IMPLEMENTABILITY AND BALANCED IMPLEMENTABILITY OF SEQUENCING RULES

PARIKSHIT DE AND MANIPUSHPAK MITRA

ABSTRACT. We address the implementation issue for sequencing problems under incomplete information. We show that rules for which any agent's job completion time is non-increasing in own waiting costs, are implementable. We call such rules NI sequencing rules. We prove that any affine cost minimizer sequencing rule is an NI sequencing rule but the converse is not true. For two agent sequencing problems we identify the complete class of NI sequencing rules that are implementable with balanced transfers. For sequencing problems with more than two agents we identify a sufficient class of NI sequencing rules that are implementable with balanced transfers.

JEL Classifications: C78, C79, D02, D82;
Keywords: sequencing rules, implementation, affine cost minimizers, balanced implementation, group priority based cost minimizers.

1. INTRODUCTION

In this paper we address the implementability and balanced implementability issues for sequencing problems. In a sequencing problem we have a finite set of agents each of whom has one job to process using one facility. The facility can only handle one job at a time. Once the processing of a job starts, it cannot be interrupted. Each job is characterized by processing time and waiting cost. The waiting cost represents the agent's disutility for waiting one unit of time. There is a fair amount of literature on sequencing problems (see De and Mitra [3], Dolan [5], Duives, Heydenreich, Mishra, Muller and Uetz [6], Hain and Mitra [7], Mitra [16], Moulin [17] and Suijs [23]). Assuming that processing time of the agents are common knowledge and waiting costs are private information, we identify the class of sequencing rules that are implementable in dominant strategies. Any sequencing rule for which any agent's job completion time is non-increasing in his own waiting cost is implementable in dominant strategies. We call such rules NI sequencing rules. This result follows from the existing literature on implementation (see Bikhchandani, Chatterjee, Lavi, Mu'alem, Nisan and Sen [1], Rochet [20] and Rockafellar [21]). More importantly, for any given NI sequencing rule,

The authors would like to thank Debasis Mishra, Suresh Mutuswami and Arunava Sen for helpful comments and suggestions.

we identify all direct mechanisms that implement it. We refer to such mechanisms as cut-off based mechanisms. Such cut-off based mechanisms were derived for scheduling problems with multiple machines and varying speed by Mishra and Mitra [12] and for multi-dimensional dichotomous domains by Mishra and Roy [14].

A classic result in mechanism design in quasi-linear set-up is the Roberts' affine maximizer theorem (see Roberts [19]) for multidimensional type spaces with finite set of alternatives. Roberts [19] showed that if there are at least three alternatives and the type space is unrestricted, then every onto implementable allocation rule is an affine maximizer. There are many papers that analyze the affine maximizer allocation rules for different allocation problems (see Carbajal, Mc-Lennan, and Tourky [2], Dobzinski and Nisan [4], Lavi, Mu'alem, and Nisan [10], Marchant and Mishra [11], Mishra and Quadir [13], Mishra and Sen [15] and Nath and Sen [18]). Sequencing problems deal with agents' cost and hence the appropriate transformed concept of affine maximizer allocation rule is the affine cost minimizer sequencing rule. In any affine cost minimizer sequencing rule, the objective is to select that order of servicing the agents (from the set of all possible orders of servicing or from some subset of it) so as to minimize the sum of an *order specific number* and the *weighted* sum of job completion time of the agents. This order specific numbers are captured by a function κ which maps from the set (or subset) of orders to the real line and we call them the *κ-functions*. All the agent specific weights are non-negative real numbers. With our restricted domain for the sequencing problem, Roberts' affine maximizer theorem (see Roberts [19]) is not true as we prove that any affine cost minimizer sequencing rule (whether onto or not onto) is an NI sequencing rule but the converse is not true. Specifically, for any sequencing problem with a given number of agents, we provide an example of NI sequencing rule that is not an affine cost minimizer.[1] That under different domain restrictions we can have implementable rules that are different from affine maximizers was also pointed out by Carbajal, Mc-Lennan, and Tourky [2], Marchant and Mishra [11] and Mishra and Quadir [13].

We then try to identify NI sequencing problems that are implementable with balanced transfers. For the sequencing problem, implementing any NI sequencing rule with balanced transfer simply ensures that the resulting utility allocation is Pareto indifferent to the utility under the sequencing rule in the absence of private information and with zero monetary transfers. For many economic environments, implementing

[1] Roberts' [19] result uses affine maximizers that are onto. Hence our result shows that class of affine cost minimizer sequencing rules, which is a generalization of Roberts' [19] class of affine maximizer allocation rules, is a strict subset of the class of NI sequencing rules.

outcome efficiency with balanced transfers is not possible (see Hurwicz [8], Hurwicz and Walker [9] and Walker [24]). However, for sequencing problems with more than two agents, it is possible to implement the outcome efficient sequencing rule with balanced transfers (see Mitra[16] and Suijs [23]) and it is also possible to implement the just sequencing rule (in the Rawlsian sense) with balanced transfers (see De and Mitra [3]). Our work establishes that there are NI sequencing rules other than the outcome efficient sequencing rule and the just sequencing rule that are implementable with balanced transfers. One obvious type of NI sequencing rules that are implementable with balanced transfers are the constant sequencing rules like the shortest processing time sequencing rule (where the shortest jobs are handled first) and the longest processing time sequencing rule (where the longer jobs are often very important and are selected first).

For sequencing problems with two agents we identify the complete class of non-constant NI sequencing rules that are implementable with balanced transfers. Specifically we show that there are exactly two types of NI sequencing rules that are implementable with balanced transfers. The first type are onto affine cost minimizers (that includes neither the outcome efficient sequencing rule nor the just sequencing rule) and the second type are NI sequencing rules that are not affine cost minimizers.

For sequencing problems with more than two agents we identify a sufficient family of NI sequencing rules that are implementable with balanced transfers. This sufficient family of NI sequencing rules include a subset of affine cost minimizer sequencing rules with constant κ-functions (normalized to zero) and also includes a subset of NI sequencing rules that are not affine cost minimizers. We refer to this family of rules as *group priority based cost minimizer* (GP-CM) sequencing rules. These sequencing rules are defined by imposing all types of priority based partition on the set of agents. This family includes the following types of sequencing rules.

1. The family of GP-CM sequencing rules includes priority based partition where all elements of the partition are singletons so that all constant sequencing rules are included.
2. The GP-CM sequencing rules also includes grand coalition as a partition and hence includes all affine cost minimizer sequencing rules for which the agent specific weights are positive and the κ-functions are constant (and the value of the constant is normalized to zero). Hence the outcome efficient sequencing rule (see Mitra [16] and Suijs [23]) and the just sequencing rule (in the Rawlsian sense) (see De and Mitra [3]) are also members of this family of GP-CM sequencing rules.

(3) There are sequencing situations where we have a well-defined priority across the set of agents. In an academic institute, faculty members may be given priority over students in using computers (or printers or photocopiers). A similar situation rises for emergency related treatment of patients where priority in treatment needs to be given based on the degree of emergency of the patients' diseases. When the number of agents to be served is known, all such situations are captured under GP-CM sequencing rules.

(4) The non-affine cost minimizers NI sequencing rules included in the family of GP-CM sequencing rules are a generalization of the affine cost minimizer sequencing rules included in this family of GP-CM sequencing rules. This generalization is done replacing any subset of agents' waiting cost with a non-linear function of the waiting cost which is increasing and onto.

The paper is organized as follows. In Section 2, we introduce the framework. In Section 2, we identify the class of all implementable sequencing rules and show that all affine cost minimizers are a strict subset of the class of implementable sequencing rules. In Section 3, we identify the class of cut-off based mechanisms that implement any NI sequencing rule. In Section 4, obtained results on implementability with balanced transfers. This is followed by an appendix where we provide the proofs of the results.

2. THE FRAMEWORK

Consider a finite set of agents $N = \{1, 2, \dots, n\}$ in need of a facility that can be used sequentially. Using this facility, the agents want to process their jobs. The job processing time can be different for different agents. Specifically, for each agent $i \in N$, the job processing time is given by $s_i > 0$. Let $\theta_i S_i$ measure the cost of job completion for agent $i \in N$ where $S_i \in \mathbb{R}_{++}$ is the job completion time for this agent and $\theta_i \in \Theta := \mathbb{R}_{++}$ denotes his constant per-period waiting cost where $\mathbb{R}_{++}$ is the positive orthant of the real line $\mathbb{R}$. Due to the sequential nature of providing the service, the job completion time S_i for agent i depends not only on his own processing time s_i but also on the processing time of the agents who precedes him in the order of service. By means of an order $\sigma = (\sigma_1, \dots, \sigma_n)$ on N, one can describe the positions of each agent in the order. Specifically, $\sigma_i = k$ indicates that agent i has the k-th position in the order. Let $\Sigma(N)$ be the set of $n!$ possible orders on N. We define $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j < \sigma_i\}$ to be the predecessor set of i in the order σ, that is, set of agents served before agent i in the order σ. Similarly, $P'_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j > \sigma_i\}$ denotes the successor set of i in the order σ, that is, set of agents served after agent i in the order σ. Let $s = (s_1, \dots, s_n) \in \mathcal{S} := \mathbb{R}^n_{++}$ denote the vector of processing time of the agents. Given a vector $s = (s_1, \ldots, s_n) \in \mathcal{S}$ and an order $\sigma \in \Sigma(N)$, the cost of job completion for agent $i \in N$ is $\theta_i S_i(\sigma)$, where the job completion time is $S_j(\sigma) = \sum_{j \in P_i(\sigma)} s_j + s_i$. The agents have quasi-linear utility of the form $U_i(\sigma, \tau_i; \theta_i); s_{-i}) = -\theta_i S_i(\sigma) + \tau_i$ where σ is the order, $\tau_i \in \mathbb{R}$ is the transfer that he receives and the parameter of the model θ_i that constitutes of the waiting cost θ_i. If the processing time vector $s \in \mathcal{S}$ is given and waiting cost is private information, then we have a sequencing problem $\Omega_N^s = (\Theta^n, s)$.

A typical profile of waiting costs is denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$. For any $i \in N$, let θ_{-i}, denote the profile $(\theta_1 \ldots \theta_{i-1}, \theta_{i+1}, \ldots \theta_n) \in \Theta^{|N \setminus \{i\}|}$ which is obtained from the profile θ by eliminating i's waiting cost where for any set X, $|X|$ denotes the cardinality of X. For a given sequencing problem Ω_N^s, a (direct revelation) mechanism is (σ, τ) that constitutes of a sequencing rule σ and a transfer rule τ. A *sequencing rule* is a function $\sigma : \Theta^n \to \Sigma(N)$ that specifies for each profile $\theta \in \Theta^n$ a unique order $\sigma(\theta) = (\sigma_1(\theta), \ldots, \sigma_n(\theta)) \in \Sigma(N)$. Because the sequencing rule is a function (and not a correspondence) we will require tie-breaking rule to reduce a correspondence to a function which, unless explicitly discussed, is assumed to be fixed. We use the following tie-breaking rule. We take the linear order $1 \succ 2 \succ \ldots \succ n$ on the set of agents N. For any sequencing rule σ and any profile $\theta \in \Theta^n$ with a tie situation between agents $i, j \in N$, we pick the order $\sigma(\theta)$ with $\sigma_i(\theta) < \sigma_j(\theta)$ if and only if $i \succ j$. A *transfer rule* is a function $\tau : \Theta^n \to \mathbb{R}^n$ that specifies for each profile $\theta \in \Theta^n$ a transfer vector $\tau(\theta) = (\tau_i(\theta), \ldots, \tau_n(\theta)) \in \mathbb{R}^n$. Specifically, given any sequencing problem Ω_N^s and given any mechanism (σ, τ), if (θ_i', θ_{-i}) is the announced profile when the true waiting cost of i is θ_i, then utility of i is $U_i(\sigma(\theta_i', \theta_{-i}), \tau_i(\theta_i', \theta_{-i}); \theta_i) = -\theta_i S_i(\sigma(\theta_i', \theta_{-i}) + \tau_i(\theta_i', \theta_{-i})$.

3. Implementability criterion for sequencing rules

Definition 1. A mechanism (σ, τ) *implements* the sequencing rule σ in dominant strategies if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i, \theta_i' \in \Theta$ and any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$,

$$U_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq U_i(\sigma(\theta_i', \theta_{-i}), \tau_i(\theta_i', \theta_{-i}); \theta_i). \tag{1}$$

Implementation of a rule σ via a mechanism (σ, τ) requires that the transfer rule τ is such that truthful reporting for any agent weakly dominates false report irrespective of other agents' report.

Definition 2. A sequencing rule σ satisfies *non-increasingness* (or NI) if for any $i \in N$ and any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, the chosen order $\sigma(\theta_i, \theta_{-i})$ for each $\theta_i \in \Theta$ is such that the job completion time $S_i(\sigma(\theta_i, \theta_{-i}))$ is non-increasing in θ_i.

Proposition 1. *If a sequencing rule σ is implementable, then it is an NI sequencing rule.*

We do not prove the converse, that is, if we have an NI sequencing rule σ, then there exists a mechanism that implements it. Note that non-increasingness is the weak monotonicity (or two-cycle monotonicity) for the sequencing problems. From Bikhchandani, Chatterjee, Lavi, Mu'alem, Nisan and Sen [1] we know that, for any deterministic rule (like the sequencing rules we have for Ω_N^s), weak monotonicity is sufficient for implementation in dominant strategies. Hence the converse is also true. In the next section we derive the complete class of mechanisms that implement any NI sequencing rule.

One can construct many examples of NI sequencing rules.

Definition 3. A sequencing rule $\bar{\sigma}$ is a *constant sequencing rule* if there is a fixed order $\bar{\sigma} \in \Sigma(N)$ such that the agents are always served in this fixed order $\bar{\sigma}$, that is, for any $\theta \in \Theta^n$, $\sigma(\theta) = \bar{\sigma}$.

There are many priority rules that are constant sequencing rules. For the constant sequencing rule with $\bar{\sigma}$ as the state independent order, for each $i \in N$ and for any given $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, the completion time of agent i is fixed at $S_i(\bar{\sigma}) = s_i + \sum_{j \in P_i(\bar{\sigma})} s_j$ for all $\theta_i \in \Theta$ implying non-increasingness in θ_i. Hence it satisfies NI. Two other NI sequencing rules from the existing literature on sequencing problems are the following.

Definition 4. A sequencing rule σ^* is *outcome efficient* if for any profile $\theta \in \Theta^n$, $\sigma^*(\theta) \in \operatorname{argmin}_{\sigma \in \Sigma(N)} \sum_{i \in N} \theta_i S_i(\sigma)$.

For each profile the outcome efficient sequencing rule selects an order to minimize the aggregate cost of completion time. Define $u_i := \theta_i / s_i$ as the urgency index of agent i which is the ratio of his waiting cost and his processing time. From Smith [22] we know that for any sequencing problem Ω_N^s a sequencing rule σ^* is outcome efficient if and only if the following condition holds.

- **(OE)** For any profile $\theta \in \Theta^n$, the selected order $\sigma^*(\theta)$ satisfies the following condition: for any $i, j \in N$, $\theta_i / s_i \geq \theta_j / s_j \Leftrightarrow \sigma_i^*(\theta) \leq \sigma_j^*(\theta)$.

Clearly, outcome efficient sequencing rule σ^* is NI. Outcome efficiency and incentives has been extensively analyzed in the sequencing literature (see Mitra [16], Suijs [23] and De and Mitra [3]).

Definition 5. A sequencing rule $\tilde{\sigma}$ is *just* if for each profile $\theta \in \Theta^n$, the chosen order $\tilde{\sigma}(\theta)$ satisfies the following property: for any $i, j \in N$ such that $\theta_i \geq \theta_j$, $\tilde{\sigma}_i(\theta) \leq \tilde{\sigma}_j(\theta)$.[2]

[2]Given the tie-breaking rule, for any profile $\theta \in \Theta^n$, both the selections $\tilde{\sigma}(\theta)$ for the just sequencing rule and $\sigma^*(\theta)$ for the outcome efficient sequencing rule satisfy profile contingent uniqueness.

Just sequencing rule was analyzed in De and Mitra [3]. Clearly, the just sequencing rule $\tilde{\sigma}$ is NI. The constant sequencing rule, the outcome efficient sequencing rule and the just sequencing rule are all affine cost minimizer sequencing rules.

