An Extreme Point Characterization of the Strategy-proof Probabilistic Rules over Binary Restricted Domains *

Hans Peters[1], Souvik Roy[2], Soumyarup Sadhukhan[2], and Ton Storcken[1]

[1]School of Business and Economics, Maastricht University

[2]Economic Research Unit, Indian Statistical Institute, Kolkata

Abstract

It is proved that every strategy-proof and unanimous probabilistic rule defined over a binary restricted domain has binary support. Moreover it is also proved that every strategy-proof and unanimous probabilistic rule defined over binary restricted domain is a probabilistic mixture of strategy-proof and unanimous deterministic rules. Important examples of binary domains consist of several types of single dipped domain (unrestricted and distance single-dipped), single peaked domain where peaks are restricted to two adjacent alternatives, restricted dichotomous domain etc.

*The authors would like to gratefully acknowledge Arunava Sen and Gopakumar Achuthankutty for their invaluable suggestions which helped improve this paper.

1 Introduction

Consider voting between two alternatives, where half of the agents are in favor of an alternative and the remaining half in favor of the other. In this situation a deterministic rule needs to choose any one of the alternative, where a probabilistic lottery seems to be more fairer. For every preference profile a probabilistic rule selects a probability distribution over the set of alternatives. In the landmark paper Gibbard (1977) (see also Sen (2011)), Gibbard provided a characterization of all strategy-proof probabilistic rules over complete domain. Additionally if the rules are unanimous also, then the probabilistic rules are probabilistic mixtures of deterministic rules. This result also shows that to analyze probabilistic rules it is sufficient to study deterministic rules only.

In Peters et al. (2014), they showed that if preferences are single peaked for finite set of alternatives then set of strategy-proof and unanimous probabilistic rules are mixture of strategy-proof and unanimous deterministic rules. The same fact is true for multi-dimensional domain with lexicographic preferences (Chatterji et al. (2012)). But it is not necessarily true for all dictatorial domains (Chatterji et al. (2014)) i.e. there are domains where all strategy-proof and unanimous deterministic probabilistic rules are dictatorial but all strategy-proof and unanimous probabilistic rules are not random dictatorial.

By binary restricted domain over two alternatives, we mean a domain of preferences where the top alternative(s) of each preference is a subset of those two alternatives. Here we are considering weak preferences so we allow preferences where agent is indifferent at top between the two alternatives. Moreover, the domain must have two preferences such that in one preference one of those two alternatives is alone at the top and the other alternative is alone at the second position, and in the other preference it is other way round.

Many well known practical domains come under this category. The single dipped domain is a binary restricted domain over the two boundary alternatives. Many restricted sub domains of single dipped domain also satisfy the criteria of binary restricted domain. General single peaked domain is not a binary restricted domain but if we consider single peaked domain with only two adjacent alternatives as peak then it is a binary restricted domain. Restricted dichotomous domain is also a well known example of binary restricted domain.

In Manjunath (2014), they studied the problem of locating a single public good along a segment, when agents preferences are single-dipped and characterized all the strategy-proof and unanimous deterministic rules there. In Barberà et al. (2012), they showed when all single dipped preferences are admissible, the range must contain two alternatives at most for deterministic rules. But this changes as they considered different sub-domains of single dipped preferences. They provided examples of sub-domains admitting strategy-proof rules with larger ranges.

In this paper we have shown that over binary restricted domain, every strategy-proof and unanimous probabilistic rule has binary support i.e. a strategy-proof and unanimous probabilistic rule will give positive probability to only two alternatives and the probabilistic rules are also tops only. If the domain is binary restricted over x and y then support will also have only x and y. We have also shown that in general if a strategy-proof probabilistic rule has binary support then we can write that rule as convex combination of two other strategy-proof probabilistic rules which are not identical. This proves that over binary restricted domain any strategy-proof and unanimous rule can be written as a convex combination of deterministic social choice rules.

The paper is organized as follows. The next section introduces the model and definitions. Subsequent sections provide necessary results and examples of existing well known binary restricted domains and conclusions.

2 Preliminaries

Let A be a finite set of alternatives and $N = \{1, \ldots, n\}$ be a finite set of agents. A complete and transitive binary relation over A is called a weak preference over A. By P and I we denote the strict and the indifference part of R. For a weak preference R by $r_k(R)$ we mean the kth ranked alternative(s) in R defined as $\{y : |\{x : xPy\}| = k - 1\}$. We denote by $\mathcal{D}$ a set of admissible weak preferences for any agent $i \in N$. As it is clear from the notation, we assume the same set of admissible weak preferences for all the agents. A preference profile, denoted by $R_N = (R_1, R_2, \ldots, R_n)$, is an element of $\mathcal{D}^n = \mathcal{D} \times \mathcal{D} \times \ldots \times \mathcal{D}$.

For $R_i \in \mathcal{D}$ by $R_i[a, b]$ we denote the set of alternatives $\{z \in A : aR_izR_ib \text{ and } z \notin \{a, b\}\}$.

For notational convenience sometimes we do not use brackets to denote singleton sets, in other words, we denote a set $\{i\}$ by i.

Definition 2.1. A *Deterministic Social Choice Function* (DSCF) is a function $f : \mathcal{D}^n \to A$.

Definition 2.2. A DSCF f is called *unanimous* if for all $R_N \in \mathcal{D}^n$

$$f(R_N) \in \cap_{i=1}^n r_1(R_i) \text{ whenever } \cap_{i=1}^n r_1(R_i) \neq \emptyset.$$

Definition 2.3. A DSCF f is *strategy-proof* if for all $i \in N$, for all $R_N \in \mathcal{D}^n$, for all $R_i' \in \mathcal{D}$ we have $f(R_N) R_i f(R_i', R_{-i})$.

