Maximal Possibility and Minimal Dictatorial Covers of Domains *

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Abstract

In line with the works of Serizawa (1995), Barberà et al. (1999) and Ching and Serizawa (2001), we introduce and characterize the notion of *maximal possibility cover* of different versions of single peaked domain of preferences - a maximal domain of preferences which includes the single peaked domain of preferences while ensuring the existence of a strategy-proof and unanimous social choice function. Further we characterize an allied dictatorial domain, which we call the *minimal dictatorial cover* of different versions of single peaked domain of preferences - a *minimal* set of preferences which is dictatorial and contains the single peaked domain of preferences. Lastly, we prove that a generalized version of the *circular* domain (Sato (2010)) - which we call *generalized circular* domain - to be a dictatorial domain.

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1 Introduction

In the wake of the celebrated Gibbard-Satterthwaite (Gibbard (1973),Satterthwaite (1975)) impossibility result, researchers are persistently interested in ways of evading such an impossibility result. It is well known that restricting the domain of admissible preferences leads to both impossibility (Kim and Roush (1980), Aswal et al. (2003), Sato (2010), Pramanik (2014) etc.) and possibility (Moulin (1980)) results. Thus there is a trade-off between the degree of manipulability admissible in the society and the existence of unanimous, strategy-proof and non-dictatorial rules. In this paper, we take an extreme view of this trade-off by looking at the maximal domain of preferences that guarantee the existence of rules that are unanimous, strategy-proof and non-dictatorial.

In this paper, we introduce the notion of a *maximal possibility cover* (MPC) of a domain $\mathcal{D}$ of preferences as the maximal domain of preferences which includes the domain $\mathcal{D}$ while ensuring the existence of a strategy-proof and unanimous social choice function. We characterize the MPCs of an immensely popular possibility domain - the domain of *single peaked* preferences where the generalized median rules (Moulin (1980), Weymark (2011)) are the only rules that are unanimous and strategy-proof. Further we introduce the notion of the *minimal dictatorial cover* of a domain $\mathcal{D}$ of preferences as the *minimal* set of preferences which is dictatorial and contains the domain $\mathcal{D}$ and provide a characterization of the MDCs of the domain of single peaked preferences. We also characterize the MPCs and the MDCs of the domain of *minimally rich* (**?**) single peaked preferences.

It is essential to contrast our work with related literature. An early seminal work of Serizawa (1995) looks at a related problem of finding the maximal domain of preferences that makes *voting by committees* strategy-proof. In a later work, Barberà et al. (1999) characterizes the maximal domain of preferences preserving strategy-proofness of generalized median rules in multidimensional social choice setting. In a more recent work, Ching and Serizawa (2001) prove that a *weakly single peaked* domain is the unique maximal domain of preferences that ensure the existence of strategy-proof, anonymous and unanimous social choice functions. Our work can be thought of as the next step in this direction as we drop the assumption of anonymity of the social choice rule when characterizing such maximal domains.

On a different note, we provide an interesting generalization of the dictatorship result in Sato (2010). We introduce a new restricted domain of preferences - which we call the *generalized circu-*

lar domain and prove that the generalized circular domain is a dictatorial domain. The generalized circular domain, except for two preferences in the *boundary* domain, relaxes the requirement of a circular domain that there must be two preferences with the same alternative at the top and places the alternatives adjacent to its top alternative either at the second rank or the last rank.

The rest of the paper is organized as follows. We describe the usual social choice framework in the Section 2. In Section 3, we introduce the notion of a *maximal possibility cover* of a domain and characterize the MPCs of the single peaked domain. In Section 4, we introduce the notion of a *minimal dictatorial cover* of a domain and characterize the MDCs of the single peaked domain. In Section 5, we generalize the dictatorship result in Sato (2010) and the last section concludes the paper.

2 The Model

Let $N = \{1, ..., n\}$ be the set of agents, who collectively have to choose an element from a finite set X of alternatives with $|X| = m \geq 3$. In what follows, we will consider the elements in the set X to be indexed as $x_1 < x_2 < \ldots < x_m$. Consider an admissible domain of preferences $\mathcal{D}$. An alternative $x \in X$ is called the k^{th} ranked alternative in a preference $P \in \mathcal{D}$ if $r_k(P) = x$. The *better than set* of an alternative $x \in X$ with respect to a preference $P \in \mathcal{D}$ is defined as $B(x, P) = \{y \in X | yPx\}$. Similarly the *worse than set* of an alternative $x \in X$ with respect to a preference $P \in \mathcal{D}$ is defined as $W(x, P) = \{y \in X | xPy\}$. Mention the notion of the rank in a preference.

Definition 2.1. A *social choice function* (SCF) f is a mapping $f : \mathcal{D}^{|N|} \rightarrow X$ where $\mathcal{D}$ is any admissible domain of preferences.

Definition 2.2. A SCF f is *manipulable* (MAN) if there exists an individual i, an admissible profile $P = (P_i)_{i \in N} \in \mathcal{D}^n$ and an admissible ordering P_i' such that $f(P_i', P_{-i}) P_i f(P)$. A SCF f is *strategyproof* (SP) if it is not manipulable.

Definition 2.3. A SCF f is *unanimous* (UN) if $f(P) = a_j$ whenever $a_j = r_1(P_i)$ for all $i \in N$.

Definition 2.4. A SCF f is *dictatorial* if there exists an individual $i \in N$ such that for all profiles P, $f(P) = r_1(P_i)$.

All throughout this paper, all our analysis would be restricted to a class of admissible domain of preferences known as *regular* domains.

Definition 2.5. A domain $\mathcal{D}$ is *regular* if, for all $x \in X$, $\exists P_i \in \mathcal{D}$ such that $r_1(P_i) = x$.

Definition 2.6. A domain $\mathcal{D}$ of preferences is *non-dictatorial* if there exists an SCF f that satisfies unanimity, strategy-proofness and non-dictatorship.

A *non-dictatorial* domain is sometimes called a *possibility* domain and we will use these terms interchangeably.

Definition 2.7. A domain $\mathcal{D}$ of preferences is *dictatorial* if there every SCF f that satisfies unanimity and strategy-proofness is dictatorial.

The following proposition comes handy whenever we attempt to show that a particular domain is dictatorial as it reduces the problem from n players to 2 players.

Proposition 2.1. *Let $\mathcal{D}$ be a regular domain. Then, the following two statements are equivalent:*

(i) $f : \mathcal{D}^2 \to X$ *is strategyproof and satisfies unanimity* $\Rightarrow f$ *is dictatorial.*

(ii) $f : \mathcal{D}^n \to X$ *is strategyproof and satisfies unanimity* $\Rightarrow f$ *is dictatorial,* $n \geq 2$.

Proof. See Aswal et al. (2003) for a proof of this proposition. ■

Definition 2.8. The domain of *left-extreme* single peaked preferences, denoted as $\mathcal{S}_l$, is the sub-domain of single peaked domain given by $\mathcal{S}_l = \{P \in \mathcal{S} | x_j < r_1(P_1) < x_k \Rightarrow x_j P x_k\}$. Similarly, the domain of *right-extreme* single peaked preferences, denoted as $\mathcal{S}_r$, is the sub-domain of single peaked preferences given by $\mathcal{S}_r = \{P \in \mathcal{S} | x_j < r_1(P_1) < x_k \Rightarrow x_k P x_j\}$.

Definition 2.9. The domain of *minimally rich* single peaked preferences is the sub-domain of single peaked preferences given by $\mathcal{S}_m = \mathcal{S}_l \cup \mathcal{S}_r$.

Definition 2.10. A *single peaked* preference ordering of agent i on X is a complete, reflexive, transitive and antisymmetric binary relation P_i on X satisfying the following property: there exists $\tau(P_i) \in X$, called the ***peak*** of P_i such that for all $x, y \in A$, if $x < y \leq \tau(P_i)$ or $x > y \geq \tau(P_i)$ then $y P_i x$. Let $\mathcal{S}$ denote the set of all single peaked preferences on X.

3 Maximal Possibility Cover of Domains

Definition 3.1. The *maximal possibility cover* (MPC) $\bar{\mathcal{D}}$ of a possibility domain $\mathcal{D}$ is a largest possibility domain that contains $\mathcal{D}$, i.e., it is a possibility domain such that $\bar{\mathcal{D}} \subsetneq \mathcal{D}'$ implies $\mathcal{D}'$ is a dictatorial domain.

Lemma 3.1. *The domain $\bar{\mathcal{D}}$ is MPC of a domain $\mathcal{D}$ for a society with 2 players iff $\bar{\mathcal{D}}$ is MPC of a domain $\mathcal{D}$ for a society with n players.*

Proof. Since $\bar{\mathcal{D}}$ is MPC of a domain $\mathcal{D}$ for a society with 2 players, this implies that for any $Q \notin \bar{\mathcal{D}}$ $\bar{\mathcal{D}} \cup Q$ is dictatorial for a society with 2 players. Using Proposition 2.1, we can claim that $\bar{\mathcal{D}} \cup Q$ is dictatorial for a society with 2 players then $\bar{\mathcal{D}} \cup Q$ is dictatorial for a society with n players as well. Similarly, $\bar{\mathcal{D}}$ is non-dictatorial for a society with 2 players then $\bar{\mathcal{D}}$ is non-dictatorial for a society with n players as well. Therefore, $\bar{\mathcal{D}}$ is MPC of the domain $\mathcal{D}$ for a society with n players as well. ■

With Lemma 3.1 in place, we can focus our analysis to the case of a society with two players.