Definition 6. A sequencing rule $\sigma^{w,\kappa} : \Theta^n \to \Sigma(N)$ is an *affine cost minimizer* (ACM) if for each $\theta \in \Theta^n$, $\sigma^{w,\kappa}(\theta) \in \arg\min_{\sigma \in \Sigma'(N)} \left\{\kappa(\sigma) + \sum_{j\in N} w_j \theta_j S_j(\sigma)\right\}$, where $\Sigma'(N) \subseteq \Sigma(N)$, $w_j \geq 0$ for all $j \in N$ and $\kappa : \Sigma'(N) \to \mathbb{R}$.

The next two examples are NI sequencing rules that are not ACM.

Example 1. Consider any sequencing problem Ω^s_N with $|N| = 2$. Define the sequencing rule σ^V such that, given any two positive numbers a_1 and a_2, it satisfies the following: For any profile $\theta = (\theta_1, \theta_2)$ such that $\theta_1 < a_1$ and $\theta_2 > a_2$, $\sigma^V(\theta) = (\sigma^V_1(\theta) = 2, \sigma^V_2(\theta) = 1)$. For all other profiles $\theta' = (\theta'_1, \theta'_2)$ such that either $\theta'_1 \geq a_1$ or $\theta'_2 \leq a_2$, $\sigma^V(\theta') = (\sigma^V_1(\theta') = 1, \sigma^V_2(\theta') = 2)$.

One can easily verify that σ^V is NI. If $\theta''_2 \leq a_2$, then for any $\theta_1 \in \Theta$, $\sigma^V_1(\theta_1, \theta''_2) = 1$ and hence $S_1(\sigma^V(\theta_1, \theta_2)) = s_1$ is non-increasing in θ_1 for any given $\theta_2 \geq a_2$. If $\theta'_2 > a_2$, then for any $\theta_1 \in (0, a_1)$, $\sigma^V_1(\theta_1, \theta'_2) = 2$ and agent 1's completion time is $S_1(\sigma^V(\theta_1, \theta'_2)) = s_2 + s_1$ and for any $\theta'_1 \geq a_1$, $\sigma^V_1(\theta'_1, \theta'_2) = 1$ and agent 1's completion time is $S_1(\sigma^V(\theta'_1, \theta'_2)) = s_1$. Hence, we have non-increasingness of completion time $S_1(\sigma^V(\theta_1, \theta'_2))$ in θ_1 for any given $\theta'_2 > a_2$. Similarly, if we fix $\theta''_1 \geq a_1$, then for any $\theta_2 \in \Theta$, $\sigma^V_2(\theta''_1, \theta_2) = 2$ and hence $S_2(\sigma^V(\theta''_1, \theta_2)) = s_1 + s_2$ is non-increasing in θ_2 for any given $\theta''_1 \geq a_1$. If $\theta_1 < a_1$, then for any $\theta_2 \in (0, a_2]$, $\sigma^V_2(\theta_1, \theta_2) = 2$ and agent 2's completion time is $S_2(\sigma^V(\theta_1, \theta_2)) = s_2 + s_1$. For any $\theta'_2 > a_1$, $\sigma^V_2(\theta_1, \theta'_2) = 1$ and agent 2's completion time is $S_2(\sigma^V(\theta_1, \theta'_2)) = s_2$. Hence $S_2(\sigma^V(\theta_1, \theta_2))$ is non-increasing in θ_2 for any given $\theta_1 < a_1$. That σ^V is not an ACM sequencing rule is established in the next proposition.

Example 2. Consider any sequencing problem Ω^s_N with $|N| \geq 3$. Define the sequencing rule σ^{NA} that satisfies the following properties:

(1) For any profile such that the urgency index of agent 1 is no smaller than the smallest urgency index of all other agents, agent 1 is served first and all other agents are served, after agent 1 completes his jobs, in the non-increasing order of their urgency indexes. Formally, let θ be a profile such that $\theta_1/s_1 \geq \min_{j \in N\setminus\{1\}}(\theta_j/s_j)$. Then $\sigma^{NA}(\theta)$ specifies that $1 = \sigma^{NA}_1(\theta) < \sigma^{NA}_j(\theta)$ for any $j \in N \setminus \{1\}$, and, for any $j, k \in N \setminus \{1\}$, $\sigma^{NA}_j(\theta) \leq \sigma^{NA}_j(\theta)$ if and only if $(\theta_j/s_j) \geq (\theta_k/s_k)$.

(2) For any profile such that the urgency index of agent 1 is smaller than the smallest urgency index of all other agents, agent 1 is served last and all other agents are served, before agent 1, according to the non-increasing order of their urgency indexes. Formally, let θ' be a profile such that $\theta'_1/s_1 < \min_{j\in N\setminus\{1\}}(\theta'_j/s_j)$. Then $\sigma^{NA}(\theta')$ specifies that $n = \sigma_1^{NA}(\theta') > \sigma_j^{NA}(\theta')$ for any $j \in N \setminus \{1\}$, and, for any $j,k \in N \setminus \{1\}$, $\sigma_j^{NA}(\theta') \leq \sigma_k^{NA}(\theta')$ if and only if $(\theta'_j/s_j) \geq (\theta'_k/s_k)$.

It is quite easy to see that the sequencing rule σ^{NA} satisfies NI. That σ^{NA} is not an affine cost minimizer will be established in the next proposition.

Let $NI(\Omega_N^s)$ denote the set of all NI sequencing rules and $ACM(\Omega_N^s)$ denote the set of all affine cost minimizer sequencing rules for any given sequencing problem Ω_N^s.

Proposition 2. *For any Ω_N^s, $ACM(\Omega_N^s) \subseteq NI(\Omega_N^s)$ and $ACM(\Omega_N^s) \neq NI(\Omega_N^s)$.*

From Roberts [19] we know that the class of mechanisms that implements any sequencing rule $\sigma^{w,\kappa} \in ACM(\Omega_N^s)$ is the generalized VCG mechanisms.

Definition 7. For any given $\sigma^{w,\kappa} \in ACM(\Omega_N^s)$, a mechanism $(\sigma^{w,\kappa}, \tau^{w,\kappa})$ is a *generalized VCG mechanism* if the transfer rule is such that for all $\theta \in \Theta^n$ and all $i \in N$,

$$\tau_i^{w,\kappa}(\theta) = \begin{cases} q_i(\theta_{-i}) & \text{if } w_i = 0, \\ q_i(\theta_{-i}) - \frac{1}{w_i}\left[\kappa(\sigma^{w,\kappa}(\theta)) + \sum\limits_{j\in N\setminus\{i\}} w_j\theta_j S_j(\sigma^{w,\kappa}(\theta))\right] & \text{if } w_i > 0, \end{cases} \tag{2}$$

where the function $q_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$ is arbitrary.

Given Proposition 2 it follows that the set of all implementable NI sequencing rules is a strict super set of the set of ACM sequencing rules. Hence, Roberts' [19] generalized VCG mechanisms that can implement any ACM sequencing rule is not defined for NI sequencing rules that are not ACM sequencing rules. In the next section, we derive the set of all mechanisms that implement any given NI sequencing rule.

4. Mechanisms implementing NI sequencing rules

Given a processing time vector $s \in \mathcal{S}$ and the sequencing problem Ω_N^s, consider an agent $i \in N$. Depending on his waiting cost $\theta_i \in \Theta$, agent i can face a maximum of 2^{n-1} (specifically, $\sum_{j=0}^{n-1}\binom{n-1}{j}$) different job completion times. But the number of different job completion time that any agent i actually faces depends on the profile $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$ and, more importantly, on the underlying sequencing rule whose implementation is under consideration. So depending on the sequencing rule and the profile θ_{-i}, the agent can face a single job completion time (like in the constant sequencing rule) or more than one job completion time (like in any outcome efficient sequencing rule).

Consider any $\sigma \in NI(\Omega_N^s)$ and consider an agent $i \in N$. Fix a profile θ_{-i} and let us assume that the number of different job completion time that agent i faces, as θ_i varies over Θ, is $T \in \{1, \ldots, 2^{n-1}\}$. Given that σ is NI, this means that either $T = 1$ or $T \geq 2$ and there exists a waiting cost cut-off vector $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$ where $0 := \theta_i^{(T)} < \theta_i^{(T-1)}(\theta_{-i}) < \ldots < \theta_i^{(2)}(\theta_{-i}) < \theta_i^{(1)}(\theta_{-i}) < \theta_i^{(0)} := \infty$ such that for any $t \in \{1, \ldots, T\}$, $S_i(\sigma(\theta_i^t, \theta_{-i})) := \bar{S}(t, \theta_{-i})$ for all $\theta_i^t \in (\theta_i^{(t)}, \theta_i^{(t-1)})$. Observe that if $T = 1$, then $\theta_i^{(1)} = \theta_i^{(T)} = 0$ implying that for all $\theta_i \in \mathbb{R}_{++}$ the completion time of agent i remains unchanged. Define $D_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - \bar{S}(t, \theta_{-i})$ and $\overline{D}_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - S_i(\sigma(\theta_i^{(t)}, \theta_{-i}))$ for any $t \in \{1, \ldots, T-1\}$. Observe that the difference in the definitions of $D_t(\theta_{-i})$ and $\overline{D}_t(\theta_{-i})$ lies in the second term. While for the $D_t(\theta_{-i})$ case, $\bar{S}(t, \theta_{-i})$ is the completion time of agent i when his waiting cost is any number θ_i^t that lies in the open interval $(\theta_i^{(t)}, \theta_i^{(t-1)})$ and for the $\overline{D}_t(\theta_{-i})$ case, $S_i(\sigma(\theta_i^{(t)}, \theta_{-i}))$ is the completion time of agent i when his waiting cost is exactly $\theta_i^{(t)}$ which is a cut-off point. Depending on the tie-breaking rule, the numbers $\bar{S}(t, \theta_{-i})$ and $S_i(\sigma(\theta_i^{(t)}, \theta_{-i}))$ may or may not be different and hence for completeness of the analysis, the distinction between $D_t(\theta_{-i})$ and $\overline{D}_t(\theta_{-i})$ is necessary.

Definition 8. Consider any $\sigma \in NI(\Omega_N^s)$ and a mechanism (σ, τ) with transfer rule $\tau : \Theta^n \to \mathbb{R}^n$. The mechanism is *cut-off based* if the transfer rule τ is obtained from the following procedure. For each $i \in N$, we first select any function $h_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ and then, given any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, we consider the waiting cost cut-off vector $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$ where $0 := \theta_i^{(T)} < \theta_i^{(T-1)}(\theta_{-i}) < \ldots < \theta_i^{(2)}(\theta_{-i}) < \theta_i^{(1)}(\theta_{-i}) < \theta_i^{(0)} := \infty$. Given the selected function $h_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$, for any profile θ_{-i} of all but agent i and the associated cut-off vector $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$, the transfer of agent i is the following:

(PI1) For any $\theta_i \in \Theta \setminus \{\theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i})\}$, $\tau_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i, \theta_{-i})$ where

(3)
$$I_i(\theta_i, \theta_{-i}) = \begin{cases} 0 & \text{if } \theta_i \in (\theta_i^{(T)}, \theta_i^{(T-1)}(\theta_{-i})), \\ \sum\limits_{r=t}^{T-1} \theta_i^{(r)}(\theta_{-i}) D_r(\theta_{-i}) & \text{if } \theta_i \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i})),\ t = \{1, \ldots, T-1\} \ \&\ T \geq 2. \end{cases}$$

(PI2) For $T \geq 2$, any $t \in \{1, \ldots, T-1\}$ and cut-off point $\theta_i^{(t)}(\theta_{-i})$, $\tau_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i})$ where the incentive payment $I_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i}) = I_i(\theta_i^t, \theta_{-i}) - \theta_i^{(t)}(\theta_{-i})\overline{D}_t(\theta_{-i}) + \theta_i^{(t)}(\theta_{-i})D_t(\theta_{-i})$ and $\theta_i^t \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i}))$.

Definition 8 specifies the following. For each agent i and each profile θ_{-i} of waiting costs of all but agent i, we get a set of cut-off points $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$ for agent i that depends on the specific NI sequencing rule. The transfers associated with cut-off based mechanism requires that each agent i gets an agent specific constant $h_i(\theta_{-i})$ that depends on the waiting cost of all other agents and, if agents i's waiting cost θ_i is greater than the smallest non-zero cut off value $\theta_i^{(T-1)}(\theta_{-i})$, agent i also has to make an *incentive payment* $I_i(\theta_i, \theta_{-i})$ that depends of the set of cut-off values that are less than the waiting costs of agent i. For each such cut-off value $\theta_i^{(r)}(\theta_{-i})$, agent i pays $\theta_i^{(r)}(\theta_{-i})D_r(\theta_{-i})$ which is the cut-off value times the absolute difference between the job completion time of agent i below and above this cut-off value. If agent i's waiting cost coincides with a cut-off point then, ceteris paribus, his incentive payment needs to be adjusted by changing the difference in completion time term $D_r(\theta_{-i})$ to the difference in completion time below and at the cut-off point $\overline{D}_r(\theta_{-i})$ only for the highest cut-off value less than the waiting cost of agent i. Whenever the dependence of the cut-off points of agent i for any given θ_{-i} is clear, we will write the cut-off vector as $(\theta_i^{(0)}, \theta_i^{(1)}, \ldots, \theta_i^{(T-1)}, \theta_i^{(T)})$ instead of $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$.

Theorem 1. *Any $\sigma \in NI(\Omega_N^s)$ is implementable via a mechanism (σ, τ) if and only if the mechanism is cut-off based.*

Such cut-off based mechanisms for multi-dimensional dichotomous preferences was derived by Mishra and Roy [14]. For scheduling problems, such cut-off based mechanisms were derived by Mishra and Mitra [12]. We discuss about the cut-off based transfers for the NI sequencing rules defined in Example 1 and Example 2. All these are NI sequencing rules that are not ACM and hence their description is important for our understanding of the cut-off based mechanisms.

Cut-off based mechanisms for the sequencing rule σ^V (Example 1): Consider the sequencing problem Ω_N^s with $|N| = 2$ and consider the sequencing rule σ^V. If we fix $\theta_2'' \leq a_2$, then for any $\theta_1 \in \Theta$, $\sigma^V(\theta_1, \theta_2'') = (\sigma_1^V(\theta_1, \theta_2'') = 1, \sigma_2^V(\theta_1, \theta_2'') = 2)$. In that case the cut-off based transfer gives $\tau_1^V(\theta_1, \theta_2'') = h_1(\theta_2'')$ for all $\theta_1 \in \Theta$ since, given θ_2'', the cut-off point for agent 1 is $\theta_1^{(1)} = \theta_1^{(T)} = 0$. Therefore, given any $\theta_2'' \leq a_2$, the incentive payment of agent 1 is $I_1^V(\theta_1, \theta_2'') = 0$ for all $\theta_1 \in \Theta$. If we fix $\theta_2' > a_2$, then for any $\theta_1 \in (0, a_1)$, $\sigma^V(\theta_1, \theta_2') = (\sigma_1^V(\theta_1, \theta_2') = 2, \sigma_2^V(\theta_1, \theta_2') = 1)$ and for any $\theta_1' \geq a_1$,

$\sigma^V(\theta_1', \theta_2') = (\sigma_1^V(\theta_1', \theta_2') = 1, \sigma_2^V(\theta_1', \theta_2') = 2)$. Hence, given θ_2', the cut-off point for agent 1 is $\theta_1^{(1)} = \theta_1^{(T-1)} = a_1$. Therefore, given any $\theta_2' > a_2$, the incentive payment of agent 1 is

$$I_1^V(\theta_1, \theta_2') = \begin{cases} 0 & \text{if } \theta_1 \in (0, a_1), \\ a_1 s_2 & \text{if } \theta_1 \geq a_1. \end{cases}$$

The cut-off based transfer for agent 1 is $\tau_1^V(\theta_1, \theta_2') = h_1(\theta_2')$ for all $\theta_1 \in (0, a_1)$ and $\tau_1^V(\theta_1', \theta_2') = h_1(\theta_2') - a_1 s_2$ for all $\theta_1' \geq a_1$.

