ON GROUP STRATEGYPROOF OPTIMAL MECHANISMS

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ABSTRACT. We consider a two agent, single indivisible good allocation problem. We focus on reasonable mechanisms that are continuous and satisfy agent sovereignty. In particular, we study optimal group strategyproof mechanisms. We provide an explicit characterization of the strategyproof mechanisms and show that there are non-affine maximizer mechanisms that do not belong to the class characterized by Roberts [15]. Further, there are no budget balanced or strong group strategyproof mechanisms in this class. Accordingly, we completely characterize the class of feasible strategyproof mechanisms satisfying a mild individual rationality axiom. We also provide a class of weak group strategyproof mechanisms. Finally, we obtain a strong negative result with respect to existence of optimal mechanisms that maximize expected transfers.

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1. INTRODUCTION

We consider the standard good allocation model where a single indivisible good is allotted to a set of agents who have a private independent non-negative valuation for the good and quasi-linear preferences over the good and money. Such a model has several practical applications where the good could be a license, a house, a plot of land or an airport landing right. We focus on reasonable mechanisms that are continuous and satisfy agent sovereignty. Our paper considers the simplest two agent case, and finds remarkably sharp results.

In particular, we provide an explicit complete characterization of the strategyproof mechanisms. However, unlike Roberts [15], this class contains non-affine maximizer mechanisms. This is clearly because in the present setting there are only two alternatives and the domain of valuations is restricted. Unlike most contemporary papers, in line with Jackson [6], we use budget balance as a yardstick for efficiency and look for strategyproof and budget balanced mechanisms. However, we find no mechanisms that are budget balanced as well as strategyproof. Hence, we completely characterize the class of feasible strategyproof mechanisms that satisfy a mild individual rationality condition. We then, look for stronger notions of non-manipulability, in terms of *strong* and *weak* (pairwise) group strategyproofness, that eliminate group level incentives to misreport. We find that there are no strong group strategyproof mechanisms. Instead, we provide a class of mechanisms that are weak group strategyproof. To the best of our knowledge, there are no papers that characterize group strategyproof mechanisms in the present setting.

This is a preliminary draft.

Finally, we look for optimal mechanisms (in terms of maximizing expected transfers for the good), in the class of feasible strategyproof mechanism that satisfy the aforementioned mild individuality condition. We fins a strong negative result. That is, there does not exist an optimal mechanism in this class, for any continuous (and possibly non-identical) distribution of valuations over any sub-domain of the positive orthant of the real number line.

The papers that are closest to the present one are Marchant and Mishra [7] and Mishra and Quadir [8]. The former paper investigates strategyproof allocation rules in a two alternative framework with quasi-linear utilities. Their results are applicable to the present paper when the number of agents is two. However, our characterization is independent of theirs. The latter paper provides characterization of strategyproof and non-bossy allocation rules in a similar setting where objects may remain unallocated and agents have strictly positive valuations. In comparison to these papers, our results provide clear functional properties of strategyproof mechanisms.

2. MODEL

Consider a 2 agent model with set of agents $N = \{1, 2\}$ and an indivisible good. Each agent i has an independent private valuation $v_i \geq 0$ for the good. A mechanism μ is a tuple (d, τ) such that at any reported profile of valuations $v \in \mathbb{R}^N_+$, each agent i is allocated a transfer $\tau_i(v) \in \mathbb{R}$ and a decision $d_i(v) \in \{0, 1\}$. $d_i(v) = 1$ implies that agent i gets a good, while $d_i(v) = 0$ stands for i not getting the good. We assume that $\sum_{i \in N} d_i(v) = 1$ for all $v \in \mathbb{R}^N_+$. Define $w(v)$ to be the agent getting the good at any profile v.[1] The utility to agent i with a true valuation of v_i at any reported profile $v' \in \mathbb{R}^N_+$, from the mechanism μ is given by $u(d_i(v'), \tau_i(v'); v_i) = v_i d_i(v') + \tau_i(v')$. Let $\forall\, i \in N$, $\forall\, S \subseteq N$, $\forall\, v \in \mathbb{R}^N_+$, $v_{-i} := (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$, $v_{-S} := (v_i)_{i \in N \setminus S}$ and $v_S := (v_i)_{i \in S}$.

