Equilibria in a Japanese English Auction with Discrete Bid Levels for the Wallet Game*

Ricardo Gonçalves[†] and Indrajit Ray[‡]

July 2015

Abstract

We consider the set up of a Japanese English Auction with exogenously fixed discrete bid levels for the Wallet Game. We prove that the standard (with continuous bid levels) equilibrium (bidding twice the private signal) is never an equlibrium in this set up. We show other cutoff equilibrium may exist and characterise such an equilibrium for a wallet game with two bid levels.

Keywords: English auctions, wallet game, discrete bid levels.

JEL Classification Numbers: D44; C72.

*We wish to thank seminar and workshop participants at Warwick and York for helpful comments.

[†]Faculdade de Economia e Gestão, Universidade Católica Portuguesa (Porto), Rua Diogo Botelho, 1327, 4169-005 Porto, Portugal. E-mail: rgoncalves@porto.ucp.pt.

[‡]*Author for Correspondences.* Economics Section, Cardiff Business School, Cardiff University, Colum Drive, Cardiff CF10 3EU, UK. E-mail: rayi1@cardiff.ac.uk; Fax: +44.2920.874419.

1 Introduction

Milgrom and Weber (1982) analysed a particular version of the English auction, the so-called Japanese English Auction (henceforth JEA) in which the price increases continuously and interested bidders must depress a button as long as they are prepared to buy the good for sale and release thereafter; the auction ends when all but one bidder release the button. Milgrom and Weber also have identified the equilibrium strategies in the JEA as the bidding limits for each participating bidder. Later, Klemperer (1998) focused on a common value auction, popularly known as the "Wallet game" as a special case of the model studied by Milgrom and Weber (1982) and illustrated the equilibrium of this game. Klemperer (1998) showed that bidding twice the value of the individual (private) signal forms the unique symmetric (Bayesian-Nash) equilibrium in this game.

Additionally, and differently from the previous literature, in the recent past, English auctions with predefined discrete bid levels have been analysed; in these English auctions, bidders have to choose among the exogenously fixed bid levels when it is their turn to bid (Rothkopf and Harstad 1994, David *et al* 2007) or at the very least, by that increment (Isaac et al., 2005).

Discrete bidding in English auctions are increasingly common in real world. Online auction sites, such as eBay, Yahoo or Amazon, use variants of such English auctions, adapted to the online world (Bajari and Hortaçsu 2004). Hence, not surprisingly, there is now a growing theoretical and experimental literature on this issue. Sinha and Greenleaf (2000) assumed discrete bidding in an independent private value English auction while Yu (1999) looked at different types of auction (first-price and second-price sealed bid, English and Dutch) with fixed bid increments. Cheng (2004) explored the relationship between discrete and continuous bidding increments in independent private value models. Gonçalves and Hey (2011) did an experiment to contrast the clock auction and an oral outcry auction with discrete and endogenous bidding in the Wallet game, following the seminal experiment by Avery and Kagel (1997) on a JEA based on the Wallet game (Klemperer, 1998).

In this short paper, we theoretically analyse the JEA in a common value environment with exogeneously specified discrete bids and fill a clear gap in the literature. This appears to be a more plausible assumption for some auctions, e.g., spectrum auctions, where jump bidding has been observed and extensively analyzed (e.g., Cramton, 1997) or online auctions where bidders may participate with the objective of reselling the good. Using a JEA in this way indeed is popular in the real world where bid levels are typically discrete. In auctions at Sotheby's or Christie's, bidding usually advances between 5% and 10% of the current price level (Rothkopf and Harstad, 1994). In Internet auctions, the auction sites usually restrict bid levels to be integers which vary according to the current price level (e.g., auctions at eBay, http://www.ebay.com). Cassady (1967) gives examples of auctions in which the bid levels are known, such as tobacco and livestock auctions in the USA.

We use the Wallet game as our background game and use a JEA with discrete bids. We first note that the usual JEA-equilibrium of bidding twice the private signal is not an equilibrium. We then show that, despite this, a cut-off equilibrium (inspired by cheap talk equilibria) may exist in a very simple setting with only two discrete bid levels. In this equilibrium, when bidders have a lower than the cut-off private signal they bid up to the low discrete bid level, whilst if their signal is higher than the cut-off, they bid up to the high bid level.