If we fix $\theta_1'' \geq a_1$, then for any $\theta_2 \in \Theta$, $\sigma^V(\theta_1'', \theta_2) = (\sigma_1^V(\theta_1'', \theta_2) = 1, \sigma_2^V(\theta_1'', \theta_2) = 2)$. In that case the cut-off based transfer gives $\tau_2^V(\theta_1'', \theta_2) = h_2(\theta_1'')$ for all $\theta_2 \in \Theta$ since, given θ_1'', the cut-off point for agent 2 is $\theta_2^{(1)} = \theta_2^{(T)} = 0$. Therefore, given any $\theta_1'' \geq a_1$, the incentive payment of agent 2 is $I_2^V(\theta_1'', \theta_2'') = 0$ for all $\theta_2 \in \Theta$. If we fix $\theta_1 < a_1$, then for any $\theta_2 \in (0, a_2]$, $\sigma^V(\theta_1, \theta_2) = (\sigma_1^V(\theta_1, \theta_2) = 1, \sigma_2^V(\theta_1, \theta_2) = 2)$ and for any $\theta_2' > a_2$, $\sigma^V(\theta_1, \theta_2') = (\sigma_1^V(\theta_1, \theta_2') = 2, \sigma_2^V(\theta_1, \theta_2') = 1)$. Hence, given $\theta_1 < a_1$, the cut-off point for agent 2 is $\theta_2^{(1)} = \theta_2^{(T-1)} = a_2$. Therefore, given any $\theta_1 < a_1$, the incentive payment of agent 2 is

$$I_2^V(\theta_1, \tilde{\theta}_2) = \begin{cases} 0 & \text{if } \tilde{\theta}_2 \in (0, a_2], \\ a_2 s_1 & \text{if } \tilde{\theta}_2 > a_2. \end{cases}$$

The cut-off based transfer for agent 2 is $\tau_2^V(\theta_1, \theta_2) = h_2(\theta_1)$ for all $\theta_2 \in (0, a_2]$ and $\tau_2^V(\theta_1, \theta_2') = h_2(\theta_1) - a_2 s_1$ for all $\theta_2' > a_2$. Therefore, from all these cases, the cut-off based transfer of the two agents is the following: For any profile $\theta \in \Theta^2$,

$$\tau_1^V(\theta) = \begin{cases} h_1(\theta_2) & \text{if } \theta_2 > a_2 \text{ and } \theta_1 \in (0, a_1), \\ h_1(\theta_2) - a_1 s_2 & \text{otherwise.} \end{cases} \tag{4}$$

$$\tau_2^V(\theta) = \begin{cases} h_2(\theta_1) - a_2 s_1 & \text{if } \theta_2 > a_2 \text{ and } \theta_1 \in (0, a_1), \\ h_2(\theta_1) & \text{otherwise.} \end{cases} \tag{5}$$

Cut-off based mechanisms for the sequencing rule σ^{NA} (Example 2): Consider the sequencing problem Ω_N^s with $|N| \geq 3$ and consider the sequencing rule σ^{NA}. To determine the transfer of agent 1, consider any profile $\theta_{-i} \in \Theta^{|N \setminus \{1\}|}$ and find the minimum urgency index for the agents $N \setminus \{1\}$, that is, find $\min_{k \in N \setminus \{1\}}(\theta_k/s_k)$. If $\theta_1 < s_1 \min_{k \in N \setminus \{1\}}(\theta_k/s_k)$, then $\sigma_1^{NA}(\theta_1, \theta_{-1}) = n$ and agent 1's transfer is $h_1(\theta_{-1})$. If $\theta_1' \geq s_1 \min_{k \in N \setminus \{1\}}(\theta_k/s_k)$, then $\sigma_1^{NA}(\theta_1', \theta_{-1}) = 1$ and agent 1 gets $h_1(\theta_{-1})$ and his payment is $s_1 \min_{k \in N \setminus \{1\}}(\theta_k/s_k)[S_1(\sigma^{NA}(\theta_1, \theta_{-1})) - S_1(\sigma^{NA}(\theta_1', \theta_{-1}))]$. Therefore, agent 1's cut-off point given the profile θ_{-1} is $\theta_1^{T-1} = s_1 \min_{k \in N \setminus \{1\}}(\theta_k/s_k)$ and his incentive payment is the following:

$$I_1^{NA}(\theta) = \begin{cases} 0 & \text{if } P_1'(\sigma^{NA}(\theta)) = \emptyset, \\ s_1 \left\{ \min\limits_{k \in N \setminus \{1\}} \left(\frac{\theta_k}{s_k} \right) \right\} \sum\limits_{j \in N \setminus \{1\}} s_j & \text{otherwise.} \end{cases}$$

Hence for agent 1, the cut-off based transfer for any profile $\theta \in \Theta^n$ is the following:

$$\tau_1^{NA}(\theta) = \begin{cases} h_1(\theta_{-1}) & \text{if } P_1'(\sigma^{NA}(\theta)) = \emptyset, \\ h_1(\theta_{-1}) - s_1 \left\{ \min\limits_{k \in N \setminus \{1\}} \left(\frac{\theta_k}{s_k} \right) \right\} \sum\limits_{j \in N \setminus \{1\}} s_j & \text{otherwise.} \end{cases} \tag{6}$$

For any agent $i \in N \setminus \{1\}$, consider any profile $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$. We can have two possibilities-(a) $(\theta_1/s_1) \geq \min_{k \in N \setminus \{1,i\}} (\theta_k/s_k)$ and (b) $(\theta_1/s_1) < \min_{k \in N \setminus \{1,i\}} (\theta_k/s_k)$. If possibility (a) holds, then $\sigma_1^{NA}(\theta_i, \theta_{-i}) = 1$ for all $\theta_i \in \Theta$. Assume that the order of the urgency indexes for the $N \setminus \{1, i\}$ agents is $u_{(1)} \geq \ldots \geq u_{(n-2)}$, then due to sequencing rule σ^{NA}, we have the following: As θ_i increases from any positive number $\theta_i^n \in (0, s_i u_{(n-2)})$ to any positive number $\theta_i^2 > s_i u_{(1)}$, the completion time $S_i(\sigma^{NA}(\theta_i, \theta_{-i}))$ weakly decreases from $S_i(\sigma^{NA}(\theta_i^n, \theta_{-i})) = s_1 + s_i + \sum_{j \in N \setminus \{1,i\}} s_j$ to $S_i(\sigma^{NA}(\theta_i^2, \theta_{-i})) = s_1 + s_i$. The cut-off points where agent i's S_i changes are the distinct numbers from the set $(u_{(1)}, \ldots, u_{(n-2)})$. Assume that there $T-1$ distinct urgency indexes in $(u_{(1)}, \ldots, u_{(n-2)})$, that is, $u_{\mu(1)} > \ldots > u_{\mu(T-1)}$. The difference in transfer between any $\theta_i^{r+1} \in (s_i u_{\mu(r+1)}, s_i u_{\mu(r)})$ and $\theta_i^r \in (s_i u_{\mu(r)}, s_i u_{\mu(r-1)})$ is

$$\tau_i^{NA}(\theta_i^{r+1}, \theta_{-i}) - \tau_i^{NA}(\theta_i^r, \theta_{-i}) = s_i u_{(r)} [S_i(\sigma^{NA}(\theta_i^{r+1}, \theta_{-i})) - S_i(\sigma^{NA}(\theta_i^r, \theta_{-i}))].$$

If possibility (b) holds, then if $\theta_i' \leq s_i(\theta_1/s_1)$, then $\sigma_i^{NA}(\theta_i, \theta_{-i}) = n$ and agent i has a transfer of $h_i(\theta_{-i})$. However, $\sigma_1^{NA}(\theta_i, \theta_{-i}) = n$ for all $\theta_i > s_i(\theta_1/s_1)$. Assume that the order of the urgency indexes for the $N \setminus \{1, i\}$ agents is $u_{(1)} \geq \ldots \geq u_{(n-2)}$, then due to sequencing rule σ^{NA}, we have the following: As θ_i increases from any positive number $\theta_i^{n-1} \in (s_i(\theta_1/s_s), s_i u_{(n-2)})$ to any positive number $\theta_i^1 > s_i u_{(1)}$, the completion time $S_i(\sigma^{NA}(\theta_i, \theta_{-i}))$ weakly decreases from $S_i(\sigma^{NA}(\theta_i^{n-1}, \theta_{-i})) = s_i + \sum_{j \in N \setminus \{1,i\}} s_j$ to $S_i(\sigma^{NA}(\theta_i^1, \theta_{-i})) = s_i$. The cut-off points where agent i's S_i changes are the distinct numbers from the set $(u_{(1)}, \ldots, u_{(n-2)})$. The remaining argument is similar to possibility (a).

From possibilities (a) and (b) we get that the incentive payment of any $i \in N \setminus \{1\}$ is the following:

$$I_i^{NA}(\theta) = \begin{cases} 0 & \text{if } P_i'(\sigma^{NA}(\theta)) = \emptyset, \\ s_i \sum\limits_{j \in P_i'(\sigma^{NA}(\theta))} \theta_j & \text{otherwise.} \end{cases}$$

Therefore, the cut-off based transfer for any $i \in N \setminus \{1\}$ and any profile $\theta \in \Theta^n$ is the following:

$$\tau_i^{NA}(\theta) = \begin{cases} h_i(\theta_{-i}) & \text{if } P_i'(\sigma^{NA}(\theta)) = \emptyset, \\ h_i(\theta_{-i}) - s_i \sum\limits_{j \in P_i'(\sigma^{NA}(\theta))} \theta_j & \text{otherwise.} \end{cases} \tag{7}$$

5. Balanced implementability

Definition 9. A sequencing rule σ is *implementable with balanced transfers* if there exists a mechanism (σ, τ) that implements it with budget balancing transfers where budget balancing transfers require that for all $\theta \in \Theta^n$, $\sum_{j \in N} \tau_j(\theta) = 0$.

Implementing a sequencing rule with balanced transfers simply means information extraction is done costlessly. That is, if any sequencing rule is implementable with balanced transfer, then, for any given profile θ, the utility of the agents are such that neither it is Pareto dominated nor it Pareto dominates the utility allocation of the agents under the same profile θ when there is complete information with no monetary transfers.

Consider any $\sigma \in NI(\Omega_N^s)$ and any cut-off based mechanism (σ, τ). For any $\theta \in \Theta^n$, the cut-off based transfer for any $i \in N$ is $\tau_i(\theta) = h_i(\theta_{-i}) - I_i(\theta)$ where $h_i(\theta_{-i})$ is the agent specific number that depends on the profile θ_{-i} of all but i and $I_i(\theta)$ is his incentive payment. Define $I(\theta) := \sum_{i \in N} I_i(\theta)$ as the *aggregate incentive payment* for the profile $\theta \in \Theta^n$. Any NI sequencing rule σ is implementable with balanced transfers if and only if there exists a cut-off based mechanism (σ, τ) with given functions $h_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ for all $i \in N$ such that for any profile $\theta \in \Theta^n$, $\sum_{i \in N} \tau_i(\theta) = I(\theta) - \sum_{i \in N} h_i(\theta_{-i}) = 0 \Leftrightarrow I(\theta) = \sum_{i \in N} h_i(\theta_{-i})$. Therefore any $\sigma \in NI(\Omega_N^s)$ is implementable with balanced transfer if and only if for any $\theta \in \Theta^n$,

$$I(\theta) = \sum_{i \in N} h_i(\theta_{-i}). \tag{8}$$

Thus for budget balance we require that the profile contingent aggregate incentive payment is $(n-1)$ type separable.

Remark 1.

(1) Any constant sequencing rule $\bar{\sigma}$ satisfies condition (8). In particular, for any profile $\theta \in \Theta^n$, the incentive payment of any $i \in N$ is $I_i(\theta) = 0$. Hence for any profile $\theta \in \Theta^n$, $I(\theta) = \sum_{i \in N} I_i(\theta) = 0$ and condition (8) holds. For any constant sequencing rule $\bar{\sigma}$, the cut-off based transfer specifies that for any $\theta \in$

Θ^n, $\tau_i(\theta) = h_i(\theta_{-i})$ for all $i \in N$. By setting the transfer $(h_i(\theta_{-i}))$ of all agents at zero we can achieve implementability with balanced transfers.

(2) Let $\sigma^{w,\kappa} \in ACM(\Omega_N^s)$ with the property that there exists $j \in N$ such that $w_j = 0$. For this $\sigma^{w,\kappa}$, the following property holds: For any $\theta_{-j} \in \Theta^{|N\setminus\{j\}}$ and any $\theta_j, \theta'_j \in \Theta$, $\sigma^{w,\kappa}(\theta_j, \theta_{-j}) = \sigma^{w,\kappa}(\theta'_j, \theta_{-j})$. Hence the incentive payment of agent j is $I_j(\theta) = 0$ for all $\theta \in \Theta^n$ and for any $i \in N \setminus \{j\}$, the incentive payment is independent of θ_j so that $I_i(\theta) := K_i(\theta_{-j})$ for any $\theta \in \Theta^N$. Therefore, for any $\theta \in \Theta^n$, $I(\theta) = \sum_{i\in N} I_i(\theta) = \sum_{i\in N\setminus\{j\}} K_i(\theta_{-j}) := K(\theta_{-j})$ and condition (8) holds. If we take the cut off based transfer such that for any $\theta \in \Theta^n$, $h_j(\theta_{-j}) = K(\theta_{-j})$ and $h_i(\theta_{-i}) = 0$ for all $i \in N \setminus \{j\}$, then we get budget balance.

An implication of budget balanced VCG mechanism for outcome efficient allocation rules was provided by Walker [24] which is better known as the Cubical Array Lemma. For any NI sequencing rule σ with cut-off based mechanisms we get something similar in terms of aggregate incentive payment which is stated in the next proposition. Before stating the next proposition we introduce some more notations. For any pair of profiles $\theta = (\theta_1, \theta_2, \ldots, \theta_n), \theta' = (\theta'_1, \theta'_2, \ldots, \theta'_n) \in \Theta^n$ and any $S \subseteq N$, let $\theta(S) = (\theta_1(S), \theta_2(S) \ldots, \theta_n(S)) \in \Theta^n$ be a profile such that

$$\theta_j(S) = \begin{cases} \theta_j & \text{if } j \notin S, \\ \theta'_j & \text{if } j \in S. \end{cases} \tag{9}$$

Observe that $\theta(S = \emptyset) = \theta, \theta(S = \{i\}) = (\theta'_i, \theta_{-i}), \theta(S = \{i, j\}) = (\theta'_i, \theta'_j, \theta_{-i-j})$ and so on, $\theta(S = N \setminus \{i\}) = (\theta_i, \theta'_{-i})$ and $\theta(S = N) = \theta'$.

Lemma 1. *For any $\sigma \in NI(\Omega_N^s)$, we can find a cut-off based mechanism (σ, τ) that implements σ with balanced transfers only if for all pairs of profiles $\theta, \theta' \in \Theta^n$,*

$$\sum_{S\subseteq N} (-1)^{|S|} I(\theta(S)) = 0. \tag{10}$$

Condition (10) in Lemma 1 states that the weighted aggregate incentive payment must add up to zero while moving from profile θ to any other profile θ' by allowing for all possible group deviations. The weights are all (-1) for groups with odd number of agents and are 1 for groups with even number of agents. The proof of Lemma 1 is similar to the proof of the Cubical Array Lemma due to Walker [24] and hence a formal proof is not provided. For $N = \{1, 2\}$ implementation of any NI sequencing rule σ with balanced transfer requires that condition (8) holds, that is for all $\theta = (\theta_1, \theta_2) \in \Theta^2$, (a) $I(\theta_1, \theta_2) = h_1(\theta_2) + h_2(\theta_1)$. Using (a) it follows that for any $\theta, \theta' \in \Theta^2$, $\sum_{S\subseteq N}(-1)^{|S|} I(\theta(S)) = I(\theta_1, \theta_2) - I(\theta'_1, \theta_2) - I(\theta_1, \theta'_2) + I(\theta'_1, \theta'_2) = [h_1(\theta_2) + h_2(\theta_1)] -$

$[h_1(\theta_2) + h_2(\theta'_1)] - [h_1(\theta'_2) + h_2(\theta_1)] + [h_1(\theta'_2) + h_2(\theta'_1)] = 0$. Lemma 1 is a generalization of this idea.

The next remark shows how Lemma 1 can help us identify necessary restrictions for implementability of any sequencing rule with balanced transfers.

Remark 2.