Definition 2.4. A *probabilistic Social Choice Function* (PSCF) is a function $\Phi : \mathcal{D}^n \to \triangle A$ where $\triangle A$ is the set of probability distributions over A. By a strict PSCF we mean a PSCF that is not a DSCF.

For $S \subseteq A$ and $R \in \mathcal{D}^n$, we define by $\Phi_S(R) = \sum_{a \in S} \Phi_a(R)$.

Definition 2.5. A PSCF Φ is called *unanimous* if for all $R_N \in \mathcal{D}^n$

$$\sum_{x \in \cap_{i=1}^n r_1(R_i)} \Phi_x(R_N) = 1 \text{ whenever } \cap_{i=1}^n r_1(R_i) \neq \emptyset$$

Definition 2.6. For any $R \in \mathcal{D}$ and $x \in A$, the *upper contour set* of x at R is defined as the set of alternatives that are weakly preferred to x in R, more formally, $B(x, R) = \{y \in X : yRx\}$.

Definition 2.7. A PSCF Φ is *strategy-proof* if for all $i \in N$, for all $R_N \in \mathcal{D}^n$, for all $R_i' \in \mathcal{D}$ and for all $x \in A$, we have

$$\sum_{y \in B(x,R_i)} \Phi_y(R_i, R_{-i}) \geq \sum_{y \in B(x,R_i)} \Phi_y(R_i', R_{-i}).$$

Definition 2.8. A PSCF Φ is *strict strategy-proof* if it is strategy-proof and for all $i \in N$, for all $R_N \in \mathcal{D}^n$, for all $R_i' \in \mathcal{D}$, $\Phi(R_i, R_{-i}) \neq \Phi(R_i', R_{-i})$ implies there is $x \in A$ such that

$$\sum_{y \in B(x,R_i)} \Phi_y(R_i, R_{-i}) > \sum_{y \in B(x,R_i)} \Phi_y(R_i', R_{-i}).$$

Definition 2.9. A PSCF Φ satisfies *non-corruptibility* if for any $i \in N$, $R_i, R_i' \in \mathcal{D}$, $R_{-i} \in \mathcal{D}^{N \setminus i}$, $\sum_{y \in B(x,R_i)} \Phi_y(R_i, R_{-i}) = \sum_{y \in B(x,R_i)} \Phi_y(R_i', R_{-i})$ and $\sum_{y \in B(x,R_i')} \Phi_y(R_i, R_{-i}) = \sum_{y \in B(x,R_i')} \Phi_y(R_i', R_{-i})$ for all $x \in A$, imply $\Phi(R_N) = \Phi(R_N')$.

Definition 2.10. *Support* of a PSCF Φ on some domain $\mathcal{D}^n$, denoted by $Supp(\Phi)$, is defined as $\{x \in A : \Phi_x(R_N) > 0 \text{ for some } R_N \in \mathcal{D}^n\}$.

Definition 2.11. A PSCF Φ is *tops only* if for all R_N and $R'_N \in \mathcal{D}^n$ such that $r_1(R_i) = r_1(R_i{}')$ for all $i \in N$ we have $\Phi(R_N) = \Phi(R'_N)$.

For PSCFs Φ' and Φ'' on a domain $\mathcal{D}^n$ and $0 \leq \lambda \leq 1$, $\lambda\Phi' + (1-\lambda)\Phi''$ is defined as the PSCF Φ such that $\Phi_x(R_N) = \lambda\Phi'_x(R_N) + (1-\lambda)\Phi''_x(R_N)$ for all $x \in A$ and $R_N \in \mathcal{D}^n$. Two PSCFs, Φ' and Φ'' on $\mathcal{D}^n$ are said to be not equal ($\Phi' \neq \Phi''$) if there exists $R_N \in \mathcal{D}^n$ such that $\Phi'(R_N) \neq \Phi''(R_N)$. We say a rule Φ is a convex combination of a set of rules $\{\Phi_k; k = 1, 2, \ldots, l\}$ if there exist $\lambda_k : k = 1, 2, \ldots, l$ with the property that $\lambda_k \geq 0$ for all k and $\sum_k \lambda_k = 1$, such that $\Phi = \sum_k \lambda_k \Phi_k$.

Definition 2.12. A domain $\mathcal{D}$ is said to be *deterministic extreme point* (DEP) domain if every strategy-proof and unanimous PSCF on $\mathcal{D}^n$ can be written as a convex combination of strategy-proof and unanimous DSCFs on $\mathcal{D}^n$ for all $n \geq 2$.

For $a \in A$, let $\mathcal{D}^a = \{R \in \mathcal{D} : r_1(R) = a\}$.

Definition 2.13 (Binary Restricted Domain)**.** A domain of weak preferences is called a *binary restricted domain* over $\{x, y\}$ where $x, y \in A$, if

1. for all $R \in \mathcal{D}$, $r_1(R) \in \{\{x\}, \{y\}, \{x, y\}\}$,
2. for all $a, b \in \{x, y\}; a \neq b$, and for each $R \in \mathcal{D}^a$, there exists $R' \in \mathcal{D}^b$ such that $R[a, b] \cap R'[b, a] = \emptyset$.

3 Results

In this section we present the main results of this paper. First we establish a necessary and sufficient condition for a domain to be deterministic extreme point in the following Theorem.