Although such an equilibrium borrows some similarities to the standard JEA equilibrium (e.g., the equilibrium strategies are weakly increasing), they differ from the latter in a significant way, namely by yielding an expected revenue always lower than that which would be obtained in a continuous bidding JEA. The rationale is relatively straightforward: in the presence of discrete bid levels, the cut-off equilibrium leads players to bid up to the lowest discrete bid level 'too' often, which reduces expected revenue compared to the continuous bidding JEA. They do this because whilst with continuous bid levels they can easily infer (from the equilibrium bid strategies) their opponent's signal and thus accurately calculate their payoff, with discrete bid levels such an accurate inference is no longer possible and bidding up to the low bid level more often provides a 'safety net' in the presence of such uncertainty.

The paper is structured in the following way. Section 2 describes the model, Section 3 presents the main results and Section 4 concludes. An appendix contains the detailed proofs of our results.

2 Model

For the sake of completeness, we first briefly describe the well-known models, namely the Wallet Game and the JEA, used as the background for our set up.

2.1 Wallet Game

We consider a common value auction model popularly known as the Wallet Game (Klemperer, 1998). There are two symmetric risk-neutral bidders $i \in \{1, 2\}$ who compete for the purchase of one single good, whose value, $\tilde{V}$, is common but *ex ante* unknown to both bidders. Each bidder privately receives an independent and uniformly distributed signal $x_i \sim U(0, 1)$, $i = 1, 2$. The common value of the good is simply the sum of the two signals: $\tilde{V} = x_1 + x_2$.

2.2 Japanese English Auction (JEA)

Milgrom and Weber's (1982) English auction is based on the rules of the Japanese auction (and we will refer to their model as a Japanese English auction). Bidders are gathered in a room and are told to depress a button if they are interested in the item for sale. The price, publicly posted on an electronic display, is raised continuously. Whoever is depressing the button at a given price level is actively bidding in the auction. If a bidder wants to drop out of the auction, all he has to do is release the button; his drop out price will then be displayed. The auction ends when only one bidder is active, who will pay a price equal to the drop out price of the penultimate bidder.

The symmetric equilibrium for the JEA yields equilibrium bid functions $b_i^*(x_i) = 2x_i$, $i = 1, 2$. This result has been derived by Klemperer (1998) and by Avery and Kagel (1997)[1]. It is a Nash equilibrium. Any of the players has no incentive whatsoever to deviate from it, given that the other player is playing that strategy. Suppose $x_1 > x_2$. The low signal bidder should bid $2x_2$. If he deviates, to win the auction, he will have to bid higher than $2x_1$, which in turn is higher than the good's true value, hence leaving him with a negative profit. The high signal bidder also has no incentive to bid less than $2x_1$ because this will have no influence on the price (it is a second-price auction). The price to pay will be $p = b_2^*(x_2) = 2x_2$, and the winning bidder's profits will be $\tilde{V}(x_1, x_2) - p = x_1 + x_2 - 2x_2 = x_1 - x_2 > 0$, because $x_1 > x_2$ by definition.

2.3 JEA with exogenously fixed discrete bid levels

We consider the Wallet game played within a JEA set up; however, we use the JEA with some exogenously fixed discrete bids. In our set up, as in the usual JEA, the price increases as the auction progresses; however the bid levels are discrete (rather than continuous) and fixed beforehand.

Formally, consider that bid levels are $A = \{a_1, ..., a_k\}$, with $0 < a_1 < ... < a_k < 2$, k finite which are common knowledge to bidders. A strategy

[1] Milgrom and Weber (1982) have derived the symmetric equilibrium in a general model; Klemperer (1998) has applied it to the 'Wallet Game'.

in this game is to choose (as in the standard JEA) a drop out price as a function of the individual signal.

Bidders are gathered in a room, and are told to depress a button if they are interested in the item for sale. The price, publicly posted on an electronic display, goes up in discrete commonly known bid levels. Whoever is depressing the button at a given bid level is actively bidding in the auction. If a bidder wants to drop out of the auction, all he has to do is release the button; his drop out price will then be displayed. The auction ends when only one bidder is active, who will pay a price equal to the drop out price of the penultimate bidder.

Therefore, we use weakly increasing strategies.

3 Results

We focus on symmetric equilibrium. We first check whether the bidding strategies equivalent to the JEA-equilibrium for the usual Wallet game form an equilibrium or not. A direct translation of the JEA bidding strategies into our setting would yield the following bidding functions: each bidder i should stay active in the auction until the bid reaches $b_i^*(x_i) = \tilde{V}(x_i, x_i) = 2x_i$ and drop after that. That is, each bidder must choose a bid level contained in A which determines his bidding limit. The associated bidding strategies would be for each bidder i to choose a bidding limit $a_{t_i} \in A$ such that $\frac{a_{t_i}}{2} \leq X_i < \frac{a_{t_i+1}}{2}$, $i = 1, 2$.