(1) For any NI sequencing rule σ^V of Example 1, Lemma 1 puts a restriction on the vector $a = (a_1, a_2)$. Specifically, for any pair of profiles $\theta = (\theta_1 = a_1 + \delta_1, \theta_2 = a_2 + \delta_2)$ and $\theta' = (\theta'_1 = a_1 - \delta_1, \theta'_2 = a_2 - \delta_2)$ with $\delta_1 \in (0, a_1)$, $\delta_2 \in (0, a_2)$, condition (10) requires that $\sum_{S \subseteq N} (-1)^{|S|} I^V(\theta(S)) = I^V(\theta_1, \theta_2) - I^V(\theta'_1, \theta_2) - I^V(\theta_1, \theta'_2) + I^V(\theta'_1, \theta'_2) = a_2 s_1 - a_1 s_2 - a_2 s_1 + a_2 s_1 = a_2 s_1 - a_1 s_2 = 0$ implying that for any σ^V to be implementable with balanced transfer it is necessary that $a_2 s_1 = a_1 s_2$. Therefore, from Lemma 1 it follows that for a sequencing problem $\Omega^{(s_1,s_2)}_{\{1,2\}}$ with two agents, any sequencing rule σ^V of Example 1, satisfying the added restriction that $a_1 s_2 \neq a_2 s_1$, is not implementable with balanced transfers.

(2) For the NI sequencing rule σ^{NA} of Example 2, Lemma 1 fails to hold. For any two profiles $\theta = (\theta_1, \ldots, \theta_n), \theta' = (\theta'_1, \ldots, \theta'_n) \in \Theta^n$ such that $\theta_1/s_1 > \ldots > \theta_n/s_n > \theta'_1/s_1 > \ldots > \theta'_n/s_n$, one can verify that $\sum_{S \subseteq N} (-1)^{|S|} I^{NA}(\theta(S)) = s_1 \left(\sum_{j \in N \setminus \{i\}} s_j\right) [(\theta'_1/s_1) - (\theta_n/s_n)] < 0$ and we have a violation of condition (10) of Lemma 1. Therefore, the NI sequencing rule σ^{NA} is not implementable with balanced transfers.

5.1. **Case 1: Two agents.** Consider any two agent sequencing problem $\Omega^{(s_1,s_2)}_{\{1,2\}}$ and consider any sequencing rule σ. Recall that in Remark 1 (1) we have already established that a constant sequencing rule is implementable with balanced transfers. In this sub-section we concentrate only on non-constant NI sequencing rules. For any $i \in \{1, 2\}$, let $A_i(\sigma) = \{\theta \in \Theta^2 \mid \sigma_i(\theta) = 1\}$ be the set of profiles such that agent i is first in the order. Clearly, for any two agent sequencing rule σ, $A_1(\sigma) \cup A_2(\sigma) = \Theta^2$. If the sequencing rule $\bar{\sigma}$ is such that $A_1(\bar{\sigma}) = \Theta^2$, then it is the constant sequencing rule with the fixed order $\bar{\sigma} = (\bar{\sigma}_1 = 1, \bar{\sigma}_2 = 2)$. If σ^* is a sequencing rule such that $A_1(\sigma^*) = \{\theta \in \Theta^2 \mid \theta_1/s_1 \geq \theta_2/s_2\}$, then it is the outcome efficient sequencing rule. If $\tilde{\sigma}$ is a sequencing rule such that $A_1(\tilde{\sigma}) = \{\theta \in \Theta^2 \mid \theta_1 \geq \theta_2\}$, then it is the just sequencing rule. For any $i \in \{1, 2\}$, an obvious consequence of any NI sequencing rule σ are as follow.

(ni): If $\theta \in A_i(\sigma)$, then $Q_i(\theta) = \{\theta' = (\theta'_1, \theta'_2) \in \Theta^2 \mid \theta'_i \geq \theta_i \ \& \ \theta'_j \leq \theta_j\} \subseteq A_i(\sigma)$. Moreover, the open set of $Q_i(\theta)$, that is, $Q'_i(\theta) = \{\theta' = (\theta'_1, \theta'_2) \in \Theta^2 \mid \theta'_i > \theta_i \ \& \ \theta'_j < \theta_j\} \subseteq A_i(\sigma)$ since $Q'_i(\theta) \subset Q_i(\theta)$.

Consider any NI sequencing rule σ and consider any pair $\theta, \theta' \in \Theta^2$ such that $\theta'_i > \theta_i$ for $i \in \{1,2\}$ and define $X_0 = \theta$, $X_1 = (\theta'_1, \theta_2)$, $X_2 = (\theta_1, \theta'_2)$ and $X_{12} = \theta'$. Given **(ni)**, for any $i, j \in \{1,2\}$ with $i \neq j$, it is *not possible* to have the following: (a) $X_0, X_{12} \in A_i(\sigma)$ and $X_i, X_j \in A_j(\sigma)$, (b) $X_0, X_i \in A_j(\sigma)$ and $X_j, X_{12} \in A_i(\sigma)$ and (c) $X_0, X_i, X_{12} \in A_j(\sigma)$ and $X_j \in A_i(\sigma)$.

Lemma 2. *If a non-constant $\sigma \in NI(\Omega^{(s_1,s_2)}{}_{\{1,2\}})$ is implementable with balanced transfers, then, for any $i, j \in \{1,2\}$ with $i \neq j$ and any pair $\theta, \theta' \in \Theta^2$ such that $\theta'_1 > \theta_1$, $\theta'_2 > \theta_1$, $X_0 = \theta$, $X_1 = (\theta'_1, \theta_2)$, $X_2 = (\theta_1, \theta'_2)$ and $X_{12} = \theta'$, the following conditions must hold.*

(B1) *If $X_0, X_i, X_{12} \in A_i(\sigma)$ and $X_j \in A_j(\sigma)$, then the cut-off point of agent i for θ'_j and the cut-off point of agent j for θ_i have the following relation: $\theta_i^{(1)}(\theta'_j)s_j = \theta_j^{(1)}(\theta_i)s_i$.*
(B2) *If $X_0, X_i \in A_i(\sigma)$ and $X_j, X_{12} \in A_j(\sigma)$, then the cut-off points of agent j for θ_i and θ'_i are equal, that is, $\theta_j^{(1)}(\theta_i) = \theta_j^{(1)}(\theta'_i)$.*

Definition 10. Let $\Omega^{(s_1,s_2)}_{\{1,2\}}$ be a two-agent sequencing problem. A sequencing rule σ^{Tx} is a *two agent balancing* (TAB) sequencing rule if there exists an agent $k \in \{1,2\}$ such that any one of the following conditions hold.

(T1) There exists $a_k > 0$ such that $a_l = (a_k s_l)/s_k > 0$ and either $A_k(\sigma^{T1a}) = \{\theta \in \Theta^2 \mid \textit{either } \theta_k \geq a_k \textit{ or } \theta_l \leq a_l\}$ or $A_k(\sigma^{T1b}) = \{\theta \in \Theta^2 \mid \textit{either } \theta_k > a_k \textit{ or } \theta_l < a_l\}$ (see Figure 1 where we have (T1a) and (T1b) for $k = 1$).
(T2) There exists a real number $a_k > 0$ such that either $A_k(\sigma^{T2a}) = \{\theta \in \Theta^2 \mid \theta_k \geq a_k\}$ or $A_k(\sigma^{T2b}) = \{\theta \in \Theta^2 \mid \theta_k > a_k\}$ (See Figure 2 where we have (T2a) and (T2b) for $k = 1$).

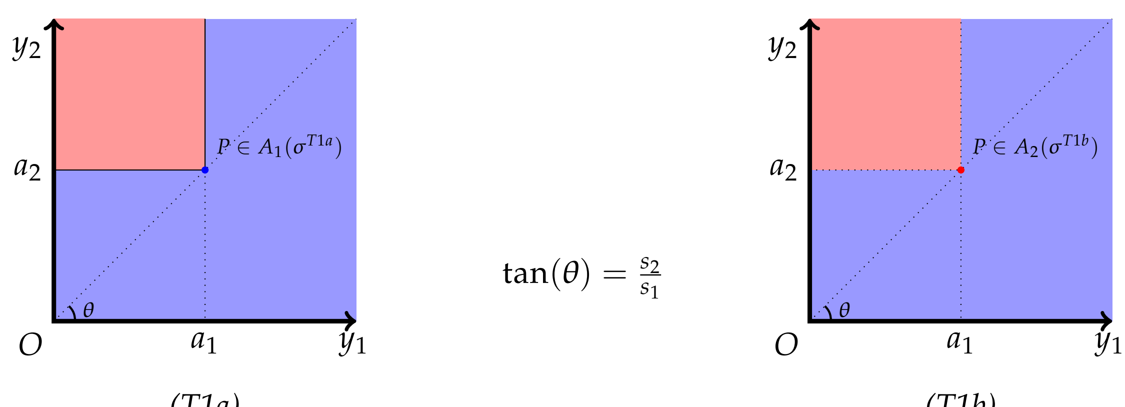

Figure 1: (T1)

(T2a) *(T2b)*

Figure 2: (T2)

Any ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_1 > 0$, $w_2 = 0$ and $\kappa(\sigma_1 = 1, \sigma_2 = 2) > \kappa(\sigma_1 = 2, \sigma_2 = 1)$ is a sequencing rule σ^{T2a} given in (T2) with $k = 1$. A sequencing rule σ^{T1a} given in (T1) is the special case of any NI sequencing rule σ^V of Example 1 with the added restriction that $a_2 s_1 = a_1 s_2$. Therefore, as established in the proof of Proposition 2, any sequencing rule defined in (T1) is not an ACM sequencing rule.

Theorem 2. *A non-constant $\sigma \in NI(\Omega^{(s_1,s_2)_{\{1,2\}}})$ is implementable with balanced transfers if and only if it is a TAB sequencing rule σ^{Tx}.*

Therefore, a consequence of Theorem 2 is that any ACM sequencing rule $\sigma^{w,\kappa}$ such that the agent specific weights w_1 and w_2 are both positive are not implementable with balanced transfers. Hence the outcome efficient sequencing rule σ^* and the just sequencing rule $\tilde{\sigma}$ are not implementable with balanced transfers.

5.2. **Case 2: More than two agents.** For any sequencing problem Ω^s_N with more than two agents it is difficult to identify the complete class of NI sequencing rules that are implementable with balanced transfers.

Consider any sequencing problem Ω^s_N with three or more agents. In Remark 1(2) we have argued that any ACM sequencing rule $\sigma^{w,\kappa}$ with the property that there exists $i \in N$ such that $w_i = 0$ is implementable with balanced transfers. What can we say about implementability with balanced transfers for any ACM sequencing rule $\sigma^{w,\kappa}$ with the property that for all $i \in N$, $w_i > 0$? We identify an ACM sequencing rule $\sigma^{w,\kappa}$ with the properties that for all $i \in N$, $w_i > 0$ and κ function is not a constant function, that cannot be implemented with balanced transfers.

Example 3. Consider any sequencing problem $\Omega_{\{1,2,3\}}^{(s_1,s_2,s_3)}$ with three agents. Consider the ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_1 = w_2 = w_3 = 1 > 0$ and

$$\kappa(\sigma) = \begin{cases} \bar{\kappa} > 0 & \text{if } \sigma^1 = (\sigma_1^1 = 1, \sigma_2^1 = 2, \sigma_3^1 = 3), \\ 0 & \text{if } \sigma \in \Sigma(\{1,2,3\}) \setminus \{\sigma^1\}. \end{cases} \tag{11}$$

Given the selection of the $\kappa(\sigma)$ function it may so happen that agent 1 has the highest urgency index, agent 2 has the second highest urgency index and agent 3 has the lowest urgency index and yet the order $(\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$ is less costly compared to the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ simply because we have an added cost of $\bar{\kappa} > 0$ associated with selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$. This aspect can be used to demonstrate that it is impossible to implement this sequencing rule with balanced transfers.

Consider any two profiles $\theta = (\theta_1, \theta_2, \theta_3), \theta' = (\theta'_1, \theta'_2, \theta'_3) \in \Theta^3$ such that $\theta_3/s_3 > \theta_2/s_2 > \theta_1/s_1 > \theta'_2/s_2 > \theta'_3/s_3 > \theta'_1/s_1$, $\theta_1 s_2 - \theta'_2 s_1 = \bar{\kappa}/2$ and $\theta'_3 = \bar{\kappa}/s_2$. We provide the chosen order and the incentive payment of the three agents for the eight possible profiles.

(a) $\sigma^{w,\kappa}(\theta_1, \theta_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)$, $I_1(\theta_1, \theta_2, \theta_3) = 0$, $I_2(\theta_1, \theta_2, \theta_3) = s_2\theta_1$ and $I_3(\theta_1, \theta_2, \theta_3) = s_3(\theta_2 + \theta_1)$.
(b) $\sigma^{w,\kappa}(\theta'_1, \theta_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)$, $I_1(\theta'_1, \theta_2, \theta_3) = 0$, $I_2(\theta'_1, \theta_2, \theta_3) = s_2\theta'_1$ and $I_3(\theta'_1, \theta_2, \theta_3) = s_3(\theta_2 + \theta'_1)$.
(c) $\sigma^{w,\kappa}(\theta_1, \theta'_2, \theta_3) = (\sigma_1 = 2, \sigma_2 = 3, \sigma_3 = 1)$, $I_1(\theta_1, \theta'_2, \theta_3) = s_1\theta'_2$, $I_2(\theta_1, \theta'_2, \theta_3) = 0$ and $I_3(\theta_1, \theta'_2, \theta_3) = s_3(\theta_1 + \theta'_2)$.
(d) $\sigma^{w,\kappa}(\theta_1, \theta_2, \theta'_3) = (\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$, $I_1(\theta_1, \theta_2, \theta'_3) = s_1\theta'_3$, $I_2(\theta_1, \theta_2, \theta'_3) = s_2(\theta_1 + \theta'_3)$ and $I_3(\theta_1, \theta_2, \theta'_3) = 0$.
(e) $\sigma^{w,\kappa}(\theta'_1, \theta'_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)$, $I_1(\theta'_1, \theta'_2, \theta_3) = 0$, $I_2(\theta'_1, \theta'_2, \theta_3) = s_2\theta'_1$ and $I_3(\theta'_1, \theta'_2, \theta_3) = s_3(\theta'_2 + \theta'_1)$.
(f) The profile $(\theta_1, \theta'_2, \theta'_3)$ shows how we cannot rely only on the urgency index when κ-function is not a constant function. In particular, we have $\theta_1/s_1 > \theta'_2/s_2$ and yet agent 1 is served after agent 2 simply because the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ less the cost of selecting the order $(\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$ equals $\bar{\kappa} + \theta'_2 s_1 - \theta_1 s_2 = \bar{\kappa} - \bar{\kappa}/2 = \bar{\kappa}/2 > 0$. Hence, $\sigma^{w,\kappa}(\theta_1, \theta'_2, \theta'_3) = (\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$. Further, the relevant cut-off point for agent 1 is $\theta_1^{(2)} = (s_1\theta'_3)/s_3$ and hence his incentive payment is $I_1(\theta_1, \theta'_2, \theta'_3) = s_1\theta'_3$. The relevant cut-off points for agent 2 are $\theta_2^{(1)} = (\theta_1 s_2 - \bar{\kappa})/s_1$ and $\theta_2^{(2)} = (s_2\theta'_3)/s_3$. Specifically, given (θ_1, θ'_3), $\theta_2^{(1)}$ is that waiting cost of agent 2 for which the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ less the cost of selecting

the order $(\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$ equals zero. Hence his incentive payment is $I_2(\theta_1, \theta_2', \theta_3') = s_2(\theta_1 + \theta_3') - \bar{\kappa}$. Finally, since agent 3 is served last, $I_3(\theta_1, \theta_2', \theta_3') = 0$.

(g) $\sigma^{w,\kappa}(\theta_1', \theta_2, \theta_3') = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$, $I_1(\theta_1', \theta_2, \theta_3') = 0$, $I_2(\theta_1', \theta_2, \theta_3') = s_2(\theta_3' + \theta_1')$ and $I_3(\theta_1', \theta_2, \theta_3') = s_3\theta_1'$.

(h) $\sigma^{w,\kappa}(\theta_1', \theta_2', \theta_3') = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$, $I_1(\theta_1', \theta_2', \theta_3') = 0$, $I_2(\theta_1', \theta_2', \theta_3') = s_2(\theta_3' + \theta_1')$ and $I_3(\theta_1', \theta_2', \theta_3') = s_3\theta_1'$.

Taking the left hand side of condition (10) of Lemma 1 and then simplifying it using (a)-(h) above we get

$$\sum_{S \subseteq N} (-1)^{|S|} I(\theta(S)) = \theta_3' s_2 + (\theta_1 s_2 - \theta_2' s_1 - \bar{\kappa}) = \bar{\kappa} + \left(\frac{\bar{\kappa}}{2} - \bar{\kappa}\right) = \frac{\bar{\kappa}}{2} \neq 0. \tag{12}$$

Condition (12) is a violation of condition (10) in Lemma 1. Hence the ACM sequencing rule $\sigma^{w,\kappa}$ with $w_1 = w_2 = w_3 = 1 > 0$ and the $\kappa(\sigma)$ function given by condition (11) is not implementable with balanced transfers.