Note that, a priori, a mechanism may have a peculiar allocation decision rule that gives the good to some agent j, whenever she reports some value v_j, irrespective of what the other agent $i \neq j$ bids. It could also be that the good is given to i, whenever j reports v_j, irrespective of what i bids. In other words, the mechanism treats some agent $i \in N$ as a dictator, whenever j reports v_j. In this paper, we exclude such arbitrary mechanisms from our purview of study. Instead, we focus on mechanisms that satisfy *agent sovereignty* in the following manner: every agent i can change the allocation decision by unilaterally changing her report, if the other agent j reports a positive value. That is, every agent can exert some influence on the mechanism allocation decision, irrespective of what other agents are bidding.[2]

Definition 1. A mechanism $\mu = (d, \tau)$ satisfies *agent sovereignty* if for all $i \neq j \in N$ and all $v \in \mathbb{R}^N_+$,

$$v_j > 0 \implies \exists\, v'_i \geq 0 \text{ such that } d(v) \neq d(v'_i, v_j)$$

Further, we impose a mild technical restriction of continuity, on the mechanisms we study. It requires that across any convergent sequence of profiles, if the good allocation decision remains unchanged, then *either* the same decision holds at the limit profile *or* the

[1] We often refer to this agent $w(v)$ as the winner at profile v in the text.

[2] This axiom has been used in similar settings by Marchant and Mishra [7] and Moulin and Shenker [10].

transfer assigned to the winner i (that is, the agent getting the good in each of the profiles of the sequence) and some other agent j, at the limit profile, is such that both are indifferent between getting the good and not getting the good.

Definition 2. A mechanism (d, τ) is *continuous* if for all $i \in N$ and all $\{v^n\} \subseteq \mathbb{R}_+^N$ such that $\{v^n\} \to \bar{v}$ and $d_i(v^n) = 1$ for all n,

$$d_i(\bar{v}) = 1 \implies \forall\, k \in \{i, w(\bar{v})\}, u(1, \tau_k(\bar{v}); \bar{v}_k) \neq u(0, \tau_k(\bar{v}); \bar{v}_k)$$

Let Γ be the class of mechanisms that satisfy non-dictatorship at all profiles and continuity. In this paper, we focus our attention on the mechanisms in Γ.

We start by defining the popular strategic axiom of strategyproofness, which eliminates any incentive to misreport on an individual level. It is defined as follows.

Definition 3. A mechanism $\mu = (d, \tau)$ satisfies *strategyproofness* (SP) if $\forall\, i \in N$, $\forall\, v_i, v_i' \in \mathbb{R}_+$, $\forall\, v_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$,

$$u(d_i(v_i, v_{-i}), \tau_i(v_i, v_{-i}); v_i) \geq u(d_i(v_i', v_{-i}), \tau_i(v_i', v_{-i}); v_i)$$

A strategyproof mechanism guarantees that revealing the true valuation is a weakly dominant strategy for each agent. But there remains the possibility of agents forming groups and misreporting together. Ideally a mechanism should also be immune to such group misreporting. Hence, we study a stronger version of strategyproofness.[3] First, we introduce the following notation. For any $v, v' \in \mathbb{R}_+^N$; v' is an S-profile of v if $\forall\, i \notin S$, $v_i = v_i'$, for any non-empty $S \subseteq N$.

Definition 4. A mechanism $\mu = (d, \tau)$ satisfies *strong pair-wise group strategyproofness* (SPGS) if $\forall\, v \in \mathbb{R}_+^N$, $\nexists\, S \subseteq N$ such that $|S| \leq 2$ and

$$u(d_i(v), \tau_i(v); v_i) \leq u(d_i(v'), \tau_i(v'); v_i), \forall\, i \in S$$
$$\text{and } u(d_j(v), \tau_j(v); v_j) < u(d_j(v'), \tau_j(v'); v_j) \text{ for some } j \in S$$

where v' is an S-profile of v.