First, the equivalent to the JEA equilibrium bidding strategies in a discrete setting is not an equilibrium.

Proposition 1 *These (symmetric) bidding strategies are not an equilibrium.*

Next, we show that cut-off strategies may be used and, under some conditions, constitute an equilibrium.

3.1 Cut-off Equilibrium with two bid levels

Suppose only two discrete bid levels exist: L (low) and H (high) Bidders must decide when to drop out of the auction or, alternatively, what is their last active bid. Assume $L + 1/2 < H < \frac{3}{4} + \frac{L}{2}$ and $L < 1/2$ which in turn also implies $H < 1$.

Now consider the following strategies $S_i \in \{L, H\}$ when signals follow the uniform distribution, $X_i \sim U(0,1)$, where $x^* \in (0,1)$ is a (common knowledge) cut-off threshold:

$$S_i = \begin{cases} L \text{ if } x_i \leq x^* \\ H \text{ if } x_i > x^* \end{cases} \tag{1}$$

Proposition 2 *Under the above assumptions on the values of L and H, these are equilibrium strategies, with $x^* = \frac{2H-1}{2(1+L-H)}$.*

Finally, it is easy to show that the above equilibrium is unique in weakly increasing strategies. To show uniqueness, note that if player 1 believes player 2 will bid L with certainty, then player 1 will also bid L (that is, bidding L is a best response) if for all x:[2]

$$\begin{aligned} u_1(L;L) - u_1(H;L) &= \frac{1}{2}\left(x_1 + \frac{1}{2} - L\right) - \left(x_1 + \frac{1}{2} - H\right) > 0 \\ &\Leftrightarrow \ x_1 < 2H - L - 1/2 = 1 - 2\left(\frac{3}{4} + \frac{L}{2} - H\right) \end{aligned} \tag{2}$$

But $1 - 2(\frac{3}{4} + \frac{L}{2} - H) < 1$ and hence this implies that there are signal realisations for bidder 1 for which bidding L is not a best response. Therefore (L, L) cannot be an equilibrium.

Second, note that if player 1 believes player 2 will bid H with certainty, then player 1 will also bid H if if for all x:[3]

$$\begin{aligned} u_1(L;H) - u_1(H;H) &= 0 - \frac{1}{2}\left(x_1 + \frac{1}{2} - H\right) < 0 \\ &\Leftrightarrow \ x_1 > H - 1/2 \end{aligned} \tag{3}$$

[2] This is equivalent to setting $x^* = 1$.
[3] This is equivalent to setting $x^* = 0$.

But $H - 1/2 > L$ and this implies that there are signal realisations for bidder 1 for which bidding H is not a best response. Therefore (H, H) cannot be an equilibrium.

In this equilibrium, seller expected revenue is given by:

$$\begin{aligned} R &= (x^*)(x^*)L + (x^*)(1-x^*)H + (1-x^*)(x^*)H + (1-x^*)(1-x^*)H \\ &= \frac{L + 4LH - 4LH^2 + 3H - 4H^2 + 4HL^2}{4(1+L-H)^2} \end{aligned} \tag{4}$$

That is, seller expected revenue is given by L when both players play L (which in occurs with probability $(x^*)^2$) and H in all other cases (i.e., when at least one bidder bids H). Interestingly, for all values of L and H which satisfy the above restrictions, seller expected revenue is lower than in a Japanese English auction with continuous bid increments ($E\left[P^{JEA}\right] = 2/3$). Figure 1 displays this result, which is similar to that obtained by Rothkopf and Harstad (1994, Proposition, p. 575) in a private values setting: expected seller revenue is strictly lower than the second highest valuation and, as such, the auction with discrete bid levels yields 'lost revenue' compared to the continuous case.

4 Conclusion

We have shown that the standard Japanese English auction equilibrium with continuous bid levels is not an equilibrium in a setting where bid levels are discrete. Nevertheless, a cut-off equilibrium (similar in nature to cheap talk equilibria) exists and, under some conditions, is an equilibrium of this auction game. Under this cut-off equilibrium, seller expected revenue is strictly lower than that which he would obtain in a continuous JEA.