Given that the agent specific weights are all positive, Example 3 shows that it is difficult to check the prospect of implementability with balanced transfers for any ACM sequencing rule with the property that the κ-function is not a constant. Keeping this difficulty in mind, we identify a sufficient family of NI sequencing rules that are implementable with balanced transfers.

Consider any sequencing problem Ω_N^s with more than two agents and let Π_N be the set of all possible *priority* partitions of the agents where the order of representing the partition is important in terms of priority. For example, if $\pi(N) = (\pi_1, \pi_2, \ldots, \pi_K)$ is any priority partition, then group π_1 is given priority over group π_2 and so on. Let $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi_N$ be any priority partition of the set of agents. The set of $\pi(N)$ induced orders is

$$\Sigma(\pi(N)) = \begin{cases} \{\sigma \in \Sigma(N) \mid \forall k \in \{1, \ldots, K-1\}, \sigma_i < \sigma_j, \forall i \in \pi_k, \forall j \in \pi_{k+1}\} & \text{if } K \geq 2, \\ \Sigma(N) & \text{if } K = 1. \end{cases} \tag{13}$$

Therefore, the set of priority partition $\pi(N)$ induced orders $\Sigma(\pi(N))$ are those orders where agents in π_1 are always served first, agents in π_2 are always served after agents in π_1 but before agents in π_3 (if any) and so on. If $K = 1$ so that $\pi(N) = (\pi_1 = \pi_K = \{N\})$, then $\Sigma(\pi(N)) = \Sigma(N)$ which is the set of all possible ordering on the set of agents N. For example, for $\Pi_{\{1,2,3\}}$, there are four types of priority partitions. These are $\pi^c = (\pi_1 = \{i\}, \pi_2 = \{j\}, \pi_3 = \{k\})$, $\pi^{21} = (\pi_1 = \{i, j\}, \pi_2 = \{k\})$,

$\pi^{12} = (\pi_1 = \{i\}, \pi_2^{12} = \{j,k\})$ and $\bar{\pi} = (\pi_1 = \{1,2,3\})$ where $i \neq j \neq k \neq i$. For π^c, $\Sigma(\pi^c) = \{(\sigma_i = 1, \sigma_j = 2, \sigma_k = 3)\}$, for π^{21}, $\Sigma(\pi^{21}) = \{\{(\sigma_i = 1, \sigma_j = 2, \sigma_k = 3)\} \cup \{(\sigma_i = 2, \sigma_j = 1, \sigma_k = 3)\}\}$, for π^{12}, $\Sigma(\pi^{12}) = \{\{(\sigma_i = 1, \sigma_j = 2, \sigma_k = 3)\} \cup \{(\sigma_i = 1, \sigma_j = 3, \sigma_k = 2)\}\}$ and finally for $\bar{\pi}$, we have the set of all possible orders on the set of agents, that is, $\Sigma(\bar{\pi}) = \Sigma(\{1,2,3\})$.

Definition 11. Consider any priority partition $\pi(N) \in \Pi_N$ and let $f = \{f_1, \dots, f_n\}$ be a set of agent specific increasing and one-to-one functions $f_j : \Theta \to \mathbb{R}_+$. The sequencing rule $\sigma^{\pi(N),f} : \Theta^n \to \Sigma(N)$ satisfies *group priority based cost minimization* (GP-CM) if for each $\theta \in \Theta^n$, $\sigma^{\pi(N),f}(\theta) \in \arg\min_{\sigma \in \Sigma(\pi(N))} \sum_{j \in N} f_j(\theta_j) S_j(\sigma)$.

By appropriately modifying the arguments used to prove that an ACM sequencing rule is NI in Proposition 2, one can easily show that any GP-CM sequencing rule $\sigma^{\pi(N),f}$ is NI. The following observations are important for our understanding of GP-CM sequencing rules.

(1) For any $\pi(N) \in \Pi_N$, any GP-CM sequencing rule $\sigma^{\pi(N),f}$ with the property that there exists an agent $j \in N$ such that $f_j(.)$ is non-linear is an NI sequencing rule which is not an ACM.
(2) For any $\pi(N) \in \Pi_N$, the GP-CM sequencing rule $\sigma^{\pi(N),f}$ where $f_j(.)$ is linear for all $j \in N$ is an ACM sequencing rule. Specifically, any ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_j > 0$ for all $j \in N$ and $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma'(N)$ and there exists a priority partition $\pi(N) \in \Pi_N$ such that $\Sigma'(N) = \Sigma(\pi(N))$ is a GP-CM sequencing rule.
(3) The GP-CM sequencing rule is not onto for any $\pi(N) = (\pi_1, \dots, \pi_K) \in \Pi_N$ such that $K \geq 2$ since, in that case, $\Sigma(N) \setminus \Sigma(\pi(N)) \neq \emptyset$ and any order $\sigma \in \Sigma(N) \setminus \Sigma(\pi(N))$ is never chosen.
(4) For $\pi(N) \in \Pi_N$ such that $K = 1$ so that the $\pi(N) = (\pi_1 = \pi_K = \{N\})$ is the grand coalition, $\Sigma(\pi(N)) = \Sigma(N)$ and any such GP-CM $\sigma^{\pi(N),f}$ is onto.
(5) A GP-CM sequencing rule $\sigma^{\pi(N),f}$ is a constant sequencing rule if $\pi(N) = (\pi_1, \dots, \pi_K)$ is such that $K = n$.
(6) A GP-CM sequencing rule $\sigma^{\pi(N),f}$ gives the outcome efficient sequencing rule σ^* if $\pi(N) = (\{N\})$ and $f_j(\theta_j) = \theta_j$ for all $j \in N$.
(7) A GP-CM sequencing rule $\sigma^{\pi(N),f}$ gives the just sequencing rule $\tilde{\sigma}$ if $\pi(N) = (\{N\})$ and $f_j(\theta_j) = (1/\prod_{k \in N \setminus \{j\}} s_k)\theta_j$ for all $j \in N$.

Remark 3. For any GP-CM sequencing rule $\sigma^{\pi(N),f}$ with the priority partition $\pi(N) \in \Pi_N$, *modified urgency index* $f_j(\theta_j)/s_j$ is used to determine the profile contingent order of serving the agents. Specifically, like Smith's [22] rule for outcome efficient sequencing rule σ^*, for any GP-CM $\sigma^{\pi(N),f}$, the selected order $\sigma^{\pi(N),f}(\theta)$ satisfies the following condition.

$$\textbf{(GP-CM)} \quad \text{For any } i, j \in \pi_k \in \pi(N),$$
$$(f_i(\theta_i)/s_i) \geq (f_j(\theta_j)/s_j) \Leftrightarrow \sigma_i^{\pi(N),f}(\theta) \leq \sigma_j^{\pi(N),f}(\theta).$$

Given the tie-breaking rule, this profile contingent selection $\sigma^{\pi(N),f}(\theta)$ is unique.

Definition 12. For any GP-CM sequencing rule $\sigma^{\pi(N),f}$ with priority partition $\pi(N) \in \Pi_N$, a mechanism $(\sigma^{\pi(N),f}, \tau^{\pi(N),f})$ is a *GP-CM-cut-off based mechanism* if the transfer rule is such that for any $\theta \in \Theta^n$ and any $i \in \pi_k \in \pi(N)$,

(14)
$$\tau_i^{\pi(N),f}(\theta) = \begin{cases} G_i(\theta_{-i}) & \text{if } P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k = \emptyset, \\ G_i(\theta_{-i}) - \sum\limits_{j \in P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k} f_i^{-1}\left(\frac{f_j(\theta_j)}{s_j}\right)(s_j s_i) & \text{if } P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k \neq \emptyset. \end{cases}$$

where the function $G_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary.

It is obvious that the incentive payment of any agent $i \in \pi_k \in \pi(N)$ under the GP-CM-cut-off based mechanism is the following:

$$I_i^{\pi(N),f}(\theta) = \begin{cases} 0 & \text{if } P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k = \emptyset, \\ \sum\limits_{j \in P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k} \sum\limits_{j \in P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k} f_i^{-1}\left(\frac{f_j(\theta_j)}{s_j}\right)(s_j s_i) & \text{if } P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k \neq \emptyset. \end{cases}$$

The GP-CM-cut-off based transfers (14) specifies that for any $i \in \pi_k \in \pi(N)$ and any $\theta_{-i} \in \Theta^{N \setminus \{i\}}$, if θ_i is such that agent i is served last among the members of the group π_k in which he belongs for the order $\sigma^{\pi(N),f}(\theta_i, \theta_{-i})$, then $\tau_i^{\pi(N),f}(\theta_i, \theta_{-i}) = G_i(\theta_{-i})$. This part of the transfer is like the cut-off based transfer for the case where agent i's type is smaller than the smallest cut-off point $\theta_i^{(T-1)}$ (see Theorem 1). If, however, θ_i' is such that agent i is not served last among the members of his group π_k under the order $\sigma^{\pi(N),f}(\theta_i', \theta_{-i})$, that is, if $P_i'(\sigma^{\pi(N),f}(\theta_i', \theta_{-i})) \cap \pi_k \neq \emptyset$, then agent i's transfer $\tau_i^{\pi(N),f}(\theta_i', \theta_{-i})$ not only has $G_i(\theta_{-i})$ but he also has to make an incentive payment $I_i(\theta_i', \theta_{-i})$. His incentive payment amount is the sum of cost that agent i inflicts on the followers from the members of his group π_k under the order $\sigma^{\pi(N),f}(\theta_i', \theta_{-i})$. This part of the incentive solving payment is nothing but the cost term $\sum_{r=t}^{T-1} \theta_i^{(r)} D_r(\theta_{-i})$ in the transfer under the cut-off based mechanism (see Theorem 1). In particular, given any $i \in \pi_k \in \pi(N)$ and given any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, the cut-off points where the order of agent i changes are set of distinct elements from the collection $\{s_i f_i^{-1}(f_j(\theta_j)/s_j)\}_{j \in \pi_k \setminus \{i\}}$ and the absolute cost difference of agent i below and above any cut-off point $s_i f_i^{-1}(f_j(\theta_j)/s_j)$

is given by $D_j(\theta_{-i}) = s_j$. Hence for each $j \in P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k$, the payment of i is $\left[s_i f_i^{-1}(f_j(\theta_j)/s_j)\right] D_j(\theta_{-i}) = f_i^{-1}(f_j(\theta_j)/s_j)(s_j s_i)$.

Theorem 3. Consider any sequencing problem Ω_N^s with more than two agents. For any priority partition $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi_N$ and for any given set of functions $f = \{f_1, \ldots, f_n\}$ that are increasing and onto, the GP-CM sequencing rule $\sigma^{\pi(N),f}$ is implementable with balanced transfers.

Till now we have obtained the following.

(1) Any ACM sequencing rule $\sigma^{w,\kappa}$ such that there exists an agent $j \in N$ such that $w_j = 0$ is implementable with balanced transfers (see Remark 1 (2)).
(2) Example 3 demonstrates the existence of an ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_i > 0$ for all $i \in N$ and κ-function is not a constant function which is not implementable with balanced transfers.
(3) Any ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_i > 0$ for all $i \in N$ and $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma'(N)$ and there exists a priority partition $\pi(N) \in \Pi_N$ such that $\Sigma'(N) = \Sigma(\pi(N))$ is a GP-CM sequencing rule and hence, by Theorem 3, is implementable with balanced transfers. Moreover, since the grand coalition $\pi(N) = (\pi_1 = \{N\})$ is also included in the set of priority partitions, any *onto* ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_i > 0$ for all $i \in N$ and $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma(N)$ is implementable with balanced transfers.
(4) The non-affine cost minimizers NI sequencing rules included in GP-AM sequencing rules are a generalization of the affine cost minimizer sequencing rules where agents' waiting cost are replaced with a non-linear function of the waiting cost which is increasing and onto. Theorem 3 shows that all these rules are also implementable with balanced transfers.

What can we say about any ACM sequencing rule $\sigma^{w,\kappa}$ such that (a) κ-function is a constant function, (b) $w_i > 0$ for all $i \in N$, and, yet, (c) it does not belong to the class of GP-CM sequencing rules? It is difficult to give a general answer and we provide one example of such an ACM sequencing rule which is not implementable with balanced transfers.

Example 4. Consider any sequencing problem $\Omega_{\{1,2,3\}}^{(s_1,s_2,s_3)}$ with three agents. Consider the ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_1 = w_2 = w_3 = 1 > 0$, $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma'(\{1,2,3\})$ and $\Sigma'(\{1,2,3\}) = \{\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \Sigma(\{1,2,3\}) \mid \sigma_1 \neq 2\}$. From the discussion about the priority partitions $\Pi_{\{1,2,3\}}$ for the three agent case, that appears before the definition of GP-CM sequencing rules, it is easy to see that there does not exists a priority partition $\pi \in \Pi_{\{1,2,3\}}$ such that $\Sigma'(\{1,2,3\}) = \Sigma(\pi(\{1,2,3\})$ and hence this ACM sequencing rule is not in the family of GP-CM sequencing rules. Consider any two profiles $\theta = (\theta_1, \theta_2, \theta_3), \theta' = (\theta'_1, \theta'_2, \theta'_3) \in \Theta^3$ such that $\theta'_1 = s_1$, $\theta'_3 = 2s_3$, $\theta_3 = 3s_3$, $\theta'_2 = 3s_2$, $\theta_2 = 4s_2$, $\theta_1 = As_1$ and A is any number in the open interval $(2 + a, \min\{2 + 2a, 3\})$ where $a = s_2/(s_2 + s_3) \in (0, 1)$. Observe that $\theta_2/s_2 = 4 > \theta'_2/s_2 = \theta_3/s_3 = 3 > \theta_1/s_1 = A > \theta'_3/s_3 = 2 > \theta'_1/s_1 = 1$. Given the tie-breaking rule, we provide the chosen order and the incentive payments under the eight possible profiles.

(a) $\sigma^{w,\kappa}(\theta_1, \theta_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and $I_1(\theta_1, \theta_2, \theta_3) = 0$. The relevant cut-off points for agent 2 are $\theta_2^{(2)} = (\theta_1(s_2 + s_3) - \theta_3 s_1)/s_1$ and $\theta_2^{(1)} = (s_2\theta_3)/s_3$. Specifically, given (θ_1, θ_3), $\theta_2^{(2)}$ is that waiting cost of agent 2 for which the cost of selecting the order $(\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)$ less the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 3, \sigma_3 = 2)$ equals zero. Hence his incentive payment is $I_2(\theta_1, \theta_2, \theta_3) = \theta_1(s_2 + s_3) - \theta_3 s_1 + s_2\theta_3$. The relevant cut-off point for agent 3 is $\theta_3^{(2)} = (\theta_1(s_2 + s_3) - \theta_2 s_1)/s_1$ and, given (θ_1, θ_2), $\theta_3^{(2)}$ is that waiting cost of agent 3 for which the cost of selecting the order $(\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ less the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ equals zero. Hence his incentive payment is $I_3(\theta_1, \theta_2, \theta_3) = \theta_1(s_2 + s_3) - \theta_2 s_1$. Therefore, $I(\theta_1, \theta_2, \theta_3) = \sum_{i \in \{1,2,3\}} I_i(\theta_1, \theta_2, \theta_3) = 2\theta_1(s_2 + s_3) - (\theta_2 + \theta_3)s_1 + \theta_3 s_2$.

(b) $\sigma^{w,\kappa}(\theta'_1, \theta_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and, like case (a), the aggregate incentive payment is $I(\theta'_1, \theta_2, \theta_3) = 2\theta'_1(s_2 + s_3) - (\theta_2 + \theta_3)s_1 + \theta_3 s_2$.

(c) $\sigma^{w,\kappa}(\theta_1, \theta'_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and the aggregate incentive payment is $I(\theta_1, \theta'_2, \theta_3) = 2\theta_1(s_2 + s_3) - (\theta'_2 + \theta_3)s_1 + \theta_3 s_2$.

(d) $\sigma^{w,\kappa}(\theta_1, \theta_2, \theta'_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and the aggregate incentive payment is $I(\theta_1, \theta_2, \theta'_3) = 2\theta_1(s_2 + s_3) - (\theta_2 + \theta'_3)s_1 + \theta'_3 s_2$.