Definition 5. A mechanism $\mu = (d, \tau)$ satisfies *weak pair-wise group strategyproofness* (WPGS) if $\forall\, v \in \mathbb{R}_+^N$, $\nexists\, S \subseteq N$ such that $|S| \leq 2$ and

$$u(d_i(v), \tau_i(v); v_i) < u(d_i(v'), \tau_i(v'); v_i), \forall\, i \in S$$

where v' is an S-profile of v.

Therefore, SPGS requires that any misreporting pair of agents either have no member strictly better off, or have at least one member strictly worse off. WPGS requires that any misreporting pair of agents have at least one member who is not strictly better off. It can easily be seen that SPGS implies WPGS, which in turn implies strategyproofness.

The following definition qualifies the participation of the agents in the mechanism. Specifically, it requires that no agent with zero valuation for the good, should get a negative utility by participating in the mechanism. Since this is a weaker version of the popular individual rationality axiom, we call it minimal individual rationality.

[3]This stronger notion of strategyproofness has also been studied by Bogomolnaia and Moulin [4], Barbera, Berga and Moreno [2], Hatsumi and Serizawa [5], Mitra and Mutuswami [9], Barbera and Jackson [3], Serizawa [16].

Definition 6. A mechanism (d, τ) satisfies *minimal individual rationality* (MIR) if for all $i \in N$ and all $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$,

$$u(d_i(0, v_{-i}), \tau_i(0, v_{-i}); 0) \geq 0$$

Finally, no mechanism should entail a wastage of resources. The following two definitions embody this idea. The first describes budget balanced mechanisms where the sum of transfers is required to be zero. The latter is a weaker condition where the sum of transfers need only be non-positive.

Definition 7. A mechanism $\mu = (d, \tau)$ satisfies *budget balance* (BB) if for all $v \in \mathbb{R}_+^N$,

$$\sum_{i \in N} \tau_i(v) = 0$$

Definition 8. A mechanism $\mu = (d, \tau)$ satisfies *feasibility* (F) if for all $v \in \mathbb{R}_+^N$,

$$\sum_{i \in N} \tau_i(v) \leq 0$$

3. Results

We start by stating the following well-known characterization of strategyproof mechanisms.

Result 1. Any mechanism $\mu = (d, \tau)$ satisfies SP if and only if $\forall\, i \in N$ and $\forall\, v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$, there exist real valued functions $K_i^\mu : \mathbb{R}_+^{N \setminus \{i\}} \mapsto \mathbb{R}$ and $T_i^\mu : \mathbb{R}^{N \setminus \{i\}} \mapsto \mathbb{R}$ such that

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > T_i^\mu(v_{-i}) \\ 0 & \text{if } v_i < T_i^\mu(v_{-i}) \end{cases} \quad \text{and} \quad \tau_i(v) = \begin{cases} K_i^\mu(v_{-i}) - T_i^\mu(v_{-i}) & \text{if } d_i(v) = 1 \\ K_i^\mu(v_{-i}) & \text{if } d_i(v) = 0 \end{cases}$$

Proof: The results follow from Proposition 9.27 in Nisan [14] and Lemma 1 in Mukherjee [12]. $\square$

Note that this result allows for arbitrary tie-breaking in allocation decision of the good at any profile $v \in \mathbb{R}_+^N$; such that there exist $i \neq j \in N$ with $v_i = T_i^\mu(v_{-i})$, $v_j = T_j^\mu(v_{-j})$ and $v_k \leq T_k^\mu(v_{-k})$ for all $k \in N \setminus \{i, j\}$. In this paper, without loss of generality, we assume a lexicographic tie-breaking rule (as in Sprumont [17]) where the linear order $1 \succ 2 \succ \ldots \succ n$ is used to break ties among the agents. That is, for any profile v, if $v_k \leq T_k^\mu(v_{-k})$ for all $k \in N$, then for all $i \in N$,

$$d_i(v) = 1 \iff i \succ j \text{ for all } j \in N \text{ such that } v_j = T_j^\mu(v_{-j})$$

The following proposition establishes a two particular properties of $T_i^\mu(.)$ functions for any continuous and strategyproof mechanism $\mu \in \Gamma$.