Theorem 3.1. *A necessary and sufficient condition for a domain $\mathcal{D}$ to be deterministic extreme point (DEP) is that every strategy-proof and unanimous strict PSCF $\Phi : \mathcal{D}^n \to \triangle A$ can be written as a convex combination of two other strategy-proof and unanimous PSCFs, Φ' and Φ'', such that $\Phi' \neq \Phi''$.*

Proof. (If part) First we will prove that the set of all strategy-proof and unanimous probabilistic rules $\mathcal{S}$ over a domain $\mathcal{D}^n$ is closed and convex. Suppose Φ is defined as $\Phi(R_N) = \alpha\Phi'(R_N) + (1-\alpha)\Phi''(R_N)$ for all $R_N \in \mathcal{D}^n$ where $0 \leq \alpha \leq 1$ and Φ', Φ'' are two strategy-proof and unanimous probabilistic rules defined over $\mathcal{D}^n$. It is obvious that Φ is unanimous as Φ' and Φ'' are unanimous. Suppose Φ is manipulable. Then there are $i \in N$, $R_N \in \mathcal{D}^n$ and R'_i such that for some $b \in A$,

$$\sum_{a \in B(b,R_i)} \Phi_a(R'_i, R_{-i}) > \sum_{a \in B(b,R_i)} \Phi_a(R_N)$$

which means

$$\sum_{a \in B(b,R_i)} \left(\alpha \Phi'_a(R'_i, R_{-i}) + (1-\alpha)\Phi''_a(R'_i, R_{-i})\right) > \sum_{a \in B(b,R_i)} \left(\alpha \Phi'_a(R_N) + (1-\alpha)\Phi''_a(R_N)\right).$$

Then either

$$\alpha \left(\sum_{a \in B(b,R_i)} \Phi'_a(R'_i, R_{-i}) \right) > \alpha \left(\sum_{a \in B(b,R_i)} \Phi_{1a}(R_N) \right)$$

or,

$$(1-\alpha) \left(\sum_{a \in B(b,R_i)} \Phi''_{2a}(R'_i, R_{-i}) \right) > (1-\alpha) \left(\sum_{a \in B(b,R_i)} \Phi''_a(R_N)) \right),$$

which in turn means either Φ' or Φ'' is manipulable, a contradiction. So, $\mathcal{S}$ is convex.

We show that $\mathcal{S}$ is closed. Consider a sequence $(\Phi^k; k \in \mathbb{N})$ of strategy-proof and unanimous probabilistic rules over $\mathcal{D}^n$ such that $\lim_{k\to\infty} \Phi^k = \Phi$ i.e. for all $x \in A$ and $R_N \in \mathcal{D}^n$, $\lim_{k\to\infty} \Phi^k_x(R_N) = \Phi_x(R_N)$. Let Φ be manipulable then there exist $i \in N$, $R_N \in \mathcal{D}^n$ and R'_i such that for some $b \in A$,

$$\sum_{a \in B(b,R_i)} \Phi_a(R'_i, R_{-i}) > \sum_{a \in B(b,R_i)} \Phi_a(R_N).$$

This means there exists $k' \in \mathbb{N}$ such that for all $k \geq k'$,

$$\sum_{a \in B(b,R_i)} \Phi_a^k(R_i', R_{-i}) > \sum_{a \in B(b,R_i)} \Phi_a^k(R_N).$$

This in turn means Φ^k is manipulable for $k \geq k'$, which is a contradiction. So, $\mathcal{S}$ is closed.

Since $\mathcal{S}$ is closed and convex, it is completely characterized by its extreme points. The statement of the Theorem says that no strategy-proof and unanimous non-deterministic probabilistic rule is an extreme point. It is also easy to see that every strategy-proof and unanimous deterministic rule is an extreme point of $\mathcal{S}$. This shows that $\mathcal{D}$ is deterministic extreme point (DEP) domain.

(Only if part) Let $\mathcal{D}$ be a deterministic extreme point domain and Φ be a strict strategy-proof and unanimous probabilistic rule defined over $\mathcal{D}^n$. Since $\mathcal{D}$ is a DEP domain, $\Phi(R_N) = \sum_{i=1}^{k} \lambda_i f_i(R_N)$ for all $R_N \in \mathcal{D}^n$ where $\lambda_i \geq 0$ $\forall i = 1, 2, \ldots, k$, $\sum_{i=1}^{k} \lambda_i = 1$, and $\forall i = 1, 2, \ldots, k$, f_i is strategy-proof and unanimous DSCF defined on $\mathcal{D}^n$ with $f_i \neq f_j$ for $i \neq j$. As Φ is strict PSCF, there are $\lambda_k > 0$ and $\lambda_l > 0$ with $k \neq l$. We define $\Phi' = \sum_{i \neq k} \frac{\lambda_i}{1-\lambda_k} f_i$ and $\Phi'' = f_k$. This means $\Phi = (1 - \lambda_k)\Phi' + \lambda_k \Phi''$. It is easy to see that $\Phi' \neq \Phi''$ and Φ' is strategy-proof and unanimous. ■

In the following theorem we show that if a strategy-proof and strict probabilistic rule has binary support, then that rule can be written as a convex combination of two other rules. Note that Theorem 3.2 does not assume the PSCF to be unanimous, however it is clear from the proof that the theorem also holds under additional condition of unanimity.