(e) $\sigma^{w,\kappa}(\theta'_1, \theta'_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and the aggregate incentive payment is $I(\theta'_1, \theta'_2, \theta_3) = 2\theta'_1(s_2 + s_3) - (\theta'_2 + \theta_3)s_1 + \theta_3 s_2$.

(f) $\sigma^{w,\kappa}(\theta_1, \theta'_2, \theta'_3) = (\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ and $I_3(\theta_1, \theta'_2, \theta'_3) = 0$. The only cut-off point for agent 1 is $\theta_1^{(1)} = ((\theta'_2 + \theta'_3)s_1)/(s_2 + s_3)$ and, given (θ'_2, θ'_3), $\theta_1^{(1)}$ is that waiting cost of agent 1 for which the cost of selecting the order $(\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ less the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ equals zero. Hence $I_1(\theta_1, \theta'_2, \theta'_3) = (\theta'_2 + \theta'_3)s_1$. The relevant cut-off point for agent 2 is $\theta_2^{(2)} = (s_2\theta'_3)/s_3$ and $I_2(\theta_1, \theta'_2, \theta'_3) = \theta'_3 s_2$. Hence, $I(\theta_1, \theta'_2, \theta'_3) = \sum_{i \in \{1,2,3\}} I_i(\theta_1, \theta'_2, \theta'_3) = (\theta'_2 + \theta'_3)s_1 + \theta'_3 s_2$.

(g) $\sigma^{w,\kappa}(\theta'_1, \theta_2, \theta'_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and the aggregate incentive payment is $I(\theta'_1, \theta_2, \theta'_3) = 2\theta'_1(s_2 + s_3) - (\theta_2 + \theta'_3)s_1 + \theta'_3 s_2$.

(h) $\sigma^{w,\kappa}(\theta'_1, \theta'_2, \theta'_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and $I(\theta'_1, \theta'_2, \theta'_3) = 2\theta'_1(s_2 + s_3) - (\theta'_2 + \theta'_3)s_1 + \theta'_3 s_2$.

Taking the left hand side of condition (10) of Lemma 1 and then simplifying it using (a)-(h) above we get

$$\sum_{S \subseteq N} (-1)^{|S|} I(\theta(S)) = (s_1 + s_2)[s_2 + 2((s_2 + s_3) - As_1)]. \tag{15}$$

For condition (10) in Lemma 1 to hold, for any $A \in (2 + a, \min\{2 + 2a, 3\})$ we must have that condition (15) must be equal to zero. This is not possible since the right hand side of condition (15) changes for different selections of A from the interval $(2 + a, \min\{2 + 2a, 3\})$. In particular, if (15) is equal to zero for some selection $a \in A$, then (15) is not equal to zero for any selection $a + \epsilon \in A$ with $\epsilon > 0$. That we can always select two distinct numbers from the interval A is immediate. If $\min\{2 + 2a, 3\} = 2 + 2a$, then select $a_1 = 2 + ((3a)/2)$ and $a_1 + \epsilon = 2 + ((7a)/4)$. Note that $a_1, a_1 + \epsilon \in A$ and $\epsilon = a/4 > 0$. If $\min\{2 + 2a, 3\} = 3$, then select $b_1 = (5/2) + (a/2)$ and $b_1 + \epsilon' = (8/3) + (a/3)$. Note that $b_1, b_1 + \epsilon' \in A$ and $\epsilon' = (1/6)(1 - a) > 0$. Therefore, this ACM sequencing rule $\sigma^{w,\kappa}$ is not implementable with balanced transfers.

Finally, there are NI sequencing rules, different both from GP-CM sequencing rules and from sequencing rules of Remark 1 (2), that are implementable with balanced transfers. For example, consider any σ such that for each agent $j \in N \setminus \{1, 2\}$, $\sigma_j(\theta) = k \in \{3, \ldots, n\}$ is fixed for all θ and for agents 1 and 2 we follow conditions specified by (T1a) for the TAB sequencing rules (ignoring the waiting costs of all other agents) to obtain their order. Clearly, this rule is implementable with balanced transfers by setting the transfer of all $j \in N \setminus \{1, 2\}$ at zero, ceteris paribus. Hence, GP-CM sequencing rules and sequencing rules of Remark 1 (2), taken together, is not necessary for implementability with balanced transfers.

6. APPENDIX

Proof of Proposition 1: Consider any sequencing rule σ. Consider any agent $i \in N$ and fix any profile θ_{-i} of all but agent i. By taking any two types θ_i and θ'_i for agent i and applying the implementability conditions we have the following inequalities: (1) $U_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq U_i(\sigma(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, \theta_{-i}); \theta_i)$ and (2) $U_i(\sigma(\theta), \tau_i(\theta); \theta'_i) \leq U_i(\sigma(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, \theta_{-i}); \theta'_i)$. From inequalities (1) and (2) we get

$$[S_i(\sigma(\theta'_i, \theta_{-i})) - S_i(\sigma(\theta))]\theta_i \geq \tau_i(\theta'_i, \theta_{-i}) - \tau_i(\theta) \geq [S_i(\sigma(\theta'_i, \theta_{-i})) - S_i(\sigma(\theta))]\theta'_i. \tag{16}$$

Consider any $\theta'_i > \theta_i$. By eliminating the difference in transfer (middle) term from inequality (16) and then applying some obvious reshuffling we get the following inequality

$$(\theta_i - \theta'_i)[S_i(\sigma(\theta'_i, \theta_{-i})) - S_i(\sigma(\theta))] \geq 0. \tag{17}$$

Given $\theta'_i > \theta_i$, from inequality (17) we have $S_i(\sigma(\theta'_i, \theta_{-i})) \leq S_i(\sigma(\theta'_i, \theta_{-i}))$ which implies non-increasingness of the completion time $S_i(\sigma(\theta_i, \theta_{-i}))$ in θ_i. Since the selection of agent i and the fixing of θ_{-i} of all but agent i were arbitrary, the necessity of NI follows. □

Proof of Proposition 2: Consider any affine maximizer sequencing rule $\sigma^{w,\kappa}$. Let $T \subset N$ be such that $w_j = 0$ for all $j \in T$. Since, by affine maximization, for any profile θ, $\sigma^{w,\kappa}_i(\theta) < \sigma^{w,\kappa}_j(\theta)$ for any $j \in T$ and any $i \in N \setminus T$ and since the tie-breaking rule fixes the order of service across agents in T, it follows that for any $j \in T$, any θ_{-j}, $S_j(\sigma(\theta_j, \theta_{-j}))$ is a constant for all $\theta_j \in \Theta$. Hence for any agent $j \in T$, the job completion time is fixed for all profiles implying non-increasingness. Consider any agent i with $w_i > 0$, any profile $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, any $\theta'_i > \theta_i$ such that $\sigma(\theta_i, \theta_{-i}) := \sigma$, $\sigma(\theta'_i, \theta_{-i}) := \sigma'$ and $\sigma \neq \sigma'$. Using affine maximization we have the following:

(I) $w_i\theta_i S_i(\sigma) + \sum_{j\in N\setminus\{i\}} w_j\theta_j S_j(\sigma) + \kappa(\sigma) \leq w_i\theta_i S_i(\sigma') + \sum_{j\in N\setminus\{i\}} w_j\theta_j S_j(\sigma') + \kappa(\sigma')$, and

(II) $w_i\theta'_i S_i(\sigma) + \sum_{j\in N\setminus\{i\}} w_j\theta_j S_j(\sigma) + \kappa(\sigma) \geq w_i\theta'_i S_i(\sigma') + \sum_{j\in N\setminus\{i\}} w_j\theta_j S_j(\sigma') + \kappa(\sigma')$.

From (I) and (II) we get

(III) $w_i\theta_i[S_i(\sigma) - S_i(\sigma')] + \sum_{j\in N\setminus\{i\}} w_j\theta_j[S_j(\sigma) - S_j(\sigma')] \leq \kappa(\sigma') - \kappa(\sigma)$, and

(IV) $w_i\theta'_i[S_i(\sigma) - S_i(\sigma')] + \sum_{j\in N\setminus\{i\}} w_j\theta_j[S_j(\sigma) - S_j(\sigma')] \geq \kappa(\sigma') - \kappa(\sigma)$.

Using (III) and (IV) it easily follows that $w_i(\theta'_i - \theta_i)[S_i(\sigma) - S_j(\sigma')] \geq 0$. Given $\theta'_i > \theta_i$ and $w_i > 0$, it follows that $S_i(\sigma') = S_i(\sigma(\theta'_i, \theta_{-i})) \leq S_i(\sigma(\theta_i, \theta_{-i})) = S_i(\sigma)$ and we have non-increasingness.

To prove the final part we first prove that the NI sequencing rule σ^V (defined in Example 1) for any two-agent sequencing problem is not an affine cost minimizer. Suppose, to the contrary, that σ^V is an affine cost minimizer. Then for any θ' such that $\theta'_1 > a_1$, the affine cost minimization must give $w_1\theta'_1 s_1 + w_2\theta'_2(s_1 + s_2) + \kappa(\sigma^1 = (\sigma_1 = 1, \sigma_2 = 2)) \leq w_1\theta'_1(s_1 + s_2) + w_2\theta'_2 s_2 + \kappa(\sigma^2 = (\sigma_1 = 2, \sigma_2 = 1))$. Therefore, we must have (I) $w_2\theta'_2 s_1 + \kappa(\sigma^1) \leq w_1\theta'_1 s_2 + \kappa(\sigma^2)$ for any θ'_2. However, if $w_2 > 0$, then this is not possible as by keeping θ'_1 fixed and taking θ'_2 very large we can always have a violation of inequality (I). Hence, we must have $w_2 = 0$. But if $w_2 = 0$, then the sequencing rule σ^V is independent of the waiting cost of agent 2 which is not the case (since for any $\theta_1 < a_1$ the sequencing rule σ^V depends on whether θ_2 is greater than a_2 or not). Hence we have the required contradiction.

To complete the proof we show that for any sequencing problem with three or more agents, σ^{NA} (defined in Example 2) is not an affine cost minimizer. Suppose, to the contrary, that σ^{NA} is an affine cost minimizer. Since σ^{NA} has the property that for any $i \in N$, any given θ_{-i}, there exists $\theta'_i > \theta_i$ such that $S_i(\theta'_i, \theta_{-i}) < S_i(\theta_i, \theta_{-i})$, it is necessary that the affine cost minimizer must be such that $w_i > 0$ for all $i \in N$. Consider the profile $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_n)$ such that $\theta_2/s_2 > \theta_1/s_1 > \theta_3/s_3 > \ldots \geq \theta_n/s_n$. Then by σ^{NA}, the order selected is $\sigma = (\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3, \ldots, \sigma_n = n)$. Moreover affine cost minimization must rule out the order $\sigma' = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2, \ldots, \sigma_n = n)$ and hence it follows that $w_1\theta_1 s_1 + w_2\theta_2(s_1 + s_2) + w_3\theta_3(s_1 + s_2 + s_3) + \kappa(\sigma) \leq w_1\theta_1(s_1 + s_2 + s_3) + w_2\theta_2 s_2 + w_3\theta_3(s_2 + s_3) + \kappa(\sigma')$. This inequality implies that $w_2\theta_2 s_1 + w_3\theta_3 s_1 \leq w_1\theta_1(s_2 + s_3) + \kappa(\sigma') - \kappa(\sigma)$. By selecting a profile $\theta' = (x_2, \theta_{-2})$ such that $x_2 > \theta_2$ we continue to have $x_2/s_2 > \theta_1/s_1 > \theta_3/s_3 > \ldots > \theta_n/s_n$ and by σ^{NA}, the order selected continues to be $\sigma = (\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3, \ldots, \sigma_n = n)$. Hence for any $x_2 > \theta_2$ we must have

$$w_2 x_2 s_1 + w_3\theta_3 s_1 \leq w_1\theta_1(s_2 + s_3) + \kappa(\sigma') - \kappa(\sigma). \tag{18}$$

But as x_2 increases, the left hand side of inequality (18) increases and the right hand side remains unchanged. Therefore, for a sufficiently large value of x_2, inequality (18) fails to hold and hence we have a contradiction to our assumption that σ^{NA} is an affine cost minimizer. □

Proof of Theorem 1: Consider any NI sequencing rule σ. We first prove that if a mechanism (σ, τ) implements σ, then it is necessarily a cut-off based mechanism. Consider any agent $i \in N$ and fix any any profile $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$. From ineqaulity (16) in Proposition 2 it follows that for all $\theta_i, \theta'_i \in \Theta$ such that $\sigma(\theta) = \sigma(\theta'_i, \theta_{-i})$, $\tau_i(\theta) = \tau_i(\theta'_i, \theta_{-i})$.

If $T = 1$, then for any $\theta_i, \theta'_i \in \Theta$, $\sigma(\theta) = \sigma(\theta'_i, \theta_{-i})$ implying $\tau_i(\theta) = \tau_i(\theta'_i, \theta_{-i}) = h_i(\theta_{-i})$ and, given (3), the mechanism is necessarily cut-off based.

Suppose $T \geq 2$. Then there exists a waiting cost cut-off vector $(\theta_i^{(1)}, \ldots, \theta_i^{(T-1)})$ where $0 := \theta_i^{(T)} < \theta_i^{(T-1)} < \ldots < \theta_i^{(2)} < \theta_i^{(1)} < \theta_i^{(0)} := \infty$ such that for any $t \in \{1, \ldots, T\}$, $S_i(\sigma(\theta_i^t, \theta_{-i})) := \bar{S}(t, \theta_{-i})$ for all $\theta_i^t \in (\theta_i^{(t)}, \theta_i^{(t-1)})$. Given that the sequencing rule is NI, $\bar{S}(1, \theta_{-i}) < \bar{S}(2, \theta_{-i}) < \ldots < \bar{S}(T-1, \theta_{-i}) < \bar{S}(T, \theta_{-i})$. Consider any pair $(\theta_i^{t+1}, \theta_i^t) \in (\theta_i^{(t+1)}, \theta_i^{(t)}) \times (\theta_i^{(t)}, \theta_i^{(t-1)})$. By applying the implementability condition (1) when the actual profile is $(\theta_i^{t+1}, \theta_{-i})$ ((θ_i^t, θ_{-i})) and the misreport of agent

i is θ_i^t (θ_i^{t+1}) we get

$$\theta_i^{t+1} D_t(\theta_{-i}) \leq \tau_i(\theta_i^{t+1}, \theta_{-i}) - \tau_i(\theta_i^t, \theta_{-i}) \leq \theta_i^t D_t(\theta_{-i}). \tag{19}$$

Since (19) must hold for all $(\theta_i^{t+1}, \theta_i^t) \in (\theta_i^{(t+1)}, \theta_i^{(t)}) \times (\theta_i^{(t)}, \theta_i^{(t-1)})$, it follows that

$$\tau_i(\theta_i^{t+1}, \theta_{-i}) - \tau_i(\theta_i^t, \theta_{-i}) = \theta_i^{(t)} D_t(\theta_{-i}). \tag{20}$$

Condition (20) must hold for all $t \in \{1, \ldots, T-1\}$. By setting $\tau_i(\theta_i^T, \theta_{-i}) = h_i(\theta_{-i})$ for any $\theta_i^T \in (\theta_i^{(T)}, \theta_i^{(T-1)})$ and then solving condition (20) recursively we get (3).