Theorem 3.1. *A domain $\bar{\mathcal{S}}$ is a MPC of the single peaked domain $\mathcal{S}$ iff $\bar{\mathcal{S}} = \{P \in \mathcal{U} | r_1(P) = x_1 \Rightarrow r_2(P) = x_2\}$ or $\bar{\mathcal{S}} = \{P \in \mathcal{U} | r_1(P) = x_m \Rightarrow r_2(P) = x_{m-1}\}$.*

Proof. We will use a forthcoming result i.e., the dictatorship result in Theorem 4.3, to prove this theorem. First we prove the necessity part. Observe that for any preference $Q \notin \bar{\mathcal{S}}$, $\bar{\mathcal{S}} \cup \{Q\} \supset \tilde{\mathcal{S}}_m$ and hence is dictatorial.

Now we proceed to prove the sufficiency part. Let $\bar{\mathcal{S}}^1 = \{P \in \mathcal{U} | r_1(P) = x_1 \Rightarrow r_2(P) = x_2\}$ and $\bar{\mathcal{S}}^2 = \{P \in \mathcal{U} | r_1(P) = x_m \Rightarrow r_2(P) = x_{m-1}\}$. Consider another MPC of the single peaked domain $\mathcal{S}$, $\mathcal{D}$, where $\mathcal{D} \neq \bar{\mathcal{S}}^1$ and $\mathcal{D} \neq \bar{\mathcal{S}}^2$. In other words, there exists $\bar{P} \in \mathcal{D} \setminus \mathcal{S}$ such that either $\bar{P} \notin \bar{\mathcal{S}}^1 \setminus \mathcal{S}$ or $\bar{P} \notin \bar{\mathcal{S}}^2 \setminus \mathcal{S}$. Observe that $(\bar{\mathcal{S}}^1 \setminus \mathcal{S}) \cup (\bar{\mathcal{S}}^2 \setminus \mathcal{S}) = (\mathcal{U} \setminus \mathcal{S})$. Therefore, it cannot be the case that $\bar{P} \notin \bar{\mathcal{S}}^1 \setminus \mathcal{S}$ and $\bar{P} \notin \bar{\mathcal{S}}^2 \setminus \mathcal{S}$. First suppose $\bar{P} \in \bar{\mathcal{S}}^1 \setminus \mathcal{S}$ and $\bar{P} \notin \bar{\mathcal{S}}^2 \setminus \mathcal{S}$. This means that $r_1(\bar{P}) = x_m$ and $r_2(\bar{P}) \neq x_{m-1}$. Since $\mathcal{D}$ is an MPC, it cannot be the case that $\mathcal{D} \supset \tilde{\mathcal{S}}$. Therefore, $P' \notin \mathcal{D}$ for any $P' \in \mathcal{U}$ with $r_1(P') = x_1$ and $r_2(P') \neq x_2$. This means that any $P' \in \mathcal{D}$ with $r_1(P') = x_1$ implies $r_2(P') = x_2$ and hence we have $\mathcal{D} \subseteq \bar{\mathcal{S}}^1$. Similarly if we assume that $\bar{P} \notin \bar{\mathcal{S}}^1 \setminus \mathcal{S}$ and $\bar{P} \in \bar{\mathcal{S}}^2 \setminus \mathcal{S}$ then using similar arguments we can establish that $\mathcal{D} \subseteq \bar{\mathcal{S}}^2$. ■

REMARK. The MPCs of the domain of *minimally rich* single peaked preferences are the same as the MPCs of the single peaked domain.

4 Minimal Dictatorial Cover of Possibility Domains

We introduce the notion of *minimal dictatorial cover* of a possibility domain.

Definition 4.1. The *minimal dictatorial cover* of a possibility domain $\mathcal{D}$, denoted by $\tilde{\mathcal{D}}$, is a dictatorial domain that satisfies the following properties:

- $\tilde{\mathcal{D}} \supset \mathcal{D}$.
- $\nexists \mathcal{D}^{'}$ with $\mathcal{D} \subset \mathcal{D}^{'} \subset \tilde{\mathcal{D}}$ such that $\mathcal{D}^{'}$ is dictatorial.

For convenience, we label $r_2(Q)$ as x_l and $r_2(Q^{'})$ as x_r. Note that $x_l \neq x_1, x_2$ and $x_r \neq x_m, x_{m-1}$. We will use the following ramification result to reduce the dimension of the problem from n players to 2 players.

Theorem 4.1. *A domain $\tilde{\mathcal{S}}$ is the minimal dictatorial cover of the domain $\mathcal{S}$ of single peaked preferences iff $\tilde{\mathcal{S}} = \mathcal{S} \cup \{Q, Q^{'}\}$ such that $r_1(Q) = x_1$, $r_2(Q) \neq x_2$, $r_1(Q^{'}) = x_m$ and $r_2(Q^{'}) \neq x_{m-1}$.*

Proof. We will prove the necessary part of theorem in two steps:

1. $\tilde{\mathcal{S}}$ is a dictatorial domain.
2. Any $\mathcal{D}$ with $\mathcal{S} \subset \mathcal{D} \subset \tilde{\mathcal{S}}$ then $\mathcal{D}$ is *not* dictatorial.

STEP 1: A forthcoming theorem, Theorem 4.3, proves that the domain $\tilde{\mathcal{S}}_m = \mathcal{S}_m \cup \{Q, Q'\}$ is dictatorial. Observe that $\tilde{\mathcal{S}} \supset \tilde{\mathcal{S}}_m$ and hence is a dictatorial domain.

STEP 2: Consider any $\mathcal{D}$ with $\mathcal{S} \subset \mathcal{D} \subset \bar{\mathcal{S}}$. It is clear that either preferences of the form $Q \notin \mathcal{D}$ or preferences of the form $Q' \notin \mathcal{D}$. Without loss of generality assume that $Q \notin \mathcal{D}$. This means for any $P_1 \in \mathcal{D}$ with $r_1(P_1) = x_1$ implies $r_2(P_1) = x_1$. Therefore, the following non-dictatorial rule is unanimous and strategy-proof when defined over $\mathcal{D}$:

$$f(P_1, P_2) = \begin{cases} x_1 \text{ if } r_1(P_1) = x_1, x_1 P_2 x_2 \\ x_2 \text{ if } r_1(P_1) = x_1, x_2 P_2 x_1 \\ r_1(P_1) \text{ otherwise} \end{cases}$$

Clearly, player 1 fails to be a dictator at a preference which places x_1 at the top. This proves that $\mathcal{D}$ is a non-dictatorial domain.

Now we prove the sufficiency part. Consider any other MDC $\mathcal{D}'$. By definition, $\mathcal{D}' \not\supset \tilde{\mathcal{S}}$ but $\mathcal{D}' \supset \mathcal{S}$. This means that either preferences of the form $Q \notin \mathcal{D}'$ or preferences of the form

$Q' \notin \mathcal{D}'$. Without loss of generality assume that $Q \notin \mathcal{D}$. This means for any $P_1 \in \mathcal{D}$ with $r_1(P_1) = x_1$ implies $r_2(P_1) = x_1$. Again the rule f given in the necessary part of the proof is unanimous, strategy-proof and non-dictatorial when defined over $\mathcal{D}'$ and therefore, any such $\mathcal{D}'$ is non-dictatorial. Hence no other domain $\mathcal{D}'$ forms an MDC of the domain of single peaked preferences $\mathcal{S}$. ■

We also characterize the MDCs of the minimally rich single peaked domain $\mathcal{S}_m$ as stated in Theorem 4.2.

Theorem 4.2. *A domain $\tilde{\mathcal{S}}_m$ is a MDC of the minimally rich single peaked domain $\mathcal{S}_m$ iff $\tilde{\mathcal{S}}_m = \mathcal{S}_m \cup \{Q, Q^{'}\}$ where $r_1(Q) = x_1$, $r_2(Q) \neq x_2$, $r_1(Q^{'}) = x_m$ and $r_2(Q^{'}) \neq x_{m-1}$.*

Proof. We will prove the necessary part of theorem in two steps:

1. $\tilde{\mathcal{S}_m}$ is a dictatorial domain.

2. Any $\mathcal{D}$ with $\mathcal{S}_m \subset \mathcal{D} \subset \bar{\mathcal{S}_m}$ then $\mathcal{D}$ is *not* dictatorial.

STEP 1: A forthcoming theorem, Theorem 4.3, proves that the domain $\tilde{\mathcal{S}}_m$ is dictatorial.

STEP 2: In the proof of Theorem 4.1, we have already proved that any domain $\mathcal{D}$ with $\mathcal{S} \subset \mathcal{D} \subset \tilde{\mathcal{S}}$ is a possibility domain. Then clearly any domain $\mathcal{D}'$ with $\mathcal{S}_m \subset \mathcal{D}' \subset \tilde{\mathcal{S}_m}$ is also a possibility domain as $\mathcal{D}' \subset \mathcal{D}$ ($\mathcal{D}' \subset \tilde{\mathcal{S}_m} \subset \mathcal{S} \subset \mathcal{D}$).

The proof of the sufficiency part of this theorem involves arguments similar to the ones used in the proof of the sufficiency part of Theorem 4.1. ■

All of the above theorems crucially depends on the following theorem which proves that the domain $\tilde{\mathcal{S}}_m = \mathcal{S}_m \cup \{Q, Q'\}$ is dictatorial.

Theorem 4.3. *The domain $\tilde{\mathcal{S}}_m = \mathcal{S}_m \cup \{Q, Q'\}$ is a dictatorial domain.*

With Proposition 2.1 in place, we only need to prove that $\tilde{\mathcal{S}}_m$ is dictatorial in the case of two players, i.e., $N = \{1, 2\}$.