Proposition 1. If a mechanism $\mu = (d, \tau) \in \Gamma$ satisfies SP, then

(1) For all $i \in N$, $T_i^\mu(.)$ is a non-decreasing continuous function.
(2) For all $x, y \geq 0$, $x = T_1^\mu(y) \iff y = T_2^\mu(x)$

Proof: To prove (1), fix y, y' such that $0 \leq y < y'$. If $T_1^\mu(y') < T_1^\mu(y)$, then for any $x \in (T_1^\mu(y'), T_1^\mu(y))$ consider the profiles (x, y) and (x, y'). By Result 1, $d_2(x, y') = 0$ and $d_2(x, y) = 1$, which contradicts Result 1 itself. Arguing similarly for agent 2, we get

that $T_i^\mu(.)$ is a non-decreasing function for both $i \in N$. Therefore, these functions must either be continuous or have jump discontinuities. W. l. o. g. consider the function $T_1^\mu(.)$ and suppose that there exists a $y \geq 0$ such that $T_1^\mu(y) < \lim_{z\to y+} T_1^\mu(y)$. Choose an $x \in \left(T_1^\mu(y), \lim_{z\to y+} T_1^\mu(y)\right)$ and consider the sequence of profiles $\{(x, y^r)\}$ such that for all r, $y^r > y$ and $\{y^r\} \to y$. By Result 1, $d(x, y^r) = (0, 1)$ for all r, but $d(x, y) = (1, 0)$. Since, $\mu \in \Gamma$ and hence, continuous, we have $x = T_1^\mu(y)$. This contradicts our choice of x and so, (1) follows.
To prove (2), fix any $x, y \geq 0$. There are two possibilities: (i) $d_1(x, y) = 1$ or (ii) $d_2(x, y) = 1$. If case (i) holds, then by Result 1, $x \geq T_1^\mu(y)$ and $y \leq T_2^\mu(x)$. If $x > T_1^\mu(y)$ and $y = T_2^\mu(x)$, then choose $\nu > 0$ such that $x > T_1^\mu(y+\nu)$ (by (1) above, such a ν exists). By Result 1, $d_1(x, y+\nu) = d_2(x, y+\nu) = 1$ and hence, a contradiction. Similarly, if $x = T_1^\mu(y)$ and $y < T_2^\mu(x)$, then choose $\nu > 0$ such that $y < T_2^\mu(x-\nu)$. As before, Result 1 implies that $d_1(x-\nu, y) = d_2(x-\nu, y) = 0$ and hence, a contradiction. Arguing in similar manner, we can establish a contradiction in case (ii), and so, the result follows. □

Proposition 2. Any mechanism $\mu = (d, \tau) \in \Gamma$ satisfies SP if and only if there exist functions, $K_i^\mu : \mathbb{R}_+^{N\setminus\{i\}} \mapsto \mathbb{R}$ and $T_i^\mu : \mathbb{R}_+^{N\setminus\{i\}} \mapsto \mathbb{R}$, such that

(1) For all $i \in N$, if $d_i(v) = 1$, then $i \in \text{argmax}_{j\in N}(v_j - T_j^\mu(v_{-j}))$.
(2) For all $i \in N$, $\tau_i(v) = \begin{cases} K_i^\mu(v_{-i}) - T_i^\mu(v_{-i}) & \text{if } d_i(v) = 1 \\ K_i^\mu(v_{-i}) & \text{if } d_i(v) = 0 \end{cases}$
(3) For all $x \geq 0$, $T_1^\mu(T_2^\mu(x)) = T_2^\mu(T_1^\mu(x)) = x$.
(4) For all $i \in N$, T_i^μ is a strictly increasing continuous function.
(5) For all $i \in N$, $T_i^\mu(0) = 0$.