To complete the proof of this part, consider any pair $\theta_i^{t+1} \in (\theta_i^{(t+1)}, \theta_i^{(t)})$ and the cut-off point $\theta_i^{(t)}$. By applying the implementability condition (1) when the actual profile is $(\theta_i^{t+1}, \theta_{-i})$ ($(\theta_i^{(t)}, \theta_{-i})$) and the misreport of agent i is $\theta_i^{(t)}$ (θ_i^{t+1}) we get

$$\theta_i^{t+1} \overline{D}_t(\theta_{-i}) \leq \tau_i(\theta_i^{t+1}, \theta_{-i}) - \tau_i(\theta_i^{(t)}, \theta_{-i}) \leq \theta_i^{(t)} \overline{D}_t(\theta_{-i}). \tag{21}$$

Using condition (20) above, we substitute $\tau_i(\theta_i^{t+1}, \theta_{-i}) = \tau_i(\theta_i^t, \theta_{-i}) + \theta_i^{(t)} D_t(\theta_{-i})$ in (21) to get

$$\theta_i^{t+1} \overline{D}_t(\theta_{-i}) \leq \tau_i(\theta_i^t, \theta_{-i}) - \tau_i(\theta_i^{(t)}, \theta_{-i}) + \theta_i^{(t)} D_t(\theta_{-i}) \leq \theta_i^{(t)} \overline{D}_t(\theta_{-i}). \tag{22}$$

Since inequality (22) must hold for all $\theta_i^{t+1} \in (\theta_i^{(t+1)}, \theta_i^{(t)})$, it follows that

$$\tau_i(\theta_i^t, \theta_{-i}) - \tau_i(\theta_i^{(t)}, \theta_{-i}) = -\theta_i^{(t)} D_t(\theta_{-i}) + \theta_i^{(t)} \overline{D}_t(\theta_{-i}). \tag{23}$$

From condition (23) we get $\tau_i(\theta_i^{(t)}, \theta_{-i}) = \tau_i(\theta_i^t, \theta_{-i}) - \theta_i^{(t)} \overline{D}_t(\theta_{-i}) + \theta_i^{(t)} D_t(\theta_{-i})$ for any $t \in \{1, \ldots, T-1\}$ with $T \geq 2$ and this completes the proof. The converse is quite easy and hence omitted. □

Proof of Lemma 2:

Proof of (B1): Without loss of generality, let $i = 1$ and let $X_0, X_1, X_{12} \in A_1$ and $X_2 \in A_2$. From the cut-off based mechanism we get $I_1(X_0) = I_1(X_1) = \theta_1^{(1)}(\theta_2)s_2$, $I_2(X_0) = I_2(X_1) = I_1(X_2) = I_2(X_{12}) = 0$, $I_2(X_2) = \theta_2^{(1)}(\theta_1)s_1$ and $I_1(X_{12}) = \theta_1^{(1)}(\theta_2')s_2$. By applying condition (10) of Lemma 1 we get $\sum_{S \subseteq \{1,2\}} I(\theta(S)) = \theta_2^{(1)}(\theta_1)s_1 - \theta_1^{(1)}(\theta_2')s_2 = 0$ and we get (B1) for $i = 1$.

Proof of (B2): Without loss of generality, let $i = 1$ and let $X_0, X_1 \in A_1$ and $X_2, X_{12} \in A_2$. From the cut-off based mechanism we get $I_1(X_0) = I_1(X_1) = \theta_1^{(1)}(\theta_2)s_2$, $I_2(X_0) = I_2(X_1) = I_1(X_2) = I_1(X_{12}) = 0$, $I_2(X_2) = \theta_2^{(1)}(\theta_1)s_1$ and $I_2(X_{12}) = \theta_2^{(1)}(\theta_1')s_1$. By applying condition (10) of Lemma 1 we get $\sum_{S \subseteq \{1,2\}} I(\theta(S)) = [\theta_2^{(1)}(\theta_1) - \theta_2^{(1)}(\theta_1')]s_1 = 0$ and we get (B2) for $i = 1$. □

Proof of Theorem 2: We first show that any NI sequencing rule which is TAB is implementable with balanced transfers. If we have the TAB sequencing rule σ^{Tx} given by (T1a) with $k = 1$, then the cut-off based transfer of agents 1 and 2 are given by (4) and (5) respectively. If we set $h_2(\theta_1) = 0$ for all $\theta_1 \in \Theta$ and if we set

$$h_1(\theta_2) = \begin{cases} 0 & \text{if } \theta_2 \leq a_2, \\ a_2 s_1 & \text{if } \theta_2 > a_2, \end{cases} \tag{24}$$

then, using $a_1 s_2 = a_2 s_1$, we get

$$\tau_1(\theta) = -\tau_2(\theta) = \begin{cases} 0 & \text{if either } \theta_1 \geq a_1 \text{ or } \theta_2 \leq a_2, \\ a_2 s_1 & \text{if } \theta_1 < a_1 \text{ and } \theta_2 > a_2. \end{cases} \tag{25}$$

For the TAB sequencing rule (T1b) with $k = 1$, the argument is similar and hence omitted. Finally, if we have the TAB sequencing rule σ^{Tx} given by (T2a) and with $k = 1$, then, by Theorem 1, we get the following cut-off based transfers. For any $\theta \in \Theta^2$,

$$\tau_1(\theta) = \begin{cases} h_1(\theta_2) & \text{if } \theta_1 < a_1, \\ h_1(\theta_2) - a_1 s_2 & \text{if } \theta_1 \geq a_1, \end{cases} \tag{26}$$

and $\tau_2(\theta) = h_2(\theta_1)$. If we set $h_1(\theta_2) = 0$ for all $\theta_2 \in \Theta$, and if we set

$$h_2(\theta_1) = \begin{cases} 0 & \text{if } \theta_1 < a_1, \\ a_1 s_2 & \text{if } \theta_1 \geq a_1, \end{cases} \tag{27}$$

then we get implementability with balanced transfers. Specifically, we have

$$\tau_1(\theta) = -\tau_2(\theta) = \begin{cases} 0 & \text{if } \theta_1 < a_1, \\ -a_1 s_2 & \text{if } \theta_1 \geq a_1. \end{cases} \tag{28}$$

For the TAB sequencing rule (T2b) with $k = 1$, the argument is similar and hence omitted. Therefore, we get implementability with balanced transfers for TAB seqeuncing rules σ^{Tx}.

We now prove the converse, that is, if a non-constant NI sequencing rule is implementable with balanced transfers, then it must be a TAB sequencing rule. We prove this in two steps. Let σ be non-constant NI sequencing rule which is implementable with balanced transfers and satisfies the following property.

(P1) There exists an $i \in \{1,2\}$ such that we can find a pair $\theta, \theta' \in \Theta^2$ with the property that $\theta'_i > \theta_i$ and $X_0, X_i, X_{12} \in A_i(\sigma)$ and $X_j \in A_j(\sigma)$ where $X_0 = \theta$, $X_1 = (\theta'_1, \theta_2)$, $X_2 = (\theta_1, \theta'_2)$ and $X_{12} = \theta'$.

Step 1: If σ is a non-constant NI sequencing rule that is implementable with balanced transfers and for which **(P1)** holds, then σ must be a TAB sequencing rule of the form (T1).

Proof of Step 1: Suppose we have $X_0, X_i, X_{12} \in A_i(\sigma)$ and $X_j \in A_j(\sigma)$, for any $i, j \in \{1,2\}$ with $i \neq j$. Without loss of generality, let $i = 1, j = 2$. Then by Lemma 2 (B1), the cut-off points $\theta_1^{(1)}(\theta'_2)$ and $\theta_2^{(1)}(\theta_1)$ are such that $\theta_1^{(1)}(\theta'_2)s_2 = \theta_2^{(1)}(\theta_1)s_1$. Since $X_0 = \theta, X_1 = (\theta'_1, \theta_2), X_{12} = (\theta'_1, \theta'_2) \in A_1(\sigma)$ we have $Q_1(\theta), Q_1(\theta'_1, \theta_2), Q_1(\theta'_1, \theta'_2) \subset A_1(\sigma)$. Also we have $X_2 = (\theta_1, \theta'_2) \in A_2(\sigma)$ hence $Q_2(\theta_1, \theta'_2) \subset A_2(\sigma)$. Since $\theta_2^{(1)}(\theta_1)$ is the cut-off point for agent 2 at θ_1, $Q'_1(F) \subset A_1(\sigma)$ and $Q'_2(F) \subset A_2(\sigma)$ where $F = (\theta_1, \theta_2^{(1)}(\theta_1))$. Similarly the cut-off point for agent 1 at θ'_2 is $\theta_1^{(1)}(\theta'_2)$. Let $E = (\theta_1^{(1)}(\theta'_2), \theta'_2)$, hence $Q'_1(E) \subset A_1(\sigma)$ and $Q'_2(E) \subset A_2(\sigma)$. Take any point $(\theta_1^P, \theta_2^P) := P \in T_1(E) := \{(\theta_1, \theta_2) \in \Theta^2 \mid \theta_1 > \theta_1^{(1)}(\theta'_2) \,\&\, \theta_2 > \theta'_2\}$ and, if possible, assume $P \in A_2(\sigma)$. As shown in Figure 3, consider the points X_0, X'_1, X'_2, P where $X_0, X'_1 \in A_1(\sigma)$ and $X'_2, P \in A_2(\sigma)$. Then by Lemma 2 (B2) the cut-off points for agent 2 at θ_1 and at θ_1^P are equal, that is, $\theta_2^{(1)}(\theta_1) = \theta_2^{(1)}(\theta_1^P)$. Given $(\theta_1^P, \theta_2^{(1)}(\theta_1)) \in A_1(\sigma), \theta_2^{(1)}(\theta_1^P) > \theta_2^{(1)}(\theta_1)$. Hence our assumption that $P \in A_2(\sigma)$ is not correct. Therefore, $P \in A_1(\sigma)$ implying that $T_1(E) \subset A_1(\sigma)$. All these facts are represented in Figure 3, where the red coloured region denotes subsets of $A_2(\sigma)$ and the blue coloured region denotes subsets of $A_1(\sigma)$.

In Figure 4, let us consider any $B := (\theta_1^B, \theta_2^B) \in S_1 = \{(\theta''_1, \theta''_2) \in \Theta^2 \mid \theta''_1 < \theta_1 \,\&\, \theta''_2 < \theta_2(\theta_1)\}$. If possible, assume $B \in A_2(\sigma)$. Consider the points B, R, X_{12}, M such that $B, M \in A_2(\sigma)$ and $R, X_{12} \in A_1(\sigma)$. Again, using Lemma 2 (B2), the cut-off points for agent 1 for θ_2^B and θ'_2, are equal, that is, $\theta_1^{(1)}(\theta_2^B) = \theta_1^{(1)}(\theta'_2)$. However, this is not the case since $\theta_1^{(1)}(\theta_2^B) \leq \theta_1$ and $\theta_1^{(1)}(\theta'_2) > \theta_1$ so that $\theta_1^{(1)}(\theta_2^B) \neq \theta_1^{(1)}(\theta'_2)$. So $B \in A_1(\sigma)$. Finally, we now show that, for any point r in the rectangle X_2ECF (see Figure 4), $r \in A_2(\sigma)$. If possible, assume $r \in A_1(\sigma)$. We consider the points X_0, T, P, X_2 where $X_0, T \in A_1(\sigma)$ and $X_2, P \in A_2(\sigma)$. Since we have assumed $r \in A_1(\sigma)$, the cut-off point for agent 2 at θ_1^r, that is, $\theta_2^{(1)}(\theta_1^r) \geq \theta_2^r$ and the cut-off point for agent 2 at θ_1 is $\theta_2^{(1)}(\theta_1)$. Since $\theta_2^{(1)}(\theta_1^r) \geq \theta_2^r > \theta_2^{(1)}(\theta_1)$, Lemma 2 (B2) is violated. Hence for any r in the rectangle X_2ECF, $r \in A_2(\sigma)$. The final result of all these arguments is depicted in Figure 5.

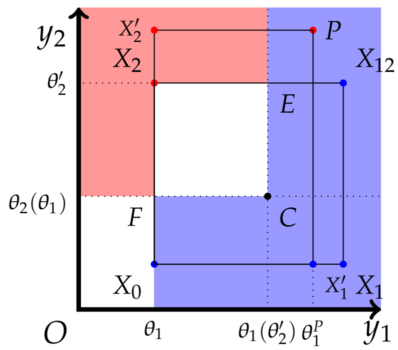

Figure 3

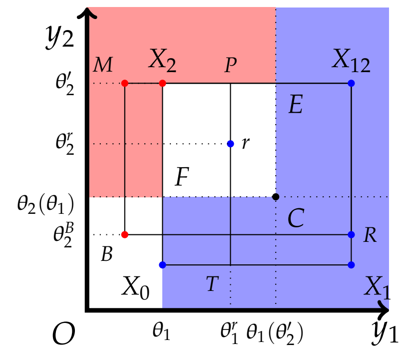

Figure 4

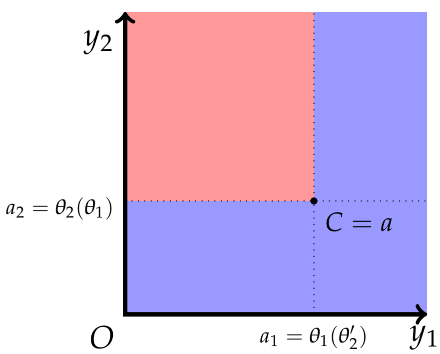

Figure 5

Define $a = (a_1 = \theta_1^{(1)}(\theta_2'), a_2 = \theta_2^{(1)}(\theta_1))$. Figure 5 shows that red coloured region, that is, the open set $Q_2'(a) \subseteq A_2(\sigma)$ and the blue coloured region, that is, the open set $\Theta^2 \setminus Q_2(a) \subseteq A_1(\sigma)$. If $a \in A_1(\sigma)$, then $A_1(\sigma) = \{\theta \in \Theta^2 \mid \textit{either } \theta_1 \geq a_1 \textit{ or } \theta_2 \leq a_2\}$ which is the TAB sequencing rule σ^{T1a}. If $a \in A_2(\sigma)$, then $A_1(\sigma) = \{\theta \in \Theta^2 \mid \textit{either } \theta_1 > a_1 \textit{ or } \theta_2 < a_2\}$ which is the TAB sequencing rule σ^{T1b}. Hence σ is a TAB sequencing rule of the form (T1) with $k = 1$ and $l = 2$. This proves Step 1.

Step 2: If σ is a non-constant NI sequencing rule that is implementable with balanced transfers and for which **(P1)** does not hold, then σ must be a TAB sequencing rule of the form (T2).

Proof of Step 2: Suppose σ is a non-constant NI sequencing rule that is implementable with balanced transfers and for which **(P1)** does not hold. Since σ is not a constant sequencing rule and satisfies NI, there exists $i \in \{1,2\}$ and a pair $\theta, \theta' \in \Theta^2$ such that $X_0, X_i \in A_i(\sigma)$ and $X_j, X_{12} \in A_j(\sigma)$ where $j \in \{1,2\}$, $j \neq i$, $X_0 = \theta$, $X_1 = (\theta_1', \theta_2)$, $X_2 = (\theta_1, \theta_2')$, $X_{12} = \theta'$, $\theta_1 < \theta_1'$ and $\theta_2 < \theta_2'$. Without loss of generality, let $i = 1, j = 2$ so that $X_0, X_1 \in A_1(\sigma)$ and $X_2, X_{12} \in A_2(\sigma)$. Then by Lemma 2 (B2), the cut-off points $\theta_2^{(1)}(\theta_1)$ and $\theta_2^{(1)}(\theta_1')$ are such that $\theta_2^{(1)}(\theta_1) = \theta_2^{(1)}(\theta_1') := \bar{\theta}_2$. Consider any $\lambda \in (0,1)$ and the profile pair $(\theta_1(\lambda), \theta_2), \theta' \in \Theta^2$ where $\theta_1(\lambda) := \lambda\theta_1 + (1-\lambda)\theta_1'$ and define $X_0^\lambda = (\theta_1(\lambda), \theta_2)$, $X_1^\lambda = (\theta_1', \theta_2)$, $X_2^\lambda = (\theta_1(\lambda), \theta_2')$, $X_{12}^\lambda = \theta'$. Observe that $X_0^\lambda, X_1^\lambda \in Q_1(\theta_1, \theta_2) \subseteq A_1(\sigma)$ and $X_2^\lambda, X_{12}^\lambda \in Q_2(\theta_1', \theta_2') \subseteq A_2(\sigma)$. Hence by applying Lemma 2 (B2) we get the cut-off points $\theta_2^{(1)}(\theta_1(\lambda))$ and $\theta_2^{(1)}(\theta_1')$ are such that

$\theta_2^{(1)}(\theta_1(\lambda)) = \theta_2^{(1)}(\theta_1') = \bar{\theta}_2$ implying that $\theta_2^{(1)}(\theta_1(\lambda)) = \theta_2^{(1)}(\theta_1) = \theta_2^{(1)}(\theta_1') = \bar{\theta}_2$ for any $\lambda \in (0,1)$. Hence by applying non-increasingness of σ we have $Q_1'(\theta_1, \bar{\theta}_2) \subseteq A_1(\sigma)$ and $Q_2'(\theta_1', \bar{\theta}_2) \subseteq A_2(\sigma)$. This is depicted in Figure 6.