We prove Theorem 4.2 *via* a series of lemmas.

Lemma 4.1. *The following statements hold:*

(i) If $x_{k+1} \in O_1(P_2)$ for some $P_2 \in \tilde{\mathcal{S}}_m$ with $r_1(P_2) = x_k$ then $x_{k+1} \in O_1(P_2)$ for all $P_2 \in \tilde{\mathcal{S}}_m$ with $r_1(P_2) = x_k$ where $2 \leq k \leq m-1$.

(ii) If $x_{k-1} \in O_1(P_2)$ for some $P_2 \in \tilde{\mathcal{S}}_m$ with $r_1(P_2) = x_k$ then $x_{k-1} \in O_1(P_2)$ for all $P_2 \in \tilde{\mathcal{S}}_m$ with $r_1(P_2) = x_k$ where $2 \leq k \leq m-1$.

Proof. We provide a proof of statement (i) and statement (ii) can be proved using analogous arguments. Consider P_2 with $r_1(P_2) = x_k$ and $r_2(P_2) = x_{k-1}$ and P_2' with $r_1(P_2') = x_k$ and $r_2(P_2') = x_{k+1}$. Suppose $x_{k+1} \in O_1(P_2)$ and $x_{k+1} \notin O_1(P_2')$. Consider P_1 such that $r_1(P_1) = x_{k+1}$ and $r_2(P_2) = x_k$. Then $f(P_1, P_2) = x_{k+1}$ and $f(P_1, P_2') = x_k$ which means that player 2 manipulates at (P_1, P_2) via P_2'. A similar argument applies in the case where $x_{k+1} \in O_1(P_2)$ and $x_{k+1} \notin O_1(P_2')$. ■

Lemma 4.2. *Consider $P_2 \in \tilde{\mathcal{S}}_m \setminus \{Q, Q'\}$ such that $r_1(P_2) = x_k$. Then the following statements hold:*

(i) For any $i < k-1$, $x_i \in O_1(P_2)$ means $x_{i+1} \in O_1(P_2)$.

(ii) For any $j > k+1$, $x_j \in O_1(P_2)$ means $x_{j-1} \in O_1(P_2)$.

Proof. We prove statement (i) and an analogous argument applies in the case of statement (ii). Assume to the contrary that for some $i < k-1$, $x_i \in O_1(P_2)$ and $x_{i+1} \notin O_1(P_2)$. Consider $P_1 \in \tilde{\mathcal{S}}_m \setminus \{Q, Q'\}$ with $r_1(P_1) = x_{i+1}$ and $r_2(P_1) = x_i$. Observe that $f(P_1, P_2) = x_i$. Now consider any $P_2' \in \tilde{\mathcal{S}}_m \setminus \{Q, Q'\}$ with $r_1(P_2') = x_{i+1}$. Clearly $f(P_1, P_2') = x_{i+1}$ and player 2 manipulates at (P_1, P_2) via P_2'. ■

Lemma 4.3. *The following statements hold:*

(i) Consider $P_2 \in \tilde{\mathcal{S}}_m$ such that $r_1(P_2) = x_l$.

(a) If $x_{l+1} \in O_1(P_2)$ then $O_1(P_2) = X$.

(b) If $x_{l-1} \in O_1(P_2)$ then $O_1(P_2) = X$.

(ii) Consider $P_2 \in \tilde{\mathcal{S}}_m$ such that $r_1(P_2) = x_r$.

(a) If $x_{r+1} \in O_1(P_2)$ then $O_1(P_2) = X$.

(b) If $x_{r-1} \in O_1(P_2)$ then $O_1(P_2) = X$.

Proof. We prove statement (i) and statement (ii) can be proved using analogous arguments.

PROOF OF PART (a): Consider P_2 with $r_1(P_2) = x_l$ and $r_2(P_2) = x_{l+1}$ and P_2' with $r_1(P_2') = x_l$ and $r_2(P_2') = x_{l-1}$. By Lemma 4.1, $x_{l+1} \in O_1(P_2)$ iff $x_{l+1} \in O_1(P_2')$.

We claim that $x_{l+1} \in O_1(P_2')$ implies $x_l \in O_1(\hat{P}_2)$ where $\hat{P}_2$ such that $r_1(\hat{P}_2) = x_2$ and $r_2(\hat{P}_2) = x_1$. By Lemma 4.2, it is enough to prove that $x_{l+1} \in O_1(P_2')$ implies $x_{l+1} \in O_1(\hat{P}_2)$. Suppose $x_{l+1} \in O_1(P_2')$ and $x_{l+1} \notin O_1(\hat{P}_2)$. Consider P_1 such that $r_1(P_1) = x_{l+1}$ and $r_2(P_1) = x_l$. Then $f(P_1, P_2') = x_{l+1}$ and $f(P_1, \hat{P}_2) \in \{x_2, \ldots, x_l\}$ which means that player 2 manipulates at (P_1, P_2') via $\hat{P}_2$.

We claim that $x_1 \in O_1(\hat{P}_2')$ where $\hat{P}_2'$ such that $r_1(\hat{P}_2') = x_2$ and $r_2(\hat{P}_2') = x_3$. By Lemma 4.1, it is enough to prove that $x_1 \in O_1(\hat{P}_2)$. Suppose not. Consider $\hat{P}_1$ with $r_1(\hat{P}_1) = x_1$ and $r_2(\hat{P}_1) = x_l$. Observe that $f(\hat{P}_1, \hat{P}_2) = x_l$ and player 2 manipulates at $(\hat{P}_1, \hat{P}_2)$ via any preference with top x_1. Therefore, $x_1 \in O_1(\hat{P}_2')$ and hence $x_1 \in O_1(P_2)$. If not, $f(\hat{P}_1', \hat{P}_2') = x_1$ and $f(\hat{P}_1', P_2) \in \{x_2, \ldots, x_l\}$ where $\hat{P}_1'$ such that $r_1(\hat{P}_1') = x_1$ and $r_2(\hat{P}_1') = x_2$. This means that player 2 will manipulate at $(\hat{P}_1', \hat{P}_2')$ via P_2 and we prove that $x_1 \in O_1(P_2)$. By a similar argument, we can prove that $x_1 \in O_1(\bar{P}_2)$ where $\bar{P}_2$ such that $r_1(\bar{P}_2) = x_m$ and $r_2(\bar{P}_2) = x_{m-1}$ and $x_1 \in O_1(\tilde{P}_2)$ where $\tilde{P}_2$ such that $r_1(\tilde{P}_2) = x_{m-1}$ and $r_2(\tilde{P}_2) = x_{m-2}$. Using Lemma 4.2 and the fact that $x_1 \in O_1(\bar{P}_2)$, $O_1(\bar{P}_2) = X$.

We claim that $x_k \in O_1(\tilde{P}_2)$ any $x_k \in X$ where $1 \leq k \leq m-1$ and $\tilde{P}_2$ with $r_1(\tilde{P}_2) = x_{m-1}$ and $r_2(\tilde{P}_2) = x_m$. By unanimity, $x_{m-1} \in O_1(P_2)$. Now we claim that $x_i \in O_1(\tilde{P}_2)$ then $x_{i-1} \in O_1(\tilde{P}_2)$ for $1 \leq i \leq m-1$. Suppose $x_i \in O_1(\tilde{P}_2)$ and $x_{i-1} \in O_1(\tilde{P}_2)$ for some i. Observe that $f(\tilde{P}_1, \tilde{P}_2) = x_i$ and $f(\tilde{P}_1, \bar{P}_2) = x_{i-1}$ where such that $r_1(\tilde{P}_1) = x_{i-1}$ and $r_2(\tilde{P}_1) = x_i$. This means that player 2 manipulates at $(\tilde{P}_1, \tilde{P}_2)$ via $\bar{P}_2$. Since $r \neq m, m-1$, $x_r \in O_1(\tilde{P}_2)$. We now claim that $x_m \in O_1(\tilde{P}_2)$. If not $f(\tilde{P}_1', \tilde{P}_2) = x_r$ where $\tilde{P}_1'$ such that $r_1(\tilde{P}_1') = x_m$ and $r_2(\tilde{P}_1') = x_r$ and player 2 manipulates at $(\tilde{P}_1', \tilde{P}_2)$ via any preference with x_m at the top. By Lemma 4.1, $x_m \in O_1(\tilde{P}_2')$.

Now we claim that $x_m \in O_1(P_2')$. If not, $f(\tilde{P}_1'', \tilde{P}_2') = x_m$ and $f(\tilde{P}_1'', P_2') \in \{x_l, \ldots, x_{m-1}\}$ where $\tilde{P}_1''$ such that $r_1(\tilde{P}_1'') = x_m$ and $r_2(\tilde{P}_1'') = x_{m-1}$. This means that player 2 will manipulate at $(\tilde{P}_1'', \tilde{P}_2')$ via P_2' and we prove that $x_m \in O_1(P_2')$. From the above arguments, we have proved that $x_1 \in O_1(P_2)$ and $x_m \in O_1(P_2')$. With unanimity and Lemma 4.2 in place, $O_1(P_2) = O_1(P_2') = X$.

Proof of Part (b): The proof for part (b) is similar to the proof for part (a) for the case $l-1 \leq r$. Hence we prove part (b) when $l-1 > r$. Consider P_2 with $r_1(P_2) = x_l$ and $r_2(P_2) = x_{l-1}$ and P_2' with $r_1(P_2') = x_l$ and $r_2(P_2') = x_{l+1}$. By Lemma 4.1, $x_{l-1} \in O_1(P_2)$ iff $x_{l-1} \in O_1(P_2')$. In what follows, we will prove that if $x_{l-1} \in O_1(P_2)$ then $x_{l+1} \in O_1(P_2)$ and then we are done by using arguments in the proof of part (a).