However, we focus in a very specific case with only two discrete bid levels. Whether a more general result can be obtained is certainly an interesting question. Also, under this cut-off equilibrium, the seller can clearly choose

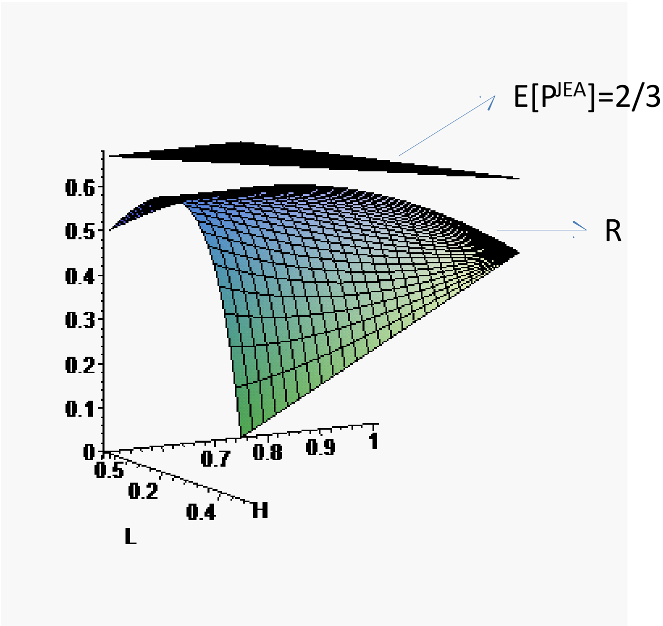

Figure 1: Seller expected revenue with discrete and continuous bid levels

the bid levels so as to maximize his expected revenue and thus obtain a second-best outcome. These are likely to be the next steps in our research.

References

[1] Avery, C. and Kagel, J. (1997), "Second-Price Auctions with Asymmetric Payoffs: An Experimental Investigation", *Journal of Economics and Management Strategy*, 6 (3), 573-603.

[2] Cheng, Harrison (2004), "Optimal auction design with discrete bidding", mimeo.

[3] Bajari, P. and Hortaçsu, A. (2004), "Economic Insights from Internet Auctions", *Journal of Economic Literature*, 42 (2), 457-486.

[4] Cassady, R., Jr. (1967), *Auctions and Auctioneering*, Berkeley: University of California Press.

[5] Cramton, P. (1997), "The FCC Spectrum Auctions: An Early Assessment", *Journal of Economics and Management Strategy*, 6, 431-495.

[6] David, E., Rogers, A., Jennings, N. R., Schiff, J., Kraus, S. and Rothkopf, M. H. (2007), "Optimal design of English auctions with discrete bid levels", *ACM Transactions on Internet Technology*, 7 (2), article 12.

[7] Gonçalves, R. and Hey, J. (2011), "Experimental evidence on English auctions: oral outcry vs. clock", *Bulletin of Economic Research,* 63 (4), 313-352.

[8] Isaac, M., Salmon, T. and Zillante, A. (2007), "A theory of jump bidding in ascending auctions", *Journal of Economic Behavior & Organization*, 62 (1), 144-164.

[9] Klemperer, Paul (1998), "Auctions with almost Common Values: The 'Wallet Game' and its Applications", *European Economic Review*, 42, 757-769.

[10] Milgrom, P. and Weber, R. (1982), "A Theory of Auctions and Competitive Bidding", *Econometrica*, 50, 1089-1122.

[11] Rothkopf, M. and Harstad, R. (1994), "On the Role of Discrete Bid Levels in Oral Auctions", *European Journal of Operational Research*, 74, 572-581.

[12] Sinha, Atanu and Greenleaf, Eric (2000), "The Impact of Discrete Bidding and Bidder Aggressiveness on Seller's Strategies in Open English Auctions: Reserves and Covert Shilling", *Marketing Science*, 19 (3), 244-265.

[13] Yu, J. 1999. Discrete approximation of continuous allocation´ mechanisms. Ph.D. thesis, California Institute of Technology, Division of Humanities and Social Science.

A Appendix - Proofs

Proof of Proposition 1

In order to show this, suppose $x_1 > x_2$ and take case 1 in Figure 2. If bidder 2 followed this bidding strategy, $t_2 = k - 1$, i.e., bidder 2 would be active at bid level a_{k-1} but would drop out at bid level a_k. If bidder 1 followed this bidding strategy, then $t_1 = k$, i.e., bidder 1 would be active at a_k but would drop out if the bidding proceeded to the next bid level. In this case, because bidder 2 drops out at a_k, then the auction ends at that bid level and this is the price bidder 1, the winning bidder, pays.