Theorem 3.2. *Suppose $\Phi : \mathcal{D}^n \to A$ is a strategy-proof strict PSCF such that $Supp(\Phi) = \{x, y\}$ for some $x, y \in A$. Then there exist strategy-proof PSCFs $\Phi', \Phi''; \Phi' \neq \Phi''$ such that $\Phi(R_N) = \frac{1}{2}\Phi'(R_N) + \frac{1}{2}\Phi''(R_N)$ for all $R_N \in \mathcal{D}^n$.*

Proof. Note that $Supp(\Phi) = \{x, y\}$ means that $\Phi_x(R)$ $\forall R \in \mathcal{D}^n$ defines Φ completely. Since Φ is a strict PSCF, there exists $R_N' \in \mathcal{D}^n$ such that $\Phi_x(R_N') = p \in (0, 1)$. Let $C = \{R_N \in \mathcal{D}^n : \Phi_x(R_N) \neq p\}$. Moreover, as C is finite set, we can find $\epsilon \in (0, p)$ small enough such that for all $R_N \in C$, $\Phi_x(R_N) \notin [p - \epsilon, p + \epsilon]$. We construct Φ' and Φ'' with support $\{x, y\}$ as

$$\Phi'_x(R_N) = \begin{cases} \Phi_x(R_N) \text{ if } R_N \in C \\ \Phi_x(R_N) + \epsilon \text{ otherwise} \end{cases}$$

$$\Phi''_x(R_N) = \begin{cases} \Phi_x(R_N) \text{ if } R_N \in C \\ \Phi_x(R_N) - \epsilon \text{ otherwise.} \end{cases}$$

It is clear that $\Phi' \neq \Phi''$ and $\Phi(R_N) = \frac{1}{2}\Phi'(R_N) + \frac{1}{2}\Phi''(R_N)$ for all $R_N \in \mathcal{D}^n$. Unanimity of Φ' and Φ'' follows from the unanimity of Φ. We show that Φ' and Φ'' are strategy proof. Assume for contradiction that Φ' is manipulable. This means without of loss of generality that there exists $i \in N$, $R_N \in \mathcal{D}^n$ and $R'_i \in \mathcal{D}$ such that xP_iy and $\Phi'_x(R'_i, R_{-i}) > \Phi'_x(R_N)$. We consider the following cases.

Case 1 Suppose $(R'_i, R_{-i}), R_N \notin C$. By the construction of Φ', if $(R'_i, R_{-i}), R_N \notin C$ then $\Phi'_x(R'_i, R_{-i}) = \Phi'_x(R_N) = p + \epsilon$, a contradiction.

Case 2 Suppose $(R'_i, R_{-i}), R_N \in C$. By the construction of Φ', if $(R'_i, R_{-i}), R_N \in C$ then $\Phi'_x(R'_i, R_{-i}) = \Phi_x(R'_i, R_{-i})$ and $\Phi'_x(R_N) = \Phi_x(R_N)$. Since xP_iy, this implies Φ is manipulable at (R_N) via R'_i by agent i which is a contradiction.

Case 3 Suppose $(R'_i, R_{-i}) \in C$, $R_N \notin C$. By the construction of Φ', if $(R'_i, R_{-i}) \in C$ and $R_N \notin C$ then $\Phi'_x(R'_i, R_{-i}) = \Phi_x(R'_i, R_{-i})$ and $\Phi'_x(R_N) = p + \epsilon = \Phi_x(R_N) + \epsilon$. Hence $\Phi'_x(R'_i, R_{-i}) > \Phi'_x(R_N)$ implies $\Phi_x(R'_i, R_{-i}) > \Phi_x(R_N)$. Since xP_iy, this means Φ is manipulable at (R_N) via R'_i by agent i, which is a contradiction.

Case 4 Suppose $(R'_i, R_{-i}) \notin C$, $R_N \in C$. By the construction of Φ', if $(R'_i, R_{-i}) \notin C$ and $R_N \in C$ then $\Phi'_x(R'_i, R_{-i}) = p + \epsilon = \Phi_x(R'_i, R_{-i}) + \epsilon$ and $\Phi'_x(R_N) = \Phi_x(R_N)$. Hence $\Phi'_x(R'_i, R_{-i}) > \Phi'_x(R_N)$ implies $\Phi_x(R'_i, R_{-i}) > \Phi_x(R_N)$ as by the construction of Φ'. However then xP_iy implies that Φ is manipulable at (R_N) via R'_i by agent i, which is a contradiction.

This shows that Φ' is strategy-proof. Similarly we can show that Φ'' is strategy-proof, which completes the proof. ∎

In the following Theorem we show that binary restricted domain ensures binary support for strategy-proof and unanimous probabilistic rules.

Theorem 3.3. *Suppose $\mathcal{D}$ is a Binary Restricted Domain over $\{x, y\}$. Then for all $n \geq 2$ and all strategy-proof and unanimous PSCF $\Phi : \mathcal{D}^n \to \triangle A$ we have that $Supp(\Phi) = \{x, y\}$.*

Proof. We prove this using a few propositions and lemmas. We first prove the result for two agents and then use induction to prove this for arbitrary number of agents.

Proposition 3.1. *Suppose $\mathcal{D}$ is a Binary Restricted Domain over $\{x, y\}$, and $\Phi : \mathcal{D}^2 \to \triangle A$ is a strategy-proof and unanimous PSCF. Then $Supp(\Phi) = \{x, y\}$.*

Proof. Consider a Binary Restricted Domain $\mathcal{D}$ over $\{x, y\}$, and a set of players $N = \{1, 2\}$, and a strategy-proof and unanimous PSCF $\Phi : \mathcal{D}^2 \to \triangle A$. We show that $\Phi_{\{x,y\}}(R) = 1$ for all $R \in \mathcal{D}$.

First note that if $r_1(R_i) = \{x, y\}$ for some $i \in N = \{1, 2\}$, then by unanimity $\Phi_{\{x,y\}}(R_i, R_j) = 1$ for all $R_j \in \mathcal{D}$. Also note that $\Phi_a(R) = 1$ if R is a unanimous profile at $a \in \{x, y\}$. So, we consider non-unanimous profiles R such that $r_1(R_i) \in \{x, y\}$ $\forall i \in \{1, 2\}$ and show $\Phi_{\{x,y\}}(R) = 1$. We complete the proof by the following two lemmas.