We claim that $x_{l-1} \in O_1(P_2)$ implies $x_l \in O_1(\hat{P}_2)$ where $\hat{P}_2$ with $r_1(\hat{P}_2) = x_m$ and $r_2(\hat{P}_2) =$

x_{m-1}. By Lemma 4.2, it is enough to prove that $x_{l-1} \in O_1(P_2)$ implies $x_{l-1} \in O_1(\hat{P}_2)$. Suppose not. Consider P_1 with $r_1(P_1) = x_{l-1}$ and $r_2(P_1) = x_l$. Observe that $f(P_1, P_2) = x_{l-1}$ and $f(P_1, \hat{P}_2) = \{x_l, \ldots, x_m\}$. This means that player 2 will manipulate at (P_1, P_2) via $\hat{P}_2$.

Now we claim that $x_l \in O_1(\hat{P}_2)$ implies $x_l \in O_1(\hat{P}'_2)$ where $\hat{P}'_2$ with $r_1(\hat{P}'_2) = x_m$ and $r_2(\hat{P}'_2) = x_r$. Suppose not. Consider P'_1 with $r_1(P'_1) = x_l$ and $r_2(P'_1) = x_{l+1}$. Observe that $f(P'_1, \hat{P}_2) = x_l$ and $f(P_1, \hat{P}_2) = \{x_{l+1}, \ldots, x_m\}$. This means that player 2 will manipulate at $(P_1, \hat{P}_2)$ via $\hat{P}'_2$.

Consider $\bar{P}_2$ with $r_1(\bar{P}_2) = x_r$ and $r_2(\bar{P}_2) = x_{r-1}$. Suppose $x_j \in O_1(\bar{P}_2)$ where $j < r$. By Lemma 4.2, $x_{r-1} \in O_1(\bar{P}_2)$. This means that $x_{r-1} \in O_1(P'_2)$. If not, $f(\bar{P}_1, \bar{P}_2) = x_{r-1}$ and $f(\bar{P}_1, P'_2) = \{x_r, \ldots, x_{l-1}\}$ where $\bar{P}_1$ such that $r_1(\bar{P}_1) = x_{r-1}$ and $r_2(\bar{P}_1) = x_r$ which means that player 2 will manipulate at $(\bar{P}_1, \bar{P}_2)$ via P'_2. Since $l - 1 > r$, $x_r \in O_1(P'_2)$ due to Lemma 4.2. Now we claim that $x_m \in O_1(P'_2)$. Consider $\bar{P}'_1$ such that $r_1(\bar{P}'_1) = x_m$ and $r_2(\bar{P}'_1) = x_r$. Observe that $f(\bar{P}'_1, P'_2) = x_r$ and player 2 manipulates at $(\bar{P}'_1, P'_2)$ via some preference with x_m at the top. Thus we prove that $x_m \in O_1(P'_2)$ and hence by Lemma 4.2, $x_{l+1} \in O_1(P'_2)$. Repeating the arguments in the proof of part (a), we can prove that $O_1(P_2) = O_1(P'_2) = X$.

In view of the arguments in the previous paragraphs, we now assume that $x_j \in O_1(\bar{P}_2)$ where $j \geq r$. We have to consider two cases here. Firstly, let us assume that $x_1 \in O_1(\hat{P}'_2)$. Consider $\hat{P}_1$ with $r_1(\hat{P}_1) = x_1$ and $r_2(\hat{P}_1) = x_2$. Then $f(\hat{P}_1, \hat{P}'_2)) = x_1$. If $f(\hat{P}_1, \bar{P}_2)) = x_r$ then player 2 will manipulate at $(\hat{P}_1, \hat{P}'_2))$ via $\bar{P}_2$. If $f(\hat{P}_1, \bar{P}_2)) = x_j$ where $j > r$ then player 2 manipulates at $(\hat{P}_1, \bar{P}_2)$ via any preference with x_1 at the top. Next assume that $x_1 \notin O_1(\hat{P}'_2)$. From the arguments in an earlier paragraph, we know that $x_l \notin O_1(\hat{P}'_2)$. Consider $\hat{P}'_1$ with $r_1(\hat{P}'_1) = x_1$ and $r_2(\hat{P}'_1) = x_l$. Then $f(\hat{P}'_1, \hat{P}'_2)) = x_l$. If $f(\hat{P}'_1, \bar{P}_2)) = x_r$ then player 2 will manipulate at $(\hat{P}'_1, \hat{P}'_2))$ via $\bar{P}_2$. If $f(\hat{P}'_1, \bar{P}_2)) = x_j$ where $j > r$ then player 2 manipulates at $(\hat{P}'_1, \bar{P}_2)$ via any preference with x_1 at the top. ■

Lemma 4.4. *Let P_2 be such that $r_1(P_2) = x_l$ and P'_2 be such that $r_1(P'_2) = x_r$. Then $O_1(P_2) = \{x_l\}$ iff $O_1(P_2) = \{x_r\}$.*

Proof. We prove this lemma in two cases.

CASE 1 ($l - 1 \leq r$): For this case, we only prove the sufficiency part as the necessary part can be proved using analogous arguments. Consider P_2 such that $r_1(P_2) = x_l$ and P'_2 be such that $r_1(P'_2) = x_r$. Assume on the contrary that $O_1(P'_2) = \{x_r\}$ and $O_1(P_2) \neq \{x_l\}$. An immediate consequence of Lemma 4.2 is that if $O_1(P_2) \neq \{x_l\}$ then either $x_{l+1} \in O_1(P_2)$ or $x_{l-1} \in O_1(P_2)$.

First consider the case where $x_{l+1} \in O_1(P_2)$. Due to Lemma 4.1, assume without loss of generality that $r_2(P_2) = x_{l-1}$. We claim that $x_{l+1} \in O_1(\bar{P}_2)$ where $r_1(\bar{P}_2) = x_2$ and $r_2(\bar{P}_2) = x_1$. Suppose not. Consider P_1 with $r_1(P_1) = x_{l+1}$ and $r_2(P_1) = x_l$. Observe that $f(P_1, P_2) = x_{l+1}$ and $f(P_1, \bar{P}_2) \in \{x_2, \ldots, x_l\}$ which means that player 2 manipulates at (P_1, P_2) via $\bar{P}_2$. Therefore, $x_{l+1} \in O_1(\bar{P}_2)$ and by Lemma 4.2, $x_l \in O_1(\bar{P}_2)$. Next we claim that $x_1 \in O_1(\bar{P}_2)$. Consider P_1' with $r_1(P_1') = x_1$ and $r_2(P_1') = x_l$. Observe that $f(P_1, \bar{P}_2) = x_l$ which means that player 2 manipulates at $(P_1, \bar{P}_2)$ via any preference that places x_1 at the top. By Lemma 4.1, $x_1 \in O_1(\bar{P}_2')$ where $r_1(\bar{P}_2') = x_2$ and $r_2(\bar{P}_2') = x_3$. Consider P_2' with $r_1(P_2') = x_r$ and $r_2(P_2') = x_{r+1}$. Notice that we have assumed in the begining that $O_1(P_2') = \{x_r\}$. Lastly we claim that $x_1 \in O_1(P_2')$. Suppose not. Consider P_1'' with $r_1(P_1'') = x_1$ and $r_2(P_1'') = x_2$. Observe that $f(P_1'', \bar{P}_2') = x_1$ and $f(P_1'', P_2') = x_r$ which means that player 2 manipulates at $(P_1'', \bar{P}_2')$ via P_2'. This is a contradiction to our assumption that $O_1(P_2') = \{x_r\}$.

Next consider the case where $x_{l-1} \in O_1(P_2)$. Due to Lemma 4.1, assume without loss of generality that $r_2(P_2) = x_{l+1}$. Recall that we start by assuming that $O_1(P_2') = \{x_r\}$. This means that $f(P_1, P_2) = x_{l-1}$ and $f(P_1, P_2') = x_r$ where $r_1(P_1) = x_{l-1}$. Since $l - 1 \leq r$, $x_r P_2 x_{l-1}$. Then player 2 manipulates at (P_1, P_2) via P_2'.

Case 2 ($l - 1 > r$): For this case, we only prove the necessary part as the sufficiency part can be proved using analogous arguments. Consider P_2 such that $r_1(P_2) = x_l$ and P_2' be such that $r_1(P_2') = x_r$. Assume on the contrary that $O_1(P_2) = \{x_l\}$ and $O_1(P_2') \neq \{x_r\}$. As argued in the earlier case, if $O_1(P_2') \neq \{x_r\}$ then either $x_{r+1} \in O_1(P_2')$ or $x_{r-1} \in O_1(P_2')$.

First consider the case where $x_{r+1} \in O_1(P_2')$. Due to Lemma 4.1, assume without loss of generality that $r_2(P_2') = x_{r-1}$. We claim that $x_{r+1} \in O_1(\bar{P}_2)$ where $r_1(\bar{P}_2) = x_1$ and $r_2(\bar{P}_2) = x_2$. Suppose not. Consider P_1 with $r_1(P_1) = x_{r+1}$ and $r_2(P_1) = x_r$. Observe that $f(P_1, P_2') = x_{r+1}$ and $f(P_1, \bar{P}_2) \in \{x_1, \ldots, x_r\}$ which means that player 2 manipulates at (P_1, P_2') via $\bar{P}_2$. Therefore, $x_{r+1} \in O_1(\bar{P}_2)$ and by Lemma 4.2, $x_2 \in O_1(\bar{P}_2)$. Next we claim $x_2 \in O_1(\bar{P}_2')$ with $r_1(\bar{P}_2') = x_1$ and $r_2(\bar{P}_2') = x_l$. Suppose not. Consider P_1' with $r_1(P_1') = x_2$ and $r_1(P_1') = x_1$. Observe that $f(P_1', \bar{P}_2) = x_2$ and $f(P_1', \bar{P}_2') = x_1$ which means that player 2 will manipulate via $(P_1', \bar{P}_2)$ via $\bar{P}_2'$. Now consider P_1'' with $r_1(P_1'') = x_l$. Observe that $f(P_1'', \bar{P}_2') = x_2$ and $f(P_1'', P_2) = x_l$ which means that player 2 will manipulate at $(P_1'', \bar{P}_2')$ via P_2. Therefore, $x_2 \in O_1(P_2)$ which contradicts our initial assumption.