Proof:
Proof of Only If: (2) and (3) follow directly from Result 1 and Proposition 1, respectively. Further, Result 1 and (3) imply that $T_i^\mu(.)$ must be strictly monotonic and so, (4) follows from Proposition 1. Further, Result 1 and (4) imply (1). Finally, if $T_1^\mu(0) > 0$, then by (3), $0 > T_2^\mu(0)$ and so, by (4), there exists $\nu > 0$ such that $0 > T_2^\mu(\nu)$. By Result 1, this implies that $d_2(\nu, y) = 1$ for all $y \geq 0$, and hence, a contradiction to the fact that $\mu \in \Gamma$. Arguing similarly for $T_2^\mu(.)$, (5) follows. □

Proof of If: By Result 1, (1)-(5) imply that μ is strategyproof and satisfies agent sovereignty. To show continuity, consider without loss of generality, a sequence of profiles $\{v^r\}$ converging to $\bar{v}$ such that $d(v^r) = (1, 0)$ for all r. Therefore, by Result 1, $v_1^r \geq T_1^\mu(v_2^r)$ and $v_2^r \leq T_2^\mu(v_1^r)$ for all r. By (4), $\bar{v}_1 \geq T_1^\mu(\bar{v}_2)$ and $\bar{v}_2 \leq T_2^\mu(\bar{v}_1)$ and so, if $d(\bar{v}) = (0, 1)$ then $\bar{v}_1 = T_1^\mu(\bar{v}_2)$ and $\bar{v}_2 = T_2^\mu(\bar{v}_1)$. This implies that for both $j \in N$, $u(0, \tau_j(\bar{v}); \bar{v}_j) = u(1, \tau_j(\bar{v}); \bar{v}_j)$. Hence, continuity of μ follows and so $\mu \in \Gamma$. □

Remark 1. Note that a special class of strategyproof mechanisms is one, where $T_i^\mu(x) = x$ for all $x \geq 0$ and all $i \in N$. This is the the popular class of VCG mechanisms that have an efficient (welfare maximizing) decision rule.[4] However, the class of mechanisms characterized in Proposition 2, allows for a mechanism μ such that $T_1^\mu(x) = x^2$ and $T_2^\mu(x) = +\sqrt{x}$.

[4] A mechanism $\mu = (d, \tau)$ is a VCG mechanism if $\forall\, v \in \mathbb{R}_+^N$, $\forall\, i \in N$,
$$d_i(v) = 1 \implies v_i \geq v_j$$

Note that this mechanism is an example of non-affine maximizer mechanism. This reinforces the popular concept that Robert's theorem (Roberts [15]) does not hold in the present restricted domain setting.

The following corollary states that there does not exist any reasonable mechanism in the present setting that satisfies SPGS.

Corollary 1. There is no continuous mechanism $\mu = (d, \tau) \in \Gamma$ that satisfies SPGS.

Proof: The result trivially follows from Result 1 and (4) in Proposition 2. □

This impossibility with respect to SPGS, was also noted by Mitra and Mutuswami [9] and Mukherjee [12],[11]. However, in all these papers, authors were looking for mechanisms satisfying SPGS, that also satisfy some notion of decision efficiency. In comparison, Corollary 1 establishes non-existence of *any* reasonable mechanism satisfying SPGS, decision efficient or otherwise. The following proposition provides a class of mechanisms that satisfy WPGS.

Proposition 3. A mechanism $\mu = (d, \tau) \in \Gamma$ satisfies WPGS if for all $i \in N$ such that for all $x \geq 0$,

$$K_i^{\mu}(x) = C_i + \min\{T_i^{\mu}(x), \eta\}$$

where $C_i \in \mathbb{R}$ for all $i \in N$ and either $\eta = \infty$ or $\eta \in \{x \geq 0 | T_i^{\mu}(x) = x, \forall\, i \in N\}$.