Lemma 3.1. *Let $R = (R_1, R_2)$ be a preference profile such that $R_1 \in \mathcal{D}^a$, $R_2 \in \mathcal{D}^b$ where $a \neq b \in \{x, y\}$, and $R_1[a, b] \cap R_2[b, a] = \emptyset$. Then $\Phi_{\{x,y\}}(R) = 1$.*

Proof. WLG we assume $R_1 \in \mathcal{D}^x$ and $R_2 \in \mathcal{D}^y$. Assume for contradiction that $\Phi_z(R) > 0$ for some $z \in A \setminus \{x, y\}$. If $z \notin R_1[x, y]$, then agent 1 manipulates at R via some $R_1' \in \mathcal{D}^y$, since by unanimity $\Phi_y(R_1', R_2) = 1$ and y are strictly preferred to z at the preference R_1 of player 1. So it must be that $z \in R_1[x, y]$. Similarly we have that $z \in R_2[y, x]$. However this contradicts our assumption that $R_1[x, y] \cap R_2[y, x] = \emptyset$ which completes the proof. ■

Lemma 3.2. *Let $R = (R_1, R_2)$ be an arbitrary preference profile with $R_1 \in \mathcal{D}^a$, $R_2 \in \mathcal{D}^b$ where $a \neq b \in \{x, y\}$. Then $\Phi_{\{x,y\}}(R) = 1$.*

Proof. WLG we assume that $R_1 \in \mathcal{D}^x$ and $R_2 \in \mathcal{D}^y$. If $R_1[x, y] \cap R_2[y, x] = \emptyset$ then we are done by Lemma 3.1. So, assume that $R_1[x, y] \cap R_2[y, x] \neq \emptyset$. This, by the definition of binary restricted domain $\mathcal{D}$, means that there exist R_1' and R_2' such that $R_1[x, y] \cap R_2'[y, x] = \emptyset$ and $R_1'[x, y] \cap R_2[y, x] = \emptyset$. By Lemma 3.1 we have $\Phi_{\{x,y\}}(R_1, R_2') = 1$ and $\Phi_{\{x,y\}}(R_1', R_2) = 1$. Let $\Phi_x(R_1, R_2') = \epsilon$ and $\Phi_x(R_1', R_2) = \epsilon'$. We claim $\epsilon = \epsilon'$. Assume for contradiction that $\epsilon > \epsilon'$. Since $R_1, R_1' \in \mathcal{D}^x$ and $R_2, R_2' \in \mathcal{D}^y$, strategy-proofness implies $\Phi_x(R_1', R_2') = \Phi_x(R_1, R_2') = \epsilon$ and $\Phi_y(R_1', R_2') = \Phi_y(R_1', R_2) = 1 - \epsilon'$. This means $\Phi_{\{x,y\}}(R_1', R_2') = \epsilon + 1 - \epsilon'$ which is not possible since $\epsilon > \epsilon'$. Using similar logic it can be shown that it is not possibile that $\epsilon < \epsilon'$. So $\epsilon = \epsilon'$. Moreover, using the facts that $R_1, R_1' \in \mathcal{D}^x$ and $R_2, R_2' \in \mathcal{D}^y$, we have by strategy-proofness that $\Phi_x(R_1, R_2) = \Phi_x(R_1', R_2) = \epsilon$ and $\Phi_y(R_1, R_2) = \Phi_y(R_1, R_2') = 1 - \epsilon$, and hence $\Phi_{\{x,y\}}(R_1, R_2) = 1$ completing the proof. ■

The above two lemmas complete the proof of Proposition 3.1. ■

In the following proposition we show that the binary restricted domains ensure binary support for strategy-proof and unanimous rules when there are arbitrary number of players.

Proposition 3.2. *Let $n \geq 3$, $\mathcal{D}$ be Binary Restricted Domain over $\{x, y\}$, and $\Phi : \mathcal{D}^n \to \triangle A$ be a strategy-proof and unanimous PSCF. Then $Supp(\Phi) = \{x, y\}$.*

Proof. We prove the result by induction. Assume that for all integers $k < n$, the following statement is true:
Induction Hypothesis (IH): Let $\mathcal{D}$ be a Binary Restricted Domain over $\{x, y\}$. If $\Phi : \mathcal{D}^k \to \triangle A$ satisfies strategy-proofness and unanimity, then $Supp(\Phi) = \{x, y\}$.

Let $N^* = \{1^*, 3, 4, \dots, n\}$ be a set of voters where $3, 4, \dots, n \in N$. Define a PSCF $g : \mathcal{D}^{n-1} \to \triangle A$ for the set of voters N^* as follows: For all $(R_{1^*}, R_3, \dots, R_n) \in \mathcal{D}^{n-1}$,

$$g(R_{1^*}, R_3, \dots, R_n) = \Phi(R_1, R_1, R_3, R_4, \dots, R_n)$$

Voter 1^* in the PSCF g is obtained by "cloning" voters 1 and 2 in N. Thus, if voters 1 and 2 in N have a common ordering R_1, then voter $1^* \in N^*$ has ordering R_{1^*}.