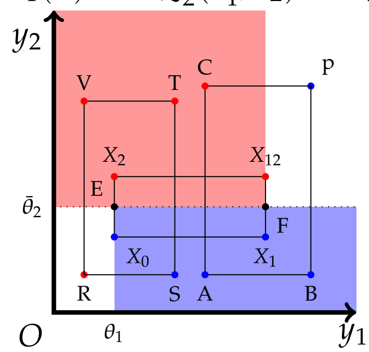

Figure 6

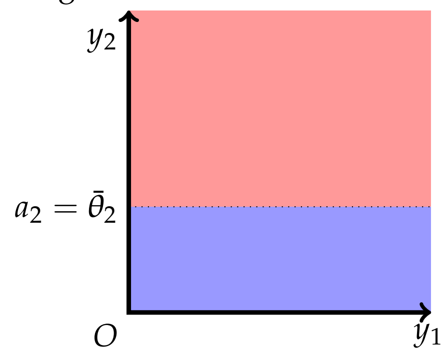

Figure 7

Consider the set $\bar{S}_2(\theta_1', \bar{\theta}_2) = \{(\theta_1'', \theta_2'') \in \Theta^2 \mid \theta_1'' > \theta_1' \;\&\; \theta_2'' > \bar{\theta}_2\}$ and any point $P \in \bar{S}_2(\theta_1', \bar{\theta}_2)$. If $P \in A_1(\sigma)$, then by selecting the rectangle $ABPC$ (see Figure 6) we find that $A, B, P \in A_1(\sigma)$ and $C \in A_2(\sigma)$ which is a violation of the fact that σ fails to satisfy Property **(P1)**. Hence $\bar{S}_2(\theta_1', \bar{\theta}_2) \subseteq A_2(\sigma)$. Similarly, consider the set $\bar{S}_1(\theta_1, \bar{\theta}_2) = \{(\theta_1'', \theta_2'') \in \Theta^2 \mid \theta_1'' < \theta_1 \;\&\; \theta_2'' < \bar{\theta}_2\}$ and any point $R \in \bar{S}_1(\theta_1, \bar{\theta}_2)$. If $R \in A_2(\sigma)$, then by selecting the rectangle $RSTV$ (see Figure 6) we find that $R, V, T \in A_2(\sigma)$ and $S \in A_1(\sigma)$ which is a violation of the fact that σ fails to satisfy Property **(P1)**. Hence $\bar{S}_1(\theta_1, \bar{\theta}_2) \subseteq A_1(\sigma)$. Therefore, we have obtained the following

(t1) $Q_1'(\theta_1, \bar{\theta}_2) \cup \bar{S}_1(\theta_1, \bar{\theta}_2) = \{(\theta_1'', \theta_2'') \in \Theta^2 \mid \theta_2'' < \bar{\theta}_2\} \subseteq A_1(\sigma)$ and
(t2) $Q_2'(\theta_1', \bar{\theta}_2) \cup \bar{S}_2(\theta_1', \bar{\theta}_2) = \{(\theta_1'', \theta_2'') \in \Theta^2 \mid \theta_2'' > \bar{\theta}_2\} \subseteq A_2(\sigma)$.

Cases (t1) and (t2) are depicted in Figure 7. By setting $\bar{\theta}_2 = a_2$, we get $\{\theta \in \Theta^2 \mid \theta_2 > a_2\} \subseteq A_2(\sigma)$ (from (t2) above) and $\Theta^2 \setminus \{\theta \in \Theta^2 \mid \theta_2 \geq a_2\} \subseteq A_1(\sigma)$ (from (t1) above). What about points on the cut-off line $y_2 = a_2 = \bar{\theta}_2$ that separates the two decision? Given non-increasingness of σ we have the following possibilities:

(i) All points of the line $y_2 = a_2$ are in $A_2(\sigma)$.
(ii) All points of the line $y_2 = a_2$ are in $A_1(\sigma)$.
(iii) There exists a $\theta_1^* > 0$ such that all points (θ_1, a_2) with $\theta_1 < \theta_1^*$ are in $A_2(\sigma)$, all points (θ_1', a_2) with $\theta_1' > \theta_1^*$ are in $A_1(\sigma)$ and (θ_1^*, a_2) belongs to either $A_1(\sigma)$ or $A_2(\sigma)$.

However, for case (iii), take the pair of profiles $\theta, \theta' \in \Theta^2$ such that $0 < \theta_1 < \theta_1^* < \theta_1'$ and $\theta_2 = a_2 < \theta_2'$ and define $X_0 = \theta$, $X_1 = (\theta_1', \theta_2)$, $X_2 = (\theta_1, \theta_2')$, $X_{12} = \theta'$. Then we have $X_0, X_2, X_{12} \in A_2(\sigma)$ and $X_1 \in A_1(\sigma)$. This violates our initial assumption that σ fails to satisfy Property **(P1)**. Hence on the cut-off line $y_2 = a_2$ either case (i) holds or case (ii) holds. If case (i) holds, then we have σ^{T2a}, and, if case (ii) holds, then we have σ^{T2b}. Hence we have the TAB sequencing rule of the form (T2) with $k = 2$ and $l = 1$.

$\square$

Proof of Theorem 3: For any partition $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi$ such that $K \geq 2$ and fix any set of increasing and onto functions $f = \{f_1, \ldots, f_n\}$ and consider the GP-CM sequencing rule $\sigma^{\pi(N),f}$. We show that any such GP-CM sequencing rule $\sigma^{\pi(N),f}$ is implementable with balanced transfer by establishing that condition (8) holds, that is the profile contingent aggregate incentive payment is $(n-1)$ type separable. For any profile $\theta \in \Theta^n$ and any $\pi_r \in \pi(N)$, define the function $z_f(\theta; \pi_r) := \sum_{j\in\pi_r} I_j^{\pi(N),f}(\theta) = \sum_{j\in\pi_r} s_j \left(\sum_{q\in P'_j(\sigma^{\pi(N),f}(\theta))\cap\pi_r} \left[s_q f_j^{-1}\left(f_q(\theta_q)/s_q\right)\right]\right)$. It is important to note that given any profile $\theta \in \Theta^n$, for any $\pi_r \in \pi(N)$, the sum of incentive payments of the group $\pi_r \in \pi(N)$ is $z_f(\theta; \pi_r)$ and it has the property that it is independent of the waiting costs of the agents $N \setminus \{\pi_r\}$. Hence for any $\theta \in \Theta^n$, for any $\pi_r \in \pi(N)$, we write $z_f(\theta; \pi_r) := z_f(\theta_{\pi_r})$. Consider the GP-CM-cut-off based mechanism with transfer (14) and select for any $\theta \in \Theta^n$ and any $i \in \pi_k \in \pi(N)$, $G_i(\theta_{-i}) = \sum_{\pi_r\in\pi(N)\setminus\{\pi_k\}}[z_f(\theta_{\pi_r})/(n - |\pi_r|)]$. Given this selection of $G_i(.)$ functions we get

$$\sum_{i\in N} \tau_i^{\pi(N),f}(\theta) = \sum_{i\in N} G_i(\theta_{-i}) - I^{\pi(N),f}(\theta)$$

$$= \sum_{i\in N} \left(\sum_{\pi_r\in\pi(N)\setminus\{\pi_k\}} \frac{z_f(\theta_{\pi_r})}{(n-|\pi_r|)}\right) - \sum_{\pi_k\in\pi(N)} \left(\sum_{j\in\pi_k} I_j^{\pi(N),f}(\theta)\right)$$

$$= \sum_{\pi_k\in\pi(N)} \left(\sum_{i\in N\setminus\{\pi_k\}} \frac{z_f(\theta_{\pi_k})}{(n-|\pi_k|)}\right) - \sum_{\pi_k\in\pi(N)} z_f(\theta_{\pi_k})$$

$$= \sum_{\pi_k\in\pi(N)} \left[\frac{z_f(\theta_{\pi_k})}{(n-|\pi_k|)} \left(\sum_{i\in N\setminus\{\pi_k\}} 1\right)\right] - \sum_{\pi_k\in\pi(N)} z_f(\theta_{\pi_k})$$

$$= \sum_{\pi_k\in\pi(N)} \left[\frac{z_f(\theta_{\pi_k})}{(n-|\pi_k|)}(n - |\pi_k|)\right] - \sum_{\pi_k\in\pi(N)} z_f(\theta_{\pi_k})$$

$$= \sum_{\pi_k\in\pi(N)} z_f(\theta_{\pi_k}) - \sum_{\pi_k\in\pi(N)} z_f(\theta_{\pi_k}) = 0.$$

Hence, for any given partition $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi$ such that $K \geq 2$, for any set of increasing and onto functions $f = \{f_1, \ldots, f_n\}$, the GP-CM sequencing rule $\sigma^{\pi(N),f}$ is implementable with balanced transfers.

For the partition $\pi(N) = (\pi_1 = \pi_K = \{n\}) \in \Pi$ such that $K = 1$ and any set of increasing and onto functions $f = \{f_1, \ldots, f_n\}$, consider the GP-CM sequencing rule

$\sigma^{\pi(N),f}$. We show that any such GP-CM sequencing rule $\sigma^{\pi(N),f}$ is implementable with balanced transfer by establishing that condition (8) holds, that is the profile contingent aggregate incentive payment is $(n-1)$ type separable.

Consider $\sigma^{\pi(N),f}(\theta)$ for the profile $\theta \in \Theta^n$ and consider agent i. Define $\Sigma^i(N) = \{\sigma \in \Sigma(N) \mid \sigma_i = n\}$ as the set of orders in $\Sigma(N)$ such that agent i is last in the order. We define the "induced" order $\sigma^{\pi(N),f}(\theta_{-i}) \in \arg\min_{\sigma \in \Sigma^i(N)} \sum_{j \in N} f_j(\theta_j) S_j(\sigma)$. Given the profile $\theta \in \Theta^n$ and any agent i, the relation between the order $\sigma^{\pi(N),f}(\theta)$ and the induced order $\sigma^{\pi(N),f}(\theta_{-i})$ is as follows:

$$\sigma_j^{\pi(N),f}(\theta_{-i}) = \begin{cases} \sigma_j^{\pi(N),f}(\theta) - 1 & \text{if } j \in P_i'(\sigma^{\pi(N),f}(\theta)), \\ \sigma_j^{\pi(N),f}(\theta) & \text{if } j \in P_i(\sigma^{\pi(N),f}(\theta)). \end{cases} \tag{29}$$

In words, $\sigma^{\pi(N),f}(\theta_{-i})$ is generated from the order $\sigma^{\pi(N),f}(\theta)$ by moving agent i in the last position and moving all agents behind him up by one position. Consider the GP-CM-cut-off based mechanism with transfer (14) and select for any $\theta \in \Theta^n$ and any $i \in N$, $G_i(\theta_{-i}) = (1/(n-2)) \sum_{j \in N \setminus \{i\}} s_j \left(\sum_{k \in P_j(\sigma^{\pi(N),f}(\theta_{-i}))} X_{jk}(\theta_k) \right)$ where $X_{jk}(\theta_k) := f_j^{-1}\left(f_k(\theta_k)/s_k\right) s_k$. Given this selection of $G_i(.)$ functions we get

$$\sum_{i \in N} \tau_i^{\pi(N),f}(\theta) = \sum_{i \in N} G_i(\theta_{-i}) - I(\theta)$$

$$= \frac{1}{(n-2)} \sum_{i \in N} \sum_{j \in N \setminus \{i\}} s_j \left(\sum_{k \in P_j'(\sigma^{\pi(N),f}(\theta_{-i}))} X_{jk}(\theta_k) \right) - I(\theta)$$

$$= \frac{1}{(n-2)} \sum_{i \in N} \left[\sum_{j \in N \setminus \{i\}} s_j \left(\sum_{k \in P_j'(\sigma^{\pi(N),f}(\theta))} X_{jk}(\theta_k) \right) - \sum_{j \in P_i(\sigma^{\pi(N),f}(\theta))} s_j X_{ji}(\theta_i) \right] - I(\theta)$$

$$= \frac{1}{(n-2)} \sum_{i \in N} \sum_{j \in N \setminus \{i\}} I_j(\theta) - \frac{1}{(n-2)} \sum_{i \in N} \left(\sum_{j \in P_i(\sigma^{\pi(N),f}(\theta))} s_j X_{ji}(\theta_i) \right) - I(\theta)$$

$$= \left(\frac{n-1}{n-2}\right) I(\theta) - \frac{1}{(n-2)} \sum_{i \in N} s_i \left(\sum_{j \in P_i'(\sigma^{\pi(N),f}(\theta))} X_{ij}(\theta_j) \right) - I(\theta)$$

$$= \left(\frac{n-1}{n-2}\right) I(\theta) - \frac{1}{(n-2)} I(\theta) - I(\theta) = 0.$$

Thus, any GP-CM sequencing rule $\sigma^{\pi(N),f}$ which is onto is implementable with balanced transfers. □

REFERENCES

[1] Bikhchandani, S., Chatterjee, Lavi, R., Mu'alem, A. Nisan, N. and Sen, A. (2006). Weak monotonicity characterizes deterministic dominant strategy implementation. Econometrica 74, 1109-1132.

[2] Carbajal, J. C., McLennan, A. and Tourky, A. (2013). Truthful Implementation and Aggregation in Restricted Domains, Journal of Economic Theory 153, 103-116.

[3] De, P. and Mitra, M. (2015). Incentives and justice for sequencing problems. *mimeo*.

[4] Dobzinski, S. and Nisan, N. (2009). A modular approach to Roberts' theorem. In Algorithmic Game Theory, pages 1423. Springer.

[5] Dolan, R. J. (1978). Incentive mechanisms for priority queuing problems. The Bell Journal of Economics 9, 421-436.

[6] Duives, J., Heydenreich, B., Mishra, D., Muller, R. and Uetz, M. (2012). On Optimal Mechanism Design for a Sequencing Problem. *mimeo*.

[7] Hain, R. and Mitra, M. (2004). Simple sequencing problems with interdependent costs. Games and Economic Behavior 48, 271-291.

[8] Hurwicz, L., (1975). On the existence of allocative systems whose manipulative Nash equilibria are Pareto optimal. Mimeo, University of Minnesota.

[9] Hurwicz, L. and Walker, M. (1990). On the generic non-optimality of dominant strategy allocation mechanisms: A general theorem that includes pure exchange economies. Econometrica 58, 683-704.

[10] Lavi, R., Mualem, A., Nisan, N. (2009). Two simplified proofs for Roberts theorem. Social Choice and Welfare 32(3), 407-423.

[11] Marchant, T. and Mishra, D. (2015). Mechanism design with two alternatives in quasi-linear environments. Social Choice and Welfare 44, 433-455.

[12] Mishra, D., and Mitra,M., "Cycle Monotonicity in Scheduling Models." In Econophysics and Economics of Games, Social Choices and Quantitative Techniques. Springer Milan, pages 10-16, 2010.

[13] Mishra, D. and Quadir, A. (2015). Non-bossy single object auctions. Economic Theory Bulletin, Forthcoming.

[14] Mishra, D. and Roy, S. (2013). Implementation in multidimensional dichotomous domains. Theoretical Economics 8, 431-466.

[15] Mishra, D. and Sen, A. (2012). Roberts' theorem with neutrality: A social welfare ordering approach. Games and Economic Behavior 75, 283-298.

[16] Mitra, M., (2002). Achieving the first best in sequencing problems. Review of Economic Design 7, 75-91.

[17] Moulin, H., (2007). On Scheduling Fees to Prevent Merging, Splitting, and Transferring of Jobs. Mathematics of Operations Research 32, 266-283.

[18] Nath, S. and Sen, A. (2015). Affine Maximizers in Domains with Selfish Valuations.

[19] Roberts, K. (1979). The characterization of implementable choice rules. In Jean-Jacques Laffont, editor, Aggregation and Revelation of Preferences. Papers presented at the first European Summer Workshop of the Econometric Society, pages 321-349, North-Holland Publications.

[20] Rochet, J. C. (1987). A necessary and sufficient condition for rationalizability in a quasi-linear context. Journal of Mathematical Economics, 16(2), 191-200.

[21] Rockafellar, R. T. (1970). Convex analysis (No. 28). Princeton university press.

[22] Smith, W. (1956). Various optimizers for single stage production. Naval Research Logistics Quarterly 3, 59-66.

[23] Suijs, J. (1996). On incentive compatibility and budget balancedness in public decision making. Economic Design 2, 193-209.

[24] Walker, M. (1980). On the non-existence of dominant strategy mechanisms for making optimal public decisions. Econometrica 48, 1521-1540.

ECONOMIC RESEARCH UNIT, INDIAN STATISTICAL INSTITUTE, KOLKATA, INDIA.
E-mail address: parikshitde_r@isical.ac.in

ECONOMICS RESEARCH UNIT-INDIAN STATISTICAL INSTITUTE, KOLKATA-700108, INDIA.
E-mail address: mmitra@isical.ac.in