Next consider the case where $x_{r-1} \in O_1(P_2)$. Due to Lemma 4.1, assume without loss of generality that $r_2(P_2) = x_{r+1}$. Recall that we start by assuming that $O_1(P_2) = \{x_l\}$. This means that $f(P_1, P_2') = x_{r-1}$ and $f(P_1, P_2) = x_l$ where $r_1(P_1) = x_{r-1}$. Since $l-1 > r$, $x_l P_2' x_{r-1}$. Then player 2 manipulates at (P_1, P_2') via P_2 which is a contradiction to our initial assumption. ■

Lemma 4.5. *Let P_2 be such that $r_1(P_2) = x_l$ and P_2' be such that $r_1(P_2') = x_r$. Then $O_1(P_2) = \{x_l\}$ or $O_1(P_2') = \{x_r\}$ implies $O_1(\bar{P}_2) = \{x_k\}$ for any $\bar{P}_2 \in \tilde{\mathcal{S}}_m$ with $r_1(\bar{P}_2) = x_k$.*

Proof. Consider P_2 be such that $r_1(P_2) = x_l$ and P_2' be such that $r_1(P_2') = x_r$. First observe that by Lemma 4.4, $O_1(P_2) = \{x_l\}$ iff $O_1(P_2') = \{x_r\}$. Therefore, we can assume without loss of generality that $O_1(P_2) = \{x_l\}$. We prove the theorem only for the case $l \leq r$. An analogous argument can be used for the case $l > r$. We prove this in three parts.

PART (A)($k < l$): Consider $\bar{P}_2$ with $r_1(\bar{P}_2) = x_k$ where $k = l-1$. Recall that we started by assuming that $O_1(P_2) = \{x_l\}$ where $r_1(P_2) = x_l$. Assume to the contrary that $O_1(\bar{P}_2) \neq \{x_{l-1}\}$. Let $r_2(\bar{P}_2) = x_{l-2}$. We claim that $x_l \in O_1(\bar{P}_2)$. Using Lemma 4.1, it is enough to prove that $x_l \in O_1(\bar{P}_2')$ with $r_1(\bar{P}_2') = x_{l-1}$ and $r_2(\bar{P}_2') = x_l$. In fact, we will prove that $O_1(\bar{P}_2') = \{x_{l-1}, x_l\}$. Suppose not, i.e., $x_i \in O_1(\bar{P}_2')$ where $i \neq l-1, l$. Consider P_1 with $r_1(P_1) = x_i$. Observe that $f(P_1, \bar{P}_2') = x_i$ and $f(P_1, P_2) = x_l$ which means that player 2 manipulates at $(P_1, \bar{P}_2')$ via P_2.

We claim that $x_l \in O_1(\tilde{P}_2)$ where $r_1(\tilde{P}_2) = x_1$ and $r_2(\tilde{P}_2) = x_2$. Suppose not. Consider P_1' with $r_1(P_1') = x_l$ and $r_2(P_1') = x_{l-1}$. Observe that $f(P_1', \bar{P}_2) = x_l$ and $f(P_1', \tilde{P}_2) = \{x_1, x_2, \ldots, x_{l-1}\}$ which means that player 2 manipulates at $(P_1', \bar{P}_2)$ via $\tilde{P}_2$. Therefore, $x_2 \in O_1(\tilde{P}_2)$ using Lemma 4.2. Next we claim that $x_2 \in O_1(\tilde{P}_2')$ with $r_1(\tilde{P}_2') = x_1$ and $r_2(\tilde{P}_2') = x_l$. Suppose not. Consider $\bar{P}_1$ with $r_1(\bar{P}_1) = x_2$ and $r_2(\bar{P}_1) = x_1$. Observe that $f(\bar{P}_1, \tilde{P}_2) = x_2$ and $f(\bar{P}_1, \tilde{P}_2') = x_1$ which means that player 2 manipulates at $(\bar{P}_1, \tilde{P}_2)$ via $\tilde{P}_2'$. Therefore, $x_2 \in O_1(\tilde{P}_2')$. Observe that $f(\bar{P}_1, \tilde{P}_2') = x_2$ and $f(\bar{P}_1, P_2) = x_l$ which means that player 2 manipulates at $(\bar{P}_1, \tilde{P}_2')$ via P_2.

Now we use induction on the set of alternatives $\{x_{l-1}, x_{l-2}, \ldots, x_1\}$. Assume that $O_1(\hat{P}_2) = \{r_1(\hat{P}_2)\}$ for any $\hat{P}_2$ with $r_1(\hat{P}_2) = x_i$ for some i such that $k \leq i \leq l$ and $k > 1$. In particular we assume that $O_1(\hat{P}_2') = \{x_k\}$ for any $\hat{P}_2'$ with $r_1(\hat{P}_2') = x_k$. Then we claim that $O_1(\bar{\bar{P}}_2) = \{r_1(\bar{\bar{P}}_2)\}$ for any $\bar{\bar{P}}_2$ with $r_1(\bar{\bar{P}}_2) = x_{k-1}$. Let $r_2(\bar{\bar{P}}_2) = x_{k-2}$. We claim that $x_k \in O_1(\bar{\bar{P}}_2)$. Using Lemma 4.1, it is enough to prove that $x_k \in O_1(\bar{\bar{P}}_2')$ with $r_1(\bar{\bar{P}}_2') = x_{k-1}$ and $r_2(\bar{\bar{P}}_2') = x_k$. In fact, we will prove that $O_1(\bar{\bar{P}}_2') = \{x_{k-1}, x_k\}$. Suppose not, i.e., k$x_j \in O_1(\bar{\bar{P}}_2')$ where $j \neq k-1, k$. Consider $\bar{P}_1'$ with $r_1(\bar{P}_1') = x_j$. Observe that $f(\bar{P}_1', \bar{\bar{P}}_2') = x_j$ and $f(\bar{P}_1', \hat{P}_2') = x_k$ which means that player 2 manipulates at $(\bar{P}_1', \bar{\bar{P}}_2')$ via $\hat{P}_2'$.

We claim that $x_k \in O_1(\tilde{P}_2)$ where $r_1(\tilde{P}_2) = x_1$ and $r_2(\tilde{P}_2) = x_2$. Suppose not. Consider $\hat{P}_1$ with $r_1(\hat{P}_1) = x_k$ and $r_2(\hat{P}_1) = x_{k-1}$. Observe that $f(\hat{P}_1, \bar{\bar{P}}_2) = x_k$ and $f(\hat{P}_1, \tilde{P}_2) = \{x_1, x_2, \ldots, x_{k-1}\}$

which means that player 2 manipulates at $(\hat{P}_1, \bar{\bar{P}}_2, \bar{P}_2)$ via $\tilde{P}_2$. Therefore, $x_2 \in O_1(\tilde{P}_2)$ using Lemma 4.2. Using arguments in an earlier paragraph, this implies $x_2 \in O_1(\tilde{P}_2')$ with $r_1(\tilde{P}_2') = x_1$ and $r_2(\tilde{P}_2') = x_l$. Observe that $f(\bar{P}_1, \tilde{P}_2') = x_2$ and $f(\bar{P}_1, P_2) = x_l$ which means that player 2 manipulates at $(\bar{P}_1, \tilde{P}_2')$ via P_2.

PART (B)($l < k < r$): Consider $\bar{P}_2$ with $r_1(\bar{P}_2) = x_k$ where $k = l + 1$. Recall that we started by assuming that $O_1(P_2) = \{x_l\}$ where $r_1(P_2) = x_l$. Assume to the contrary that $O_1(\bar{P}_2) \neq \{x_{l+1}\}$. Let $r_2(\bar{P}_2) = x_l$. We claim that $x_l \in O_1(\bar{P}_2)$. Using Lemma 4.1, it is enough to prove that $x_l \in O_1(\bar{P}_2')$ with $r_1(\bar{P}_2') = x_{l+1}$ and $r_2(\bar{P}_2') = x_l$. In fact, we will prove that $O_1(\bar{P}_2') = \{x_l, x_{l+1}\}$. Suppose not, i.e., $x_i \in O_1(\bar{P}_2')$ where $i \neq l+1, l$. Consider P_1 with $r_1(P_1) = x_i$. Observe that $f(P_1, \bar{P}_2') = x_i$ and $f(P_1, P_2) = x_l$ which means that player 2 manipulates at $(P_1, \bar{P}_2')$ via P_2.