Lemma 3.3. *The PSCF g has binary support i.e. $Supp(g) = \{x, y\}$.*

Proof. It is trivial to verify that g satisfies unanimity. We now show that g also satisfies strategy-proofness so the result follows from IH. It is easy to see that agents other than 1^* can not manipulate g since Φ is strategy-proof. Let $(R_1, R_3, \dots, R_n) \in \mathcal{D}^{n-1}$ and $\bar{R}_1 \in \mathcal{D}$. For all $b \in A$, we have

$$\begin{aligned}
\sum_{a \in B(b, R_1)} g_a(R_{1^*}, R_3, \dots, R_n) &= \sum_{a \in B(b, R_1)} \Phi_a(R_1, R_1, R_3, \dots, R_n) \\
&\geq \sum_{a \in B(b, R_1)} \Phi_a(\bar{R}_1, R_1, R_3, \dots, R_n) \qquad (1) \\
&\geq \sum_{a \in B(b, R_1)} \Phi_a(\bar{R}_1, \bar{R}_1, R_3, \dots, R_n) \qquad (2) \\
&= \sum_{a \in B(b, R_1)} g_a(\bar{R}_{1^*}, R_3, \dots, R_n).
\end{aligned}$$

If inequality (1) does not hold, agent 1 with ordering R_1 manipulates Φ at $(R_1, R_1, R_3, \dots, R_n)$ via $\bar{R}_1$. If inequality (2) does not hold, agent 2 with ordering R_1 manipulates Φ at $(\bar{R}_1, R_1, R_3, \dots, R_n)$ via $\bar{R}_1$. Therefore g satisfies strategy-proofness.

■

Note that by Lemma 3.3 we have $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ with $R_1 = R_2$. Our next lemma shows that the same holds if $r_1(R_1) = r_1(R_2)$.

Lemma 3.4. *Let R_N be a preference profile such that $r_1(R_1) = r_1(R_2)$. Then $\Phi_{\{x,y\}}(R_N) = 1$.*

Proof. Note that if $r_1(R_1) = r_1(R_2) = \{x, y\}$, then the result straightforwardly follows by Lemma 3.3 and strategy-proofness. This is because, if $\Phi_{\{x,y\}}(R_N) < 1$ for some R_1, R_2 with $r_1(R_1) = r_1(R_2)$ then player 1 can manipulate at R_N via R_2 since by Lemma 3.3 $\Phi_{\{x,y\}}(R_2, R_2, R_{N\setminus\{1,2\}}) = 1$.

In view of the above argument, it is enough to consider $r_1(R_1) = r_1(R_2) \in \{x, y\}$. WLG we assume $r_1(R_1) = r_1(R_2) = x$. By Lemma 3.3 we have $\Phi_{\{x,y\}}(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_{\{x,y\}}(R_2, R_2, R_{N\setminus\{1,2\}}) = 1$. Moreover, since $r_1(R_1) = r_1(R_2) = x$ we have by strategy-proofness $\Phi_x(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_x(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_x(R_2, R_2, R_{N\setminus\{1,2\}}) = \epsilon$ (say).

Note that if $r_1(R_i) \neq y$ for all $i \in N \setminus \{1, 2\}$ then by unanimity $\Phi_{\{x,y\}}(R_N) = \Phi_x(R_N) = 1$. So, suppose there is $i \in N \setminus \{1, 2\}$ such that $r_1(R_i) = y$. Let $R_y \in \mathcal{D}$ be such that $R_y[y, x] \cap R_1[x, y] = \emptyset$. Such an R_y always exists by the definition of $\mathcal{D}$. Consider the preference profile $\bar{R}_{N\setminus\{1,2\}}$ of the players in $N \setminus \{1, 2\}$ as follows: for all $i \in N \setminus \{1, 2\}$

$$\bar{R}_i = \begin{cases} R_y \text{ if } r_1(R_i) = y \\ R_i \text{ otherwise.} \end{cases}$$

Using Lemma 3.3 we have $\Phi_{\{x,y\}}(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \Phi_{\{x,y\}}(R_2, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Since $r_1(R_1) = r_1(R_2) = x$, we have by strategy-proofness $\Phi_x(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \Phi_x(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = \Phi_x(R_2, R_2, \bar{R}_{N\setminus\{1,2\}})$. We show $\Phi_x(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \epsilon$. First we claim that $\Phi_y(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_1, \bar{R}_{N\setminus\{1,2\}})$. To see this, consider a player $i \in N \setminus \{1, 2\}$ such that $R_i \neq \bar{R}_i$. Note that by construction we have $r_1(R_i) = r_1(\bar{R}_i) = y$, hence by strategy-proofness we have $\Phi_y(R_1, R_1, R_i, R_{N\setminus\{1,2,i\}}) = \Phi_y(R_1, R_1, \bar{R}_i, R_{N\setminus\{1,2,i\}})$. Continuing in this manner we have $\Phi_y(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_1, \bar{R}_{N\setminus\{1,2\}})$ and hence by the fact that $\Phi_{\{x,y\}}(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = 1$ we have $\Phi_x(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \epsilon$.

Using similar logic it follows that $\Phi_y(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_2, \bar{R}_{N\setminus\{1,2\}})$. We complete the proof by showing $\Phi_y(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1 - \epsilon$. Using the fact that $\Phi_x(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = \epsilon$, it suffices to show that $\Phi_{\{x,y\}}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$.

We claim $\Phi_{\{x,y\}}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Assume for contradiction that $\Phi_z(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) > 0$ for some $z \neq x, y$. Note that if $z \notin R_1[x, y]$ then agent 1 manipulates at $(R_1, R_2, \bar{R}_{N\setminus\{1,2\}})$ via R_2

since $\Phi_{\{x,y\}}(R_2, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. So, $z \in R_1[x,y]$. Now we show that $z \in R_y[y,x]$. Assume for contradiction that $z \notin R_y[y,x]$. Consider $i \in N \setminus \{1,2\}$ such that $\bar{R}_i = R_y$. Let R'_i be such that $r_1(R'_i) = x$. Then by strategy-proofness $\Phi_x(R_1, R_2, R'_i, \bar{R}_{N\setminus\{1,2,i\}}) \leq \Phi_{\{x,y\}}(R_1, R_2, R_i, \bar{R}_{N\setminus\{1,2\}}) < 1$. By sequentially changing the preferences of the players in $N \setminus \{1,2\}$ with y at the top in this manner we construct a preference profile $\hat{R}$ that is unanimous at x and for which $\Phi_x(\hat{R}) < 1$ which is a contradiction to unanimity, hence $z \in R_y[y,x]$. However, this means $R_x[x,y] \cap R_y[y,x] \ni z$ which is contradiction to $R_x[x,y] \cap R_y[y,x] = \emptyset$. This completes the proof of Lemma 3.4. ■

In the following lemma we show that $\Phi_{\{x,y\}}(R_N) = 1$ for arbitrary profile $R_N \in \mathcal{D}^n$.

