f = 0$. Let $S = \{1, 2, 3\}$. We assume that if $i \in S$, then $\tau_{i,j} = 0.5$ for all firms $j \in N \setminus \{i\}$

and if $i' \in N \setminus S$, then $\tau_{i',j'} = 0.01$ for all firms $j' \in N \setminus \{i'\}$. Then, network g drawn in Figure 7 is the unique pair-wise equilibrium network.

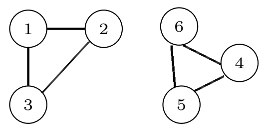


Figure 7: Partition of the collaboration network

We now present a result in line with Proposition 2. It provides properties concerning links that are formed and links that are not formed in pair-wise equilibrium networks. We assume that for all $i, j \in N$, $\tau_{i,j} \in ((1/n)\tau_{j,i}, n\tau_{j,i})$. In other words, the reduction in the expected marginal cost of firms cannot be very different. Under this condition, Proposition 9 does not prevent some links from occurring. Wlog we assume in the next proposition that $\tau_{i,j} < \tau_{j,i}$ and $\tau_{i',j'} < \tau_{j',i'}$. We state that it is not possible that there exists a pair-wise collaborative link between i and j in g while the collaborative link between i' and j' has not been formed in g if

1. firms i' and j' have a flow degree which is higher than the flow degree of firms i and j;
2. and the expected process innovation value obtained by i', $\tau_{i',j'}$, and j', $\tau_{j',i'}$, from the collaborative link between i' and j' belongs to $(\tau_{i,j}, \tau_{j,i})$, *i.e.*, the expected process innovation values obtained by i' and j' are bounded by the minimal and maximal expected process innovation values associated with the collaborative link ij.

Proposition 10 *Assume that for all $k, k' \in N$, $\tau_{k,k'}/\tau_{k',k} \in (1/n, n)$. Let g be a pair-wise equilibrium network such that $ij \in L(g)$ and $i'j' \notin L(g)$. Then, $[\tau_{i',j'}, \tau_{j',i'}] \not\subset (\tau_{i,j}, \tau_{j,i})$ or $\min\{\mathcal{T}_i(g), \mathcal{T}_j(g)\} > \min\{\mathcal{T}_{i'}(g), \mathcal{T}_{j'}(g)\}$.*

Proof Recall that for all $k \in N$, $\mathcal{T}(g_{-k}) = \mathcal{T}(g) - \mathcal{T}_k(g)$. To introduce a contradiction, assume a pair-wise equilibrium network g where both conditions are not satisfied. Wlog let $\mathcal{T}_i(g) = \min\{\mathcal{T}_i(g), \mathcal{T}_j(g)\}$ and $\mathcal{T}_{i'}(g) = \min\{\mathcal{T}_{i'}(g), \mathcal{T}_{j'}(g)\}$. To sum up, we have $[\tau_{i',j'}, \tau_{j',i'}] \subset (\tau_{i,j}, \tau_{j,i})$ and $\mathcal{T}_{i'}(g) \geq \mathcal{T}_i(g)$. Since $[\tau_{i',j'}, \tau_{j',i'}] \subset (\tau_{i,j}, \tau_{j,i})$ we have $n\tau_{i,j} - \tau_{j,i} \leq n\tau_{i',j'} - \tau_{j',i'}$. Moreover, $\mathcal{B}'(\tau_{i',j'} + \tau_{i,j}) - \mathcal{C}'(\tau_{j',i'} + \tau_{j,i}) = \mathcal{C}'(n(\tau_{i',j'} + \tau_{i,j}) - (\tau_{j',i'} + \tau_{j,i})) > 0$ since $n(\tau_{i',j'} + \tau_{i,j}) \geq 2n\tau_{i,j} > 2\tau_{j,i} \geq \tau_{j',i'} + \tau_{j,i}$. The strict inequality comes from the fact that $\tau_{k,k'} > n\tau_{k',k}$ for all

$k, k' \in N$. It follows that $\mathcal{A}' + \mathcal{B}'(2\mathcal{T}_i(g) - \tau_{i,j}) - \mathcal{C}'\left(2\sum_{j \in N_{-i}} \mathcal{T}_j(g) - \tau_{j,i}\right) < \mathcal{A}' + \mathcal{B}'(2\mathcal{T}_{i'}(g) + \tau_{i',j'}) - \mathcal{C}'\left(2\sum_{j \in N_{-i'}} \mathcal{T}_j(g) + \tau_{j',i'}\right)$ since $\mathcal{T}_i(g) \leq \mathcal{T}_{i'}(g)$, and $\mathcal{B}'(\tau_{i',j'} + \tau_{i,j}) - \mathcal{C}'(\tau_{j',i'} + \tau_{j,i}) > 0$. We conclude that $\xi^+_{i'}(\{i', j'\}, g) > \xi^-_i(\{i, j\}, g) \geq f$. Consequently, firms i' and j' have an incentive to form the link $i'j'$, a contradiction. □

Note that in contrast to Proposition 2, in Proposition 10, when we deal with a collaborative link ij we have to take into account the fact that the positive impact of the link on the competitiveness of firm j can be a disincentive for firm i to form this link.