Proof: There can be only two types of deviations by the pair $\{1, 2\}$: (i) decision *preserving* deviations where the allocation decision remains same before and after deviation and (ii) decision *changing* deviations where the allocation decision changes after deviation. Note that for all $i \in N$ and all $z \geq 0$, if $\eta = 0$ (which is possible because Proposition 2 states that $T_i^{\mu}(0) = 0$ for all $i \in N$), then $K_i^{\mu}(z) = C_i$; and if $\eta = \infty$ then $K_i^{\mu}(z) = C_i + T_i^{\mu}(z)$. In each of these cases, it is easy to see that no $\{1, 2\}$-deviation, whether decision preserving or not, can violate WPGS. Hence, we focus on a finite $\eta \in (0, \infty)$ such that $T_i^{\mu}(\eta) = \eta$ for all i, and show the sufficiency with respect to each possible kind of deviation as a separate case.

Case(i): Suppose there exists a decision preserving deviation from (x, y) to (x', y') that violates WPGS. If $d(x, y) = d(x', y') = (1, 0)$, then $0 \leq y' \leq y \leq T_2^{\mu}(x) < \eta < T_1^{\mu}(y) \leq x \leq x'$. However, by Proposition 2 and the fact that $T_i^{\mu}(\eta) = \eta$ for all $i \in N$; it follows from $T_2^{\mu}(x) < \eta < T_1^{\mu}(y)$ that $x < \eta < y$ and hence, contradiction. Further, if $d(x, y) = d(x', y') = (0, 1)$, then $0 \leq x' \leq x < T_1^{\mu}(y) < \eta < T_2^{\mu}(x) \leq y \leq y'$. As before, by Proposition 2 and the fact that $T_i^{\mu}(\eta) = \eta$ for all $i \in N$; it follows from $T_1^{\mu}(y) < \eta < T_2^{\mu}(x)$ that $y < \eta < x$, and hence, a contradiction again. Therefore, there can be no decision preserving $\{1, 2\}$-deviation that can violates WPGS.

Case(ii): Suppose there exists a decision changing deviation from (x, y) to (x', y') that violates WPGS. Without loss of generality, suppose that $d(x, y) = (1, 0)$ and $d(x', y') = 1$. Define $\Delta_1 := u_1(d_1(x, y), \tau_1(x, y); x) - u_1(d_1(x', y'), \tau_1(x', y'); x)$ and $\Delta_2 := u_2(d_2(x, y), \tau_2(x, y); y) -$

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and

$$\tau_i(b) = \sum_{j \neq i} (d_j(v) - d_j(v_{-i}))v_j + h_i(v_{-i}) \text{ where } h_i : \mathbb{R}_+^{N \setminus \{i\}} \mapsto \mathbb{R} \text{ is an arbitrary function of } v_{-i}.$$

$u_2(d_2(x', y'), \tau_2(x', y'); y)$. By supposition, $\Delta_i < 0$ for all $i \in N$. However, if $\Delta_1 = x + K_1^\mu(y) - T_1^\mu(y) - K_1^\mu(y') < 0$, then by Result 1, $y' > y$, $T_1^\mu(y) < \eta$ and $x < \eta$. By Proposition 2 and the fact that $T_2^\mu(\eta) = \eta$, $T_2^\mu(x) < \eta$ and so, $K_2^\mu(x) = T_2^\mu(x) + C_2$. Now, if $x' \leq \eta$, then by Proposition 2 and the fact that $T_2^\mu(\eta) = \eta$, $K_2^\mu(x') = T_2^\mu(x') + C_2$. So, by Result 1, $\Delta_2 = K_2^\mu(x) - \{y + K_2^\mu(x') - T_2^\mu(x')\} \geq 0$, which implies a contradiction to our supposition. On the contrary, if $x' > \eta$, then arguing as before, $T_2^\mu(x') > \eta$ and so $K_2^\mu(x') = \eta + C_2$. Therefore, by Result 1, $\Delta_2 = (T_2^\mu(x) - y) + (T_2^\mu(x') - \eta) \geq 0$ and so, a contradiction to our supposition. Thus, there can be no decision changing $\{1,2\}$-deviation that can violates WPGS. □

Now, we study the efficient mechanisms in the class characterized by Proposition 2. However, contrary to contemporary literature, we use the notion of budget balance instead of decision efficiency. Our motivation for doing so is on the lines of Jackson [6], where he argues that in absence of Pareto efficiency, budget balance should be treated as an equally important yardstick of efficient mechanisms, as decision efficiency. In fact, there is no paper in our knowledge that looks at budget balanced and strategyproof mechanisms in the present setting.[5] As the following corollary shows, there are no reasonable mechanisms that are budget balanced and strategyproof.