Lemma 3.5. *Let $R_N \in \mathcal{D}^n$ be an arbitrary preference profile. Then $\Phi_{\{x,y\}}(R_N) = 1$.*

Proof. In view of Lemma 3.4, it is enough to consider $r_1(R_1) \neq r_1(R_2)$. Note that if $r_1(R_i) = \{x,y\}$ for some $i \in \{1,2\}$ and $\Phi_z(R_N) > 0$ for some $z \notin \{x,y\}$, then player i manipulates at R_N via R_j where $j \in \{1,2\}, j \neq i$ since by Lemma 3.4 we have $\Phi_{\{x,y\}}(R_j, R_j, R_{N\setminus\{1,2\}}) = 1$. So it is enough to consider the case where $r_1(R_i) \in \{x,y\}$ for all $i \in \{1,2\}$. WLG we assume $r_1(R_1) = x$ and $r_1(R_2) = y$.

Suppose $R_1[x,y] \cap R_2[y,x] = \emptyset$. Assume for contraction that $\Phi_z(R_N) > 0$ for some $z \notin \{x,y\}$. If $z \notin R_y[y,x]$, then player 2 manipulates at R_N via R_1 since by Lemma 3.4 $\Phi_{\{x,y\}}(R_1, R_1, R_{N\setminus\{1,2\}}) = 1$. So $z \in R_y[y,x]$. Using similar logic we have $z \in R_x[x,y]$, this contradicts the assumption that $R_1[x,y] \cap R_2[y,x] = \emptyset$. Hence $\Phi_z(R_N) = 0$ for some $z \notin \{x,y\}$.

Now suppose $R_1[x,y] \cap R_2[y,x] \neq \emptyset$. By the definition of $\mathcal{D}$ we know there exist $R'_1 \in \mathcal{D}^x$ and $R'_2 \in \mathcal{D}^y$ such that $R_1[x,y] \cap R'_2[y,x] = \emptyset$ and $R'_1[x,y] \cap R_2[y,x] = \emptyset$. Since $r_1(R_1) = r_1(R'_1) = x$ and $r_1(R_2) = r_1(R'_2) = y$, by strategy-proofness we have $\Phi_x(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_x(R'_1, R_2, R_{N\setminus\{1,2\}})$ and $\Phi_y(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R'_2, R_{N\setminus\{1,2\}})$. Moreover, by using similar logic as in the proof of Lemma 3.2 we have $\Phi_x(R_1, R'_2, R_{N\setminus\{1,2\}}) = \Phi_x(R'_1, R_2, R_{N\setminus\{1,2\}})$. Hence, $\Phi_{\{x,y\}}(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_{\{x,y\}}(R_1, R'_2, R_{N\setminus\{1,2\}})$. However, $\Phi_{\{x,y\}}(R_1, R'_2, R_{N\setminus\{1,2\}}) = 1$ since $R_1[x,y] \cap R'_2[y,x] = \emptyset$, which completes the proof of Lemma 3.5. ■

These three lemmas complete the proof of Proposition 3.2. ■

Finally the proof of Theorem 3.3 is complete by Proposition 3.1 and Proposition 3.2. ■

Theorem 3.4. *Every binary restricted domain is a deterministic extreme point (DEP) domain.*

Proof. Consider a binary restricted domain over $\{x, y\}$ and a strategy-proof and unanimous strict probabilistic rule Φ on it. By Theorem 3.3, the support of Φ is $\{x, y\}$. Moreover, by Theorem 3.2, there exist strategy-proof and unanimous PSCFs Φ' and Φ'' such that $\Phi = \frac{1}{2}\Phi' + \frac{1}{2}\Phi''$. Finally, by Theorem 3.1 we conclude that binary restricted domain is a deterministic extreme point domain. ■

3.1 Characterization of Strategy-proof and Unanimous rules

In this section we give a complete characterization of the strategy-proof and unanimous PSCFs defined over a binary restricted domain. In view Theorem 3.4, it is sufficient to give a characterization of strategy-proof and unanimous DSCFs on a binary restricted domain.

Let $\mathcal{D}$ be a binary restricted domain over $\{x, y\}$. For $R_N \in \mathcal{D}^n$, by $N^a(R_N)$ we denote the set of agents $i \in N$ such that $r_1(R_i) = a$, and by $C(R_N) = \{i \in N : r_1(R_i) = \{x, y\}\}$ we denote the set of agents who are indifferent between x and y at the top of their preferences in R_N. Let $\bar{\mathcal{D}} = \{R \in \mathcal{D} : r_1(R) = \{x, y\}\}$ be the set of preferences where x and y are indifferent at the top, and let $\hat{\mathcal{D}}^n = \mathcal{D}^n \setminus \bar{\mathcal{D}}^n$.

Definition 3.1. A function $g : \hat{\mathcal{D}}^n \to \mathcal{P}_0(\mathcal{P}_0(N))$ is called a minimal winning coalition function if

1. for all $i \in N$, $C \subseteq N$, and $R_N \in \hat{\mathcal{D}}^n$, $i \in C \in g(R_N)$ implies $r_1(R_i) \in \{x, y\}$,

2. for all $R_N \in \hat{\mathcal{D}}^n$, $C \neq C' \in g(R_N)$ implies $C \not\subseteq C'$, and

3. $g(R_N) = g(R'_N)$ for all R_N and R'_N with the property that $C(R_N) = C(R'_N)$ and $R_i = R'_i$ if $i \in C(R_N)$.