We claim that $x_l \in O_1(\tilde{P}_2)$ where $r_1(\tilde{P}_2) = x_m$ and $r_2(\tilde{P}_2) = x_{m-1}$. Suppose not. Consider P_1' with $r_1(P_1') = x_l$ and $r_2(P_1') = x_{l+1}$. Observe that $f(P_1', \bar{P}_2) = x_l$ and $f(P_1', \tilde{P}_2) = \{x_{l+1}, x_{l+2}, \ldots, x_m\}$ which means that player 2 manipulates at $(P_1', \bar{P}_2)$ via $\tilde{P}_2$. Therefore, $x_{m-1} \in O_1(\tilde{P}_2)$ using Lemma 4.2. Next we claim that $x_{m-1} \in O_1(\tilde{P}_2')$ with $r_1(\tilde{P}_2') = x_m$ and $r_2(\tilde{P}_2') = x_r$. Suppose not. Consider $\bar{P}_1$ with $r_1(\bar{P}_1) = x_{m-1}$ and $r_2(\bar{P}_1) = x_m$. Observe that $f(\bar{P}_1, \tilde{P}_2) = x_{m-1}$ and $f(\bar{P}_1, \tilde{P}_2') = x_m$ which means that player 2 manipulates at $(\bar{P}_1, \tilde{P}_2)$ via $\tilde{P}_2'$. Therefore, $x_{m-1} \in O_1(\tilde{P}_2')$. Recall that $O_1(P_2') = \{x_r\}$ where $r_1(P_2') = x_r$. Observe that $f(\bar{P}_1, \tilde{P}_2') = x_{m-1}$ and $f(\bar{P}_1, P_2') = x_r$ which means that player 2 manipulates at $(\bar{P}_1, \tilde{P}_2')$ via P_2'.

Now we use induction on the set of alternatives $\{x_{l+1}, x_{l+2}, \ldots, x_r\}$. Assume that $O_1(\hat{P}_2) = \{r_1(\hat{P}_2)\}$ for any $\hat{P}_2$ with $r_1(\hat{P}_2) = x_i$ for some i such that $l \leq i \leq k$ and $k < r$. In particular we assume that $O_1(\hat{P}_2') = \{x_k\}$ for any $\hat{P}_2'$ with $r_1(\hat{P}_2') = x_k$. Then we claim that $O_1(\bar{\bar{P}}_2) = \{r_1(\bar{\bar{P}}_2)\}$ for any $\bar{\bar{P}}_2$ with $r_1(\bar{\bar{P}}_2) = x_{k+1}$. Let $r_2(\bar{\bar{P}}_2) = x_{k+2}$. We claim that $x_k \in O_1(\bar{\bar{P}}_2)$. Using Lemma 4.1, it is enough to prove that $x_k \in O_1(\bar{\bar{P}}_2')$ with $r_1(\bar{\bar{P}}_2') = x_{k+1}$ and $r_2(\bar{\bar{P}}_2') = x_k$. In fact, we will prove that $O_1(\bar{\bar{P}}_2') = \{x_k, x_{k+1}\}$. Suppose not, i.e., $x_j \in O_1(\bar{\bar{P}}_2')$ where $j \neq k, k+1$. Consider $\bar{P}_1'$ with $r_1(\bar{P}_1') = x_j$. Observe that $f(\bar{P}_1', \bar{\bar{P}}_2') = x_j$ and $f(\bar{P}_1', \hat{P}_2') = x_k$ which means that player 2 manipulates at $(\bar{P}_1', \bar{\bar{P}}_2')$ via $\hat{P}_2'$.

We claim that $x_k \in O_1(\tilde{P}_2)$ where $r_1(\tilde{P}_2) = x_m$ and $r_2(\tilde{P}_2) = x_{m-1}$. Suppose not. Consider $\hat{P}_1$ with $r_1(\hat{P}_1) = x_k$ and $r_2(\hat{P}_1) = x_{k+1}$. Observe that $f(\hat{P}_1, \bar{\bar{P}}_2) = x_k$ and $f(\hat{P}_1, \tilde{P}_2) = \{x_{k+1}, x_{k+2}, \ldots, x_m\}$ which means that player 2 manipulates at $(\hat{P}_1, \bar{\bar{P}}_2, \bar{P}_2)$ via $\tilde{P}_2$. Therefore, $x_{m-1} \in O_1(\tilde{P}_2)$ using Lemma 4.2. Using arguments in an earlier paragraph, this implies $x_{m-1} \in O_1(\tilde{P}_2')$ with $r_1(\tilde{P}_2') = x_m$ and $r_2(\tilde{P}_2') = x_r$. Observe that $f(\bar{P}_1, \tilde{P}_2') = x_{m-1}$ and $f(\bar{P}_1, P_2) = x_r$

which means that player 2 manipulates at $(\bar{P}_1, \tilde{P}'_2)$ via P_2.

PART (C)($k > r$): For this case, we can use arguments similar to the ones used in PART (B) to establish our claim. ■

Lemma 4.6. *Consider any P_2 such that $r_1(P_2) = x_l$ and any P'_2 be such that $r_1(P'_2) = x_r$. Then $O_1(P_2) = X$ or $O_1(P'_2) = X$ implies $O_1(\bar{P}_2) = X$ for any $\bar{P}_2 \in \tilde{\mathcal{S}}_m$.*

Proof. Assume that $O_1(P_2) = X$ where for any P_2 with $r_1(P_2) = x_l$. We can use similar arguments if we start with the assumption $O_1(P'_2) = X$ where for any P'_2 with $r_1(P'_2) = x_r$. Consider $\hat{P}_2$ with $r_1(\hat{P}_2) = x_l$ and $r_2(\hat{P}_2) = x_{l+1}$. By our assumption, we know $O_1(\hat{P}_2) = X$. In particular, we know that $x_m \in O_1(\hat{P}_2)$. We claim that $x_m \in O_1(\bar{P}_2)$ for any $\bar{P}_2 \in \mathcal{S}_m$. Suppose not. Consider P_1 with $r_1(P_1) = x_m$ and $r_2(P_1) = x_{m-1}$. Let $r_1(\bar{P}_2) = x_k$. Observe that $f(P_1, \hat{P}_2) = x_m$ and $f(P_1, \bar{P}_2) \in \{x_k, x_{k+1}, \ldots, x_{m-1}\}$ which means that player 2 will manipulate at $(P_1, \hat{P}_2)$ via $\bar{P}_2$. Using similar arguments, one can prove that $x_1 \in O_1(\bar{P}_2)$ for any $\bar{P}_2 \in \mathcal{S}_m$ as $x_1 \in O_1(\hat{P}'_2)$ where $r_1(\hat{P}'_2) = x_l$ and $r_2(\hat{P}'_2) = x_{l-1}$. Now using Lemma 4.2, one can claim that $O_1(\bar{P}_2) = X$ for any $\bar{P}_2 \in \mathcal{S}_m$.

Now we claim that $O_1(Q_2) = X$ where $r_1(Q_2) = x_1$ and $r_2(Q_2) = x_l$. Using the arguments in the above paragraph, $O_1(\tilde{P}_2) = X$ where $r_1(\tilde{P}_2) = x_1$ and $r_2(\tilde{P}_2) = x_1$. Observe that $x_1 \in O_1(Q_2)$ due to unanimity. First we claim that $x_1 \in O_1(Q_2)$ implies $x_2 \in O_1(Q_2)$. Suppose not. Consider P'_1 with $r_1(P'_1) = x_2$ and $r_2(P'_1) = x_1$. Observe that $f(P'_1, \tilde{P}_2) = x_2$ and $f(P'_1, Q_2) = x_1$ which means that player 2 will manipulate at $(P_1, \tilde{P}_2)$ via Q_2. Now we use induction to complete the proof. Assume that $x_k \in O_1(Q_2)$. We claim that $x_k \in O_1(Q_2)$ implies $x_{k+1} \in O_1(Q_2)$. Suppose not. Consider $\bar{P}_1$ with $r_1(\bar{P}_1) = x_{k+1}$ and $r_2(\bar{P}_1) = x_k$. Observe that $f(\bar{P}_1, \tilde{P}_2) = x_{k+1}$ and $f(\bar{P}_1, Q_2) = x_k$ which means that player 2 will manipulate at $(\bar{P}_1, \tilde{P}_2)$ via Q_2. Thus, we have proved that $O_1(Q_2) = X$ where $r_1(Q_2) = x_1$ and $r_2(Q_2) = x_l$. Using similar arguments, one can prove that $O_1(Q'_2) = X$ where $r_1(Q'_2) = x_m$ and $r_2(Q'_2) = x_r$. ■

Proof of Theorem 4.3. Consider P_2 with $r_1(P_2) = x_l$. Using Lemma 4.2-4.3, we know that $O_1(P_2) = \{x_l\}$ or X. If $O_1(P_2) = \{x_l\}$ then by Lemma 4.5, we know that $O_1(P'_2) = \{r_1(P'_2)\}$ for any $P'_2 \in \tilde{\mathcal{S}}_m$. This means that player 1 is the dictator. Now suppose that $O_1(P_2) = X$. By Lemma 4.6, we know that $O_1(P'_2) = X$ for any $P'_2 \in \tilde{\mathcal{S}}_m$. This means that player 2 is the dictator. Therefore, the domain $\tilde{\mathcal{S}}_m$ is dictatorial. ■

5 Generalized Circular Domain

In this section, we generalize the idea of a circular domain.

Definition 5.1. A regular domain $\mathcal{C}$ is a *generalized circular* domain if:

1. $\exists P, P' \in \mathcal{C}$ such that $r_1(P) = x_1$, $r_2(P) = x_2$ and $r_m(P) = x_m$ and $r_1(P') = x_1$ and $r_2(P') = x_m$.
2. $\exists P'', P''' \in \mathcal{C}$ such that $r_1(P'') = x_m$, $r_2(P'') = x_{m-1}$ and $r_m(P'') = x_1$ and $r_1(P''') = x_m$ and $r_2(P''') = x_1$.
3. $\exists \bar{P}, \tilde{P} \in \mathcal{C}$ such that $r_1(\bar{P}) = x_k$, $r_2(\bar{P}) = x_{k+1}$ and $x_m \bar{P} x_{k-1}$ and $r_1(\tilde{P}) = x_k$, $r_2(\tilde{P}) = x_{k-1}$ and $x_1 \tilde{P} x_{k+1}$ for $k \neq 1, m$.