Given that bidder 2 follows this bidding strategy, there are signal realisations for bidder 1 for which this bidding strategy is not a best reply. Bearing in mind that $x_1 > x_2$ is assumed, note that bidder 1's payoff is given by:

$$\pi_1 = x_1 + E\left[X_2 | \frac{a_{t_2}}{2} \leq X_2 < \frac{a_{t_2+1}}{2}\right] - \frac{a_{t_2+1}}{2} \tag{5}$$

Bidder 1, by observing that he has won at bid level a_k, infers that bidder 2 has chosen $t_2 = k - 1$, i.e., bidder 2 has chosen a bidding limit of a_{k-1}, in which case bidder 2's signal (if he is following the above bidding strategy) belongs to the interval $\frac{a_{k-1}}{2} \leq X_2 < \frac{a_k}{2}$. In this case, bidder 1's profit is:

$$\begin{aligned}
\pi_1 &= x_1 + E\left[X_2 | \frac{a_{k-1}}{2} \leq X_2 < \frac{a_k}{2}\right] - \frac{a_k}{2} \\
&= x_1 + \frac{\frac{a_{k-1}}{2} + \frac{a_k}{2}}{2} - a_k \\
&= x_1 + \frac{a_{k-1}}{4} - \frac{3}{4}a_k
\end{aligned} \tag{6}$$

Now suppose that bidder 1's signal is 'low', i.e., it is located very close to the left of the $\left[\frac{a_{t_1}}{2}, \frac{a_{t_1+1}}{2}\right)$ interval; suppose, say, that $x_1 = a_{t_1}/2$. If he

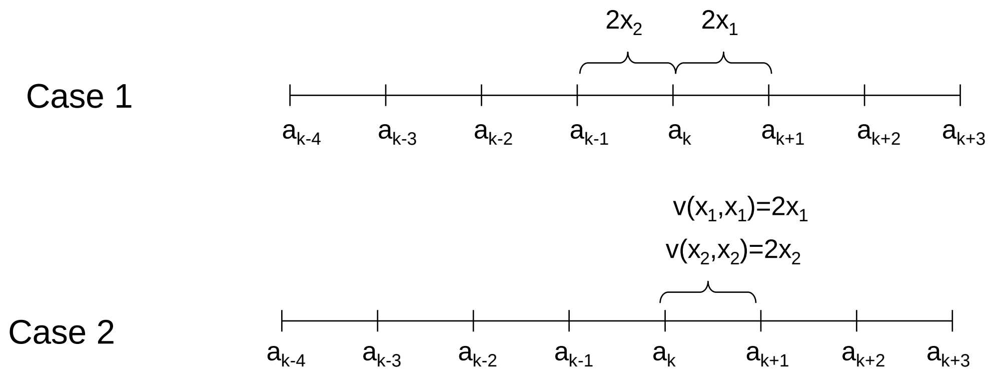

Figure 2:

were to follow the above strategy, bidder 1 would choose $t_1 = k$ and stay active until the bid reached a_k. Given bidder 2's strategy, he would win the auction. His payoff, however, would be negative:

$$\begin{aligned}\pi_1 &= x_1 + \frac{a_{k-1}}{4} - \frac{3}{4}a_k \\ &= \frac{a_k}{2} + \frac{a_{k-1}}{4} - \frac{3}{4}a_k \\ &= \frac{a_{k-1} - a_k}{4} < 0 \end{aligned} \tag{7}$$

In this case, bidder 1's signal realisation is 'too low' within the $\left[\frac{a_{t_1}}{2}, \frac{a_{t_1+1}}{2}\right)$ interval and given that he infers bidder 2's signal realisation from his bidding strategy, he will find the expected value of the good to be lower than a_k, thus yielding negative profits. An alternative interpretation is that in this case, bidder 1 would win at *too high a price*, which would result in negative profits. In this case, bidder 1 is better off if he deviates from the above strategy - which therefore cannot be an equilibrium strategy.

A similar reasoning holds for bidder 2: suppose bidder 1 follows the above strategy (choosing a bidding limit $a_{t_i} \in A$ such that $\frac{a_{t_i}}{2} \leq X_i < \frac{a_{t_i+1}}{2}$); does bidder 2 want to do the same? Take, again, case 1 in Figure 2. Bidder 2, conditional on winning, would find a_{k-1} to be *too low a price* and would rather deviate. Thus, both players have incentives for deviation. Therefore the JEA-equivalent bidding strategies are not an equilibrium.