6 Discussion

In the theoretical literature on network formation, several papers examine heterogeneity (see for instance Galeotti, Goyal, and Kamphorst, 2005 ; Billand, Bravard and Sarangi, 2011, 2013). These papers highlight the fact that heterogeneity in cost of links plays a crucial role in the emergence of new equilibrium architectures. Hence, it is important to discuss the differences between our framework and a framework without uncertainty, that is $\tau_{i,j} = 1$ for all $i \in N$, $j \in N \setminus \{i\}$, but with cost heterogeneity. In the latter framework, each firm i incurs a cost $f_{i,j}$ when it forms a collaborative link with firm j. In our framework, in equilibrium there is no possibility of formation of a link between firms i and j when these firms obtain additional benefits from this link that are too different. More precisely, we have shown (see Proposition 9) that if $\tau_{i,j} > n\tau_{j,i}$, then firm j will not consent to form a link with firm i even if the costs of forming the link, f, is very low. In other words, while the cost of link formation acts like a participation constraint, the formation of the link ij will not depend on the benefits associated with this link but on *the difference of benefits* associated with this link. If this difference is too important in favor say of i, then the link ij will decrease the equilibrium quantity, and the equilibrium profit (gross of the cost of forming links) of firm j, and the latter firm will never have an incentive to form this link. By contrast, in a framework where heterogeneity concerns the costs associated with the formation of the link ij, the cost margin or the difference between $f_{i,j}$ and $f_{j,i}$ does not play any role in the formation of ij. Indeed, in that case, since $\tau_{i,j} = \tau_{j,i} = 1$, the formation of the link ij will always increase the equilibrium quantity, and the equilibrium profit (gross of the cost of forming links) of both firms i and j. Moreover, this increase for firm i for instance will not depend on the identity of j; it will depend on the number of links formed by i and the total number of links formed by other firms. Therefore, firms i and j will act in the usual way and compare the additional benefits associated with the additional link with the cost of a link. Firm i will not take into account relative costs or consider the cost firm j will incur when it forms the link ij.

To sum up, our framework allows us to account for the fact that firm i chooses specific firms for creating collaborative links. Indeed, firm i knows that if the collaborative link ij succeeds, then it will not obtain the same benefits from this successful link depending on the choice of its partner j. If j is able to improve its competitiveness too much relative to the improvement obtained by i because of link ij, then i will not consent to form the collaborative link ij. A framework with only cost heterogeneity and no uncertainty cannot account for such important and realistic considerations.

Appendix

Meaning of (P2). Property (P2) states that the strict convexity of φ in its first argument has to be sufficiently high to compensate for differences between the probabilities of success of the collaborative links that exist in g. Indeed, we have:

$$\zeta_i^-(\{ij_{k+1}\}, g^{ij_k}) = \rho_{ij_{k+1}} \frac{\varphi(\mathcal{U}_i(g^{ij_k}), \mathcal{U}(g_{-i}^{ij_k})) - \varphi(\mathcal{U}_i(g^{ij_{k+1}}), \mathcal{U}(g_{-i}^{ij_k}))}{\rho_{ij_{k+1}}}$$

Due to the strict convexity of φ in its first argument, $\Phi_i(g^{ij_k}) = [\varphi(\mathcal{U}_i(g^{ij_k}), \mathcal{U}(g_{-i}^{ij_k})) - \varphi(\mathcal{U}_i(g^{ij_{k+1}}), \mathcal{U}(g_{-i}^{ij_k}))]/\rho_{ij_{k+1}}$ is strictly increasing. It follows that $\Phi_i(g^{ij_k}) > \Phi_i(g^{ij_{k+1}})$. Moreover, by construction, $\rho_{i,j_k} < \rho_{i,j_{k+1}}$ for all $k \in \{1, \ldots, m\}$. To sum up, (P2) means that the difference between $\Phi_i(g^{ij_k})$ and $\Phi_i(g^{ij_{k+1}})$ compensates the difference between ρ_{i,j_k} and $\rho_{i,j_{k+1}}$. Obviously, when $\rho_{i,j} = \rho$ for all $i, j \in N$, (P2) is satisfied.

Additional explanations for Example 1. We set $H = \{12, 13, 45\}$. First, it is clear that only the links formed in g or g' can be formed in a pair-wise equilibrium network. Second, we establish that no link has to be removed in g: we have $\varphi(1.8, 0.9) - \varphi(0, 0.9) - f > \varphi(0.9, 1.8) - \varphi(0, 1.8) - f \simeq 2.8 \times 10^{-6}$. Moreover, firms 2 and 3 have no incentive to form a link together in g: $\varphi(0.9 + \rho', 1.8) - \varphi(0.9, 1.8) - f = -2.31 \times 10^{-6}$. We now establish that firms 2 and 3 have an incentive to form a link in g when each link in H has a probability equal to $\kappa\rho$: $\varphi(\kappa\rho + \rho', 2\kappa\rho) - \varphi(\kappa\rho, 2\kappa\rho) - f \simeq 1.5 \times 10^{-5}$. Finally, given that the link 23 is formed, we establish that firm 4 has an incentive to remove the link 45: $\varphi(\kappa\rho, 2\kappa\rho + \rho') - \varphi(0, 2\kappa\rho + \rho') - f \simeq -0.31$.

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