Definition 3.2. An agent $i \in N$ is called a dummy agent at R_N for a minimal winning coalition function g if $r_1(R_i) \in \{x, y\}$ and $i \notin C$ for all $C \in g(R_N)$. An agent i is called a non dummy agent if $r_1(R_i) \in \{x, y\}$ and $i \in C$ for some $C \in g(R_N)$.

Definition 3.3. A minimal winning coalition function g satisfies independence of dummy agents (IDA) property if i is a dummy player at R_N for g implies $g(R_N) = g(R'_i, R_{N \setminus i})$ for all R'_i with $r_1(R'_i) = \{x, y\}$.

Definition 3.4. A minimal winning coalition function g satisfies responsive to non dummy agents (RNDA) property if i is a non dummy player at R_N for g, and $r_1(R'_i) = \{x, y\}$ imply

1. $C \in g(R'_i, R_{N\setminus i})$ for all $C \in g(R_N)$ with $i \notin C$, and

2. for all $C \in g(R'_i, R_{N\setminus i})$ such that $C \notin g(R_N)$ there is $C' \in g(R_N)$ with $i \in C'$ such that $C' \setminus i \subseteq C$.

In the following lemma we show that whenever an agent has exactly one top alternative, that agent cannot change the outcome of a strategy-proof and unanimous rule by changing his/her preference without changing the top alternative.

Lemma 3.6. *Let $\mathcal{D}$ be a Binary Restricted Domain over $\{x, y\}$. Suppose $R_N, R'_N \in \mathcal{D}^n$ are such that $N^a(R_N) = N^a(R'_N)$ for all $a \in \{x, y\}$, and $R_i = R'_i$ for all $i \notin N^x(R_N) \cup N^y(R_N)$. Then for any strategy-proof and unanimous DSCF f on $\mathcal{D}^n$, $f(R_N) = f(R'_N)$.*

Proof. Let $\mathcal{D}$ be a Binary Restricted Domain over $\{x, y\}$ and f be an arbitrary strategy-proof and unanimous DSCF on $\mathcal{D}^n$. Suppose $R_N, R'_N \in \mathcal{D}^n$ are such that $N^a(R_N) = N^a(R'_N)$ for all $a \in \{x, y\}$, and $R_j = R'_j$ for all $j \notin N^x(R_N) \cup N^y(R_N)$. Assume WLG that $f(R_N) = x$. Note that if agents in N^x change their preferences keeping x at the top then by strategy-proofness the outcome must not change. Moreover, since the outcome will be either x or y, by strategy-proofness outcome must not change if the agents in N^y change their preferences keeping y at the top. This completes the proof. ■

In the following theorem we provide a characterization of the strategy-proof and unanimous DSCFs. Let $\mathcal{D}$ be a binary restricted domain over $\{x, y\}$.

Theorem 3.5. *A deterministic social choice function $f : \mathcal{D}^n \to A$ is strategy-proof and unanimous if and only if there exists a minimal winning coalition function g satisfying IDA and RNDA such that for all $R_N \in \hat{\mathcal{D}}$, $N^x(R_N) \supseteq C$ for some $C \in g(R_N)$ implies $f(R_N) = x$.*

Proof. *(If part)* Let g be a minimal winning coalition function satisfying IDA and RNDA. Consider a social choice function f such that $R_N \in \hat{\mathcal{D}}^n$ and $N^x(R_N) \supseteq C$ for some $C \in g(R_N)$ implies $f(R_N) = x$. We show that f is unanimous and strategy-proof.

We first show that f is unanimous. Consider a profile $R_N \in \hat{\mathcal{D}}^n$ such that $\cap_{i\in N} r_1(R_i) \neq \emptyset$. If $r_1(R_i) = \{x, y\}$ for all $i \in N$ then unanimity holds trivially. Assume WLG that $\cap_{i\in N} r_1(R_i) = x$. Let $C \subsetneq N$ be such that $r_1(R_i) = \{x, y\}$ if and only if $i \in C$. Note that $N^x(R_N) = N \setminus C$. Since $g(R_N) \subseteq \mathcal{P}_0(N \setminus C)$, this means $f(R_N) = x$.

Now we show that f is strategy-proof. Consider a profile $R_N \in \hat{\mathcal{D}}^n$. Note that the agents in $C(R_N)$ have no incentive to manipulate. Let i be a dummy agent and $R'_N = (R'_i, R_{N\setminus i})$. Then, if $r_1(R'_i) \in \{x, y\}$ then by definition of g we have $f(R_N) = f(R'_N)$. Moreover if $r_1(R'_i) = \{x, y\}$ then by the IDA property, $f(R_N) = f(R'_N)$. Hence agent i cannot manipulate f. Now consider a non dummy agent i. Let $i \in \bar{C} \in g(R_N)$. Suppose $r_1(R_i) = y$ and $f(R_N) = x$. This means there is $C \in g(R_N)$ such that $r_1(R_j) = x$ for all $j \in C$. Now, if $r_1(R'_i) = x$ then by the definition of g we have $g(R_N) = g(R'_N)$ which means $C \in g(R'_N)$ and hence $f(R_N) = f(R'_N)$. On the other hand, if $r_1(R'_i) = \{x, y\}$ then by RNDA $C \in g(R'_N)$ and hence $f(R_N) = f(R'_N)$. Now, suppose $r_1(R_i) = x$ and