Corollary 2. There is no mechanism $\mu = (d, \tau) \in \Gamma$ that satisfies SP and BB.

Proof: Suppose there exists a mechanism μ that is budget balanced and strategyproof. Therefore, from Result 1, it follows that $\forall\ (x, y) \geq 0$,

$$\begin{array}{ll} K_1^\mu(y) + K_2^\mu(x) = T_1^\mu(y) & \text{if } d(x,y) = (1,0) \\ K_1^\mu(y) + K_2^\mu(x) = T_2^\mu(x) & \text{if } d(x,y) = (0,1) \end{array}$$

By Proposition 2, $T_i^\mu(0) = 0$ for all $i \in N$. Therefore, by Result 1, $d(z, 0) = (1,0)$ and $d(0, z) = (0,1)$, for all $z > 0$. Hence, the equations above imply that for all $z > 0$, $K_i^\mu(z)$ is constant for all i. Therefore, $T_i^\mu(z)$ is constant for all $z > 0$ and all i, and hence, a contradiction to Proposition 2. Thus, the result follows. □

Since Corollary 2 shows that there are no reasonable mechanisms that are strategyproof and budget balanced, in the following proposition, we look at feasible strategyproof mechanisms that are fair in the sense that; any agent with zero valuation must not get a negative utility by participating in the mechanism. We first define a class of *pairs* of functions:

$$\mathcal{F} := \left\{ (f, g) \middle| \begin{array}{l} f(.), g(.) \textit{ are strictly increasing continuous bijections over domain } [0, \infty) \textit{ such} \\ \textit{that } f(0) = g(0) = 0 \textit{ and } f(g(x)) = g(f(x)) = x \textit{ for all } x \geq 0. \end{array} \right\}$$

Proposition 4. A $\mu = (d, \tau) \in \Gamma$ satisfies SP, MIR and F, if and only if for all $i \in N$ and all $v \in \mathbb{R}_+^N$,

$$d_i(v) = 1 \implies i \in \text{argmax}_{j \in N}(v_j - T_j^\mu(v_{-j}))$$

and

$$\tau_i(v) = \begin{cases} -T_i^\mu(v_{-i}) & \text{if } d_i(v) = 1 \\ 0 & \text{if } d_i(v) = 0 \end{cases}$$

where $\left(T_1^\mu(.), T_2^\mu(.)\right) \in \mathcal{F}$.

[5] In comparison, there are several papers like Ashlagi and Serizawa [1] and Mukherjee [12], that characterize the strategyproof and decision efficient mechanisms in the present setting.

Proof: The proof of sufficiency is easy to check. We prove necessity by showing $K_i^\mu(z) = 0$ for all $z \geq 0$ in the following manner. We begin by noting that Result 1 implies that $\big(T_1^\mu(.), T_2^\mu(.)\big) \in \mathcal{F}$. Further, it follows from Result 1 and MIR that $K_i^\mu(z) \geq 0$ for all $i \in N$ and all $z \geq 0$. Therefore, by Proposition 2, feasibility at profile $(0,0)$ implies that: $0 \leq K_1^\mu(0) + K_2^\mu(0) \leq 0$. Thus, $K_i^\mu(0) = 0$ for all $i \in N$. Now, consider a profile $(x, T_2^\mu(y))$ where $x > y \geq 0$. By Result 1, $d(x, T_2^\mu(y)) = (1,0)$. Therefore, by feasibility, $0 \leq K_1^\mu(T_2^\mu(y)) - T_1^\mu(T_2^\mu(y)) + K_2^\mu(x) \leq 0$. Further, by Proposition 2, $0 \leq K_1^\mu(T_2^\mu(y)) + K_2^\mu(x) \leq y$. Note that this equation holds for all $y \in [0, x)$ and so, by Proposition 2, $\lim_{T_2^\mu(y)\to 0} K_1^\mu(T_2^\mu(y)) + K_2^\mu(x) = 0$. Recall that for all $z \geq 0$ and all $i \in N$, $K_i^\mu(z) \geq 0$. Hence, $\lim_{T_2^\mu(y)\to 0} K_1^\mu(T_2^\mu(y)) \geq 0$ and so, $K_2^\mu(x) = 0$. Arguing similarly, for the profile $(T_1^\mu(y), x)$, we get that $K_1^\mu(x) = 0$. Thus, the result follows. □