Theorem 5.1. *The generalized circular domain $\mathcal{C}$ is a dictatorial domain.*

We will consider an example that would aid us to intuitively understand the generalized circular domain.

Example 5.1. Consider $X = \{a, b, c, d, e, f, g\}$ and the generalized circular domain $\mathcal{D} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}\}$ as illustrated below:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
a	**a**	**b**	**b**	**c**	**c**	**d**	**d**	**e**	**e**	**f**	**f**	**g**	**g**
b	**g**	**c**	**a**	**d**	**b**	**e**	**c**	**f**	**d**	**g**	**e**	**a**	**f**
c	d	e	f	**g**	f	**g**	**a**	b	**a**	c	**a**	e	d
f	e	**g**	e	f	**a**	a	g	c	c	b	**g**	c	b
d	c	d	d	e	e	b	b	**g**	b	d	d	d	e
e	f	**a**	g	a	g	f	f	a	g	a	b	b	c
g	**b**	f	**c**	**b**	**d**	**c**	**e**	**d**	**f**	**e**	c	**f**	**a**

Explain the example.

Definition 5.2. Consider any admissible domain $\mathcal{D}$. A social choice function $f : \mathcal{D}^n \to X$ is *monotone* if for any two profiles P_N and $P_N^{'} \in \mathcal{D}^n$ with $B(f(P_N), P_i) \supseteq B(f(P_N'), P_i) \forall i \in N$, we have $f(P_N) = f(P_N^{'})$.

Proposition 5.1. *A SCF f is strategy-proof only if f is monotone.*

Proof. See Reny (2001) for a proof of this proposition. ∎

Now we proceed to prove Theorem 5.1. Our first step in the proof, Proposition 2.1, reduces the dimension of the problem from an arbitrary number of individuals to two individuals and is of independent interest.

The following lemmas would aid us in proving this theorem.

Lemma 5.1. *Consider $P_2 \in \mathcal{C}$ such that $r_1(P_2) = x_k$. Then for all $l \in \{k-1, k+1\}$, $x_l \in O_1(P_2)$ implies $x_l \in O_1(P_2')$ for all $P_2' \in \mathcal{C}$ with $r_1(P_2') = x_k$.*

Proof. Assume for contradiction that there exists $P_2, P_2' \in \mathcal{C}$ with $r_1(P_2) = r_1(P_2') = x_k$ such that $x_l \in O_1(P_2)$ and $x_l \notin O_1(P_2')$ for some $l \in \{k-1, k+1\}$. Consider $P_1 \in \mathcal{C}$ such that $r_1(P_1) = x_l$ and $r_2(P_1) = x_k$. Then $f(P_1, P_2) = x_l$ and $f(P_1, P_2') = x_k$. This means player 2 manipulates at (P_1, P_2) via P_2'. ∎

Lemma 5.2. *Let P_2, P_2' be such that $r_1(P_2) = x_1$ and $r_1(P_2') = x_m$. We show $O_1(P_2) = \{x_1\}$ implies $O_1(P_2') = \{x_m\}$.*

Proof. Assume for contradiction that $O_1(P_2) = \{x_1\}$ and $O_1(P_2') \neq \{x_m\}$. Consider $\hat{P}_2, \bar{P}_2$ such that $r_1(\hat{P}_2) = x_1$, $r_2(\hat{P}_2) = x_m$ and $r_1(\bar{P}_2) = x_m$, $r_2(\bar{P}_2) = x_1$. Note that $O_1(P_2) = \{x_1\}$ implies $O_1(\hat{P}_2) = \{x_1\}$, and $O_1(P_2') \neq \{x_m\}$ implies $O_1(\bar{P}_2) \neq \{x_m\}$. We claim $O_1(\bar{P}_2) = \{x_1, x_m\}$. It is enough to show that $x_j \notin O_1(\bar{P}_2)$ for all $j \neq 1, 2$. Assume to the contrary that $x_j \in O_1(\bar{P}_2)$ for some $j \neq 1, m$. Consider P_1 such that $r_1(P_1) = x_j$. Then $f(P_1, \bar{P}_2) = x_j$ and $f(P_1, \hat{P}_2) = x_1$ which means player 2 manipulates at $(P_1, \bar{P}_2)$ via $\hat{P}_2$.

Next we claim that $x_{m-1} \in O_1(\tilde{P}_2)$ with $\tilde{P}_2$ such that $r_1(\tilde{P}_2) = x_m$ and $r_m(\tilde{P}_2) = x_{m-1}$. Suppose not. By Lemma 5.1, $x_1 \in O_1(\tilde{P}_2)$. Let P_1' be such that $r_1(P_1') = x_{m-1}$ and $r_m(P_1') = x_m$. Observe that $f(P_1', \tilde{P}_2) \in X \setminus \{x_m, x_{m-1}\}$. This means that player 2 will manipulate at $(P_1', \tilde{P}_2)$ via any preference with x_{m-1} at the top.

By Lemma 5.1, $x_{m-1} \in O_1(\bar{P}_2)$ which contradicts our initial assumption. ∎

Lemma 5.3. *Consider $P_2 \in \mathcal{C}$ such that $r_1(P_2) = x_1$, $r_2(P_2) = x_2$ and $r_m(P_2) = x_m$, or $r_1(P_2) = x_m$, $r_2(P_2) = x_{m-1}$ and $r_m(P_2) = x_1$. Then the option set $O_1(P_2)$ is either $\{r_1(P_2)\}$ or X*

Proof. We prove this lemma for the case where $r_1(P_2) = x_1$, $r_2(P_2) = x_2$ and $r_m(P_2) = x_m$. The proof for the case where $r_1(P_2) = x_m$, $r_2(P_2) = x_{m-1}$ and $r_m(P_2) = x_1$ is analogous.

Take $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_1$ and $r_2(\bar{P}_2) = x_m$. Suppose $O_1(\bar{P}_2)$ is not $\{x_1\}$. We show $x_m \in O_1(\bar{P}_2)$. Assume for contradiction that $x_m \notin O_1(\bar{P}_2)$. Consider $P_1 \in \mathcal{C}$ such that $r_1(P_1) = x_m$ and $r_m(P_1) = x_1$. Then $f(P_1, \bar{P}_2) = x_j$ for some $x_j \neq x_1, x_m$. Now consider $P_2' \in \mathcal{C}$ such that $r_1(P_2') = x_m$. By unanimity $f(P_1, P_2') = x_m$ which means player 2 manipulates at $(P_1, \bar{P}_2)$ via P_2'.

Now we show $x_m \in O_1(P_2)$. Suppose to the contrary that $x_m \notin O_1(P_2)$. Consider P_1 such that $r_1(P_1) = x_m$ and $r_2(P_2) = x_1$. Then $f(P_1, P_2) = x_1$ and $f(P_1, \bar{P}_2) = x_m$ where $\bar{P}_2$ is as defined in the above paragraph. This means player 2 manipulates at $(P_1, \bar{P}_2)$ via P_2.

By virtue of Lemma 5.2, $O_1(\bar{P}_2)$ is not $\{x_1\}$ means $O_1(P_2')$ is not $\{x_m\}$ where $r_1(P_2') = x_m$ and $r_m(P_2') = x_1$. By an analogous argument in the previous paragraph, $x_1 \in O_1(P_2')$.

We finally show that $O_1(P_2) = X$. We show this by induction. Assume that $x_{k+1} \in O_1(P_2)$ where $k \neq 1, m$. We show that $x_k \in O_1(P_2)$. First consider $\hat{P}_2$ such that $r_1(\hat{P}_2) = x_{k+1}$ and $r_2(\hat{P}_2) = x_k$. We show $x_k \in O_1(\hat{P}_2)$. Suppose not. As $x_1 \in O_1(P_2')$ and $x_1 = r_m(P_2')$, monotonicity implies that $x_1 \in O_1(\hat{P}_2)$. Consider $P_1 \in \mathcal{C}$ such that $r_1(P_1) = x_k$ and $x_1 P_1 x_{k+1}$. Then $f(P_1, \hat{P}_2) \in B(x_1, \hat{P}_2)$ (observe that $x_{k+1} \notin B(x_1, \hat{P}_2)$). However then player 2 manipulates at $(P_1, \hat{P}_2)$ via some preference that has x_k at the top. Now we show that $x_k \in O_1(P_2)$ as well. Suppose not. Consider P_1' such that $r_1(P_1') = x_{k+1}$ and $r_2(P_1') = x_k$. Then $f(P_1', P_2) = x_{k+1}$ and $f(P_1', \hat{P}_2) = x_k$, which means player 2 manipulates at $(P_1', \hat{P}_2)$ via P_2. ■

Lemma 5.4. *Suppose $O_1(P_2) = \{x_k\}$ for all P_2 with $r_1(P_2) = x_k$ for some k such that $1 < k < m$. Then $O_1(P_2') = \{x_{k+1}\}$ for P_2' such that $r_1(P_2') = x_{k+1}$ for some $k \in \{1, 2, \dots, m-1\}$.*

Proof. Assume for contradiction that $O_1(P_2) = \{x_k\}$ and $O_1(P_2') \neq \{x_{k+1}\}$. Using arguments similar to Lemma 5.2, we have $O_1(\bar{P}_2) = \{x_k, x_{k+1}\}$ where $r_1(\bar{P}_2) = x_{k+1}$ and $r_2(\bar{P}_2) = x_k$. This means that $x_k \in O_1(\hat{P}_2)$ where $\hat{P}_2$ is such that $r_1(\hat{P}_2) = x_{k+1}$ and $x_m \hat{P}_2 x_1$. Now consider $\tilde{P}_2$ where $r_1(\tilde{P}_2) = x_m$ and $r_2(\tilde{P}_2) = x_1$. By Lemma 5.2, $O_1(\tilde{P}_2) = \{x_m\}$. Let P_1 be such that $r_1(P_1) = x_k$ and $x_m P_1 x_1$. Then $f(P_1, \hat{P}_2) = x_k$ and $f(P_1, \tilde{P}_2) = x_m$. But this means player 2 manipulates at $(P_1, \hat{P}_2)$ via $\tilde{P}_2$. ■

Proof of Theorem 5.1. Consider P_2 where $r_1(P_2) = x_1$ and $r_2(P_2) = x_2$. By Lemma 5.3, we have $O_1(P_2)$ is singleton or X. Suppose $O_1(P_2)$ is singleton. Then due to Lemma 5.2 it follows $O_1(P_2')$ is singleton for all $P_2' \in \mathcal{C}$ and player 2 is a dictator.