Proof of Proposition 2

If players follow these strategies, their signal realisation (compared to x^*) dictates the strategy choice. For each bidder, conditional on their receiving a private signal x_i and under the assumption that the opponent follows the above strategy, payoffs are given by:

	L	H
L	$\frac{1}{2}\left(x_1+\frac{x^*}{2}-L\right);\frac{1}{2}\left(\frac{x^*}{2}+x_2-L\right)$	$0;\left(\frac{x^*}{2}+x_2-H\right)$
H	$\left(x_1+\frac{x^*}{2}-H\right);0$	$\frac{1}{2}\left(x_1+\frac{1+x^*}{2}-H\right);\frac{1}{2}\left(\frac{1+x^*}{2}+x_2-H\right)$

Note, in particular, that bidder 1's payoff for each strategy profile depends on his signal (x_1) and on the *expected signal* for bidder 2. When bidder 2 has a low signal ($x_2 \leq x^*$) and bids L, bidder 1 does not know the exact signal realisation and, for cutoff x^*, expects bidder 2 to have a signal realisation equal to $x^*/2$ (under the uniform signal distribution). The same is true when bidder 2 has a high signal ($x_2 > x^*$) and bids H: bidder 1 does not know the exact signal realisation and, for cutoff x^*, expects bidder 2 to have a signal realisation equal to $(1+x^*)/2$ (under the uniform signal distribution).

Assuming that bidder 2 follows the above strategy, he plays L with probability x^* (the probability that $x_2 \leq x^*$) and H with probability $(1-x^*)$, that is, bidder 2 plays $\sigma_2 = (x^*; 1-x^*)$. Bidder 1's expected payoffs are given by:

$$u_1(L;\sigma_2) = x^*\frac{1}{2}\left(x_1+\frac{x^*}{2}-L\right) \quad (8)$$

$$u_1(H;\sigma_2) = x^*\left(x_1+\frac{x^*}{2}-H\right)+(1-x^*)\frac{1}{2}\left(x_1+\frac{1+x^*}{2}-H\right) \quad (9)$$

Setting $u_1(L;\sigma_2) = u_1(H;\sigma_2)$, we obtain:

$$x^* = \frac{2x_1 + 1 - 2H}{2(H - L)} \tag{10}$$

That is, when $x_1 = x^*$, the two payoffs are equal provided:

$$x^* = \frac{2H - 1}{2(1 + L - H)} \tag{11}$$

Substituting this cutoff x^* in equations (8) and (9), we obtain:

$$\begin{aligned} u_1(L;\sigma_2) - u_1(H;\sigma_2) &= \frac{1}{4}\frac{2H - 1 - 2x_1(1 + L - H)}{1 + L - H} \\ &= \frac{1}{2}(x^* - x_1) \end{aligned} \tag{12}$$

If, for bidder 1, $x_1 > x^*$, we have $u_1(H;\sigma_2) > u_1(L;\sigma_2)$, that is, with a high signal realisation (above x^*), bidder 1 prefers to bid H, and when $x_1 \leq x^*$, we have $u_1(L;\sigma_2) > u_1(H;\sigma_2)$, that is, with a low signal realisation (below x^*), bidder 1 prefers to bid L.

In order to have $x^* \in (0, 1)$, we must have:

$$\begin{aligned} x^* &> 0 \Leftrightarrow H > 1/2 && (13) \\ x^* &< 1 \Leftrightarrow H < \frac{3}{4} + \frac{L}{2} && (14) \end{aligned}$$

In addition, payoffs cannot be negative (otherwise bidder 1 would prefer not to bid), that is, the participation constraint must be satisfied. Because $u_1(L;\sigma_2)$ is increasing in x_1, we need to ensure that:

$$u_1(L;\sigma_2)\big|_{x_1=0} = \frac{(1 - 2H)(1 + 2L)(2L + 1 - H)}{16(H - L - 1)^2} > 0 \tag{15}$$

The denominator is always positive. For the numerator to be positive we must have $H < 1/2$ and $H < L + 1/2$, which we disregard because it would not yield a positive cutoff x^*; or we must have $H > 1/2$ and $H > L + 1/2$. Combining this with the earlier restrictions, we must have

$L + 1/2 < H < \frac{3}{4} + \frac{L}{2}$ and $L < 1/2$ in order for payoffs to be positive and for the cutoff $x^* \in (0, 1)$. Under these conditions, the above strategies are an equilibrium in this game.