Corollary 3. If $\mu = (d, \tau) \in \Gamma$ satisfies SP, MIR and F, then it satisfies WPGS.

Proof: The proof easily follows from Proposition 3. □

The following proposition studies optimal (expected revenue maximizing) strategyproof mechanisms satisfying MIR in the class Γ, under the distributional assumption that: valuation of each agent i, v_i is distributed according to a differentiable distribution function $F_i(.)$ over the interval $[0, \infty)$. Note that we allow for stochastically dependent distributions. The proposition shows there are no optimal strategyproof mechanisms satisfying MIR in Γ.[6]

Proposition 5. In the class of mechanisms $\mu = (d, \tau) \in \Gamma$ satisfying SP and MIR, there exists no mechanism that is optimal.

Proof: Fix any $y > 0$ and define $\theta_y := T_1^\mu(y)$. Note that by MIR, $K_i^\mu(y) \geq 0$ for all i. Further, it is obvious from Result 1 that the exact functional form of $K_i^\mu(.)$ functions does not affect the strategyproofness of mechanisms. Therefore, an optimal mechanism must set $K_i^\mu(y) = 0$ for all i. Also, Result 1 implies that $\theta_y := inf\{x \geq 0 | d(x,y) = (1,0)\}$. Therefore, by Proposition 1, if agent 2 bids x then seller's expected revenue, for a particular value of θ_y, is $\theta_y(1 - F_1(\theta_y)) + \int_0^{\theta_y} T_2^\mu(y) f_1(x)dx$. Hence, seller's objective is to maximize this expression by choosing θ_y. The first order necessary condition for maximum implies that

$$\theta_y = T_2^\mu(y) + \frac{1 - F_1(\theta_y)}{f_1(\theta_y)}$$

and so, an optimal mechanism should it exist, must have for all $y \geq 0$,

$$T_1^\mu(y) = T_2^\mu(y) + \frac{1 - F_1(T_1^\mu(y))}{f_1(T_1^\mu(y))}$$

However, Proposition 2 requires that $T_1^\mu(0) = 0 = T_2^\mu(0)$ and so, it must be that $\frac{1-F_1(0)}{f_1(0)} = 0$, which is impossible[7] and hence, we have a contradiction. □

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[6] It can be shown that non-existence result continues to hold if the lower bound of the support of distribution, is positive.

[7] This is impossible because $f_1(0) < \infty$ and $F_1(0) < 1$.

Remark 2. Unlike Myerson [13], we look at optimal mechanisms that are truthful in the sense of dominant strategy incentive compatibility. Proposition 5 shows that the candidates for optimal strategyproof mechanisms satisfying MIR (which is a weaker version of the individual rationality axiom used in Myerson [13]) must be those not belonging to Γ. That is, such mechanisms must either be discontinuous or allow the good to be allocated dictatorially at some profile of reported valuations. Note that this impossibility may also be a result of the restriction implicit in present setting, which requires that the good must be allocated at all profiles. This rules out usage of reserve prices that were shown by Myerson [13] to be essential components of the optimal strategyproof and individually rational mechanism. Are there any optimal strategyproof mechanisms satisfying some form of individual rationality, if goods are not allocated at all profiles? This appears to be an interesting area of future research.

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