Now suppose $O_1(P_2) = X$. We claim $O_2(P_1)$ is singleton where $r_1(P_1) = x_m$. Assume for contradiction that $x_k \in O_2(P_1)$ for some $k \neq m$. Take $\bar{P}_2$ such that $r_1(\bar{P}_2) = x_k$. Then $f(P_1, P_2) =$

x_m and $f(P_1, \bar{P}_2) = x_k$ which means player 2 manipulates at (P_1, P_2) via $\bar{P}_2$. Using symmetric argument this means $O_2(P_1)$ is a singleton for all $P_1 \in \mathcal{C}$ implying that player 1 is a dictator. ■

6 Conclusion

In this paper, we introduce the notions of a *maximal possibility cover* (MPC) of a domain and a *minimal dictatorial cover* (MDC) of a domain. We characterize the MPCs and MDCs of a celebrated possibility domain of preferences - the domain of *single peaked* preferences. We also generalize the dictatorship result in Sato (2010) by introducing the notion of a *generalized circular* domain and by proving that it is a dictatorial domain. Further we are working on characterizing the MPCs and MDCs of other popular domains of preferences such as the domain of single-dipped preferences and the domain of single-crossing preferences. The paper poses several open questions for future research. A very challenging but intriguing question would be to provide a general characterization of the MPCs and MDCs of any given domain of preferences. It is worthwhile to note that there is only one rule that is unanimous, strategy-proof and non-dictatorial in each of the MPCs of the single peaked domain. An interesting question to ask would be that whether, in general, can we claim that there is only one unanimous, strategy-proof and non-dictatorial rule in the MPC of a domain? We plan to tackle these issues in future research.

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Appendix

Appendix A - Novelty of our results

6.1 Novelty of our result

6.1.1 β-domain

Closely related to our work, Pramanik (9) offers a generalization of the linked domain - β-domain.

Definition 6.1. A pair of alternatives x_j, x_k is weakly connected, denoted by $x_j \overset{w}{\sim} x_k$, if there exists $A \subset X$ (possibly empty) and $P_i, \bar{P}_i, P_i^{'}, P_i^{''}$ such that:

1. $r_1(P_i) = r_1(\bar{P}_i) = x_j$ and $r_1(P_i^{'}) = r_1(P_i^{''}) = x_k$.

2. $A = M(x_j, x_k, P_i)$ and $A \subset W(x_k, \bar{P}_i)$.

3. $A = M(x_k, x_j, P_i^{'})$ and $A \subset W(x_k, P_i^{''})$.

Here $W(x_k, P_i) = \{y \in X | x_k P_i y\}$ and $M(x_j, x_k, P_i) = \{y \in X | x_j P_i y P_i x_k\}$.

Definition 6.2. Let $B \subset A$ and let $x_j \notin B$. Then x_j is linked to B if there exists $x_k, x_r \in B$ such that $x_j \overset{w}{\sim} x_k$ and $x_j \overset{w}{\sim} x_r$.

Definition 6.3. A domain $\mathcal{D}$ is a β-domain if there exists a one-to-one function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that:

1. $x_{\sigma(1)} \overset{w}{\sim} x_{\sigma(2)}$.

2. x_j is linked to $\{x_{\sigma(1)}, \ldots, x_{\sigma(j-1)}\}$.

Definition 6.4. A pair of alternatives x_j, x_k is strongly connected, denoted by $x_j \approx x_k$, if $x_j \overset{w}{\sim} x_k$ and for all $x_r \neq x_j, x_k$, there exists $P_i, P_i^{'}$ such that:

1. $r_1(P_i) = x_j$ and $x_r P_i x_k$.

2. $r_1(P_i^{'}) = x_k$ and $x_r P_i^{'} x_j$.

Fix a domain $\mathcal{D}$. Construct the graph $\bar{G}(\mathcal{D})$ with the set of vertices in $\bar{G}(\mathcal{D})$ is X and there is an edge $\{x_j, x_k\}$ iff $x_j \approx x_k$.

Definition 6.5. A domain $\mathcal{D}$ is a γ domain if $\bar{G}(\mathcal{D})$ is connected.

The following example illustrates that the generalized circular domain is not a β-domain.

Example 6.1. Consider $X = \{a, b, c, d, e\}$ and the domain $\mathcal{D} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}\}$ as illustrated below:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
a	a	b	b	c	c	d	d	e	e
b	e	c	a	d	b	e	c	d	a
c	d	d	c	e	a	c	b	c	d
d	c	e	d	b	d	b	a	b	c
e	b	a	e	a	e	a	e	a	b

It is easy to verify that $\mathcal{D}$ is a generalized circular domain. If $\mathcal{D}$ is a β-domain then $a \overset{w}{\sim} b$, $b \overset{w}{\sim} c$ but $a \overset{w}{\not\sim} c$ (there doesn't exist a set A according to the Definition 6.1). Therefore, $\mathcal{D}$ is not a β-domain and hence not a linked domain as well (as β-domain generalizes the notion of a linked domain). Also observe that for $d, e \in X$ there doesn't exist preferences in $\mathcal{D}$ which satisfies conditions 1-2 in Definition 6.4. Hence $\mathcal{D}$ is not a γ-domain as well.

6.1.2 Dictatorial Tops Only Domains

It is well known that dictatorial domains are tops-only domains but the converse is not always true. In the case of two agents, Chatterji and Sen (3) identifies a necessary property - which they call Property T - as necessary for a domain to be tops-only. They identify an additional property - Property T' - which makes a tops-only domain dictatorial.

The central notion behind their characterization is the notion of *connections* as introduced in Aswal et al. (1).

Definition 6.6. Fix a domain $\mathcal{D}$. We say that alternatives $x, y \in X$ are *connected* if there exist P_i , $P_i' \in \mathcal{D}$ such that $x = r_1(P_i) = r_2(P_i')$ and $y = r_2(P_i) = r_1(P_i')$.

According to the definition x and y are connected if there exists an admissible ordering where x and y are ranked first and second respectively and another ordering where y and x are ranked first and second respectively. If x and y are connected, we denote it by $x \sim y$.

Definition 6.7. The domain $\mathcal{D}$ satisfies Property T if $\forall P_i \in \mathcal{D}$ and $x \in X \setminus r_1(P_i)$ there exists $y \in X \setminus x$ such that yP_ix and $y \sim x$.

Property T requires the following. For any alternative in an admissible order (which is not the most preferred alternative of that order) there must exist another alternative which is better than it and to which it is connected. Now we turn to strengthening Property T to obtain impossibility results.

Definition 6.8. Let $B \subset A$ such that $m > |B| > 1$ and let $x \in B$. The domain $\mathcal{D}$ satisfies Property T' if there exists P_i, $P_i' \in \mathcal{D}$ and $y, z \in A$ such that

1. $y \in B, z \in A \setminus B$ and $y \sim z$
2. $r_1(P_i) = r_1(P_i') = x$, yP_iz and $zP_i'y$

Property T' expresses a reversality property. Pick a partition $(B, A \setminus B)$ of the set A such that B has at least two elements and $A \setminus B$ at least one and let $x \in B$. Then, there exists $y \in B$ and

$z \in A \setminus B$ which are connected and for which an appropriate reversal exists; in particular, there exists admissible orderings which have x as the peak and for which the preferences for y and z are reversed. Informally, every non-trivial partition of A must have a reversal.

Chatterji and Sen (3) states and proves the following result.

Theorem 6.1. *The domain that satisfies property T and property T' is a dictatorial domain.*

The following example illustrates that our result is independent from their characterization.

Example 6.2. Consider $X = \{a, b, c, d, e\}$ and the domain $\mathcal{D} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}\}$ as illustrated below:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
a	a	b	b	c	c	d	d	e	e
b	e	c	a	d	b	e	c	d	a
d	c	e	d	b	d	b	a	b	c
c	d	d	c	e	a	c	b	c	d
e	b	a	e	a	e	a	e	a	b

One can quickly observe that $\mathcal{D}$ is a generalized circular domain. The domain $\mathcal{D}$ violates Property T at all the preferences except for P_5 and P_6 in the case of the third ranked alternative. For instance, in the preference P_4 the third ranked alternative d is not connected to the first ranked alternative b as well as the second ranked alternative a. This means that our domain is an exception to their characterization of tops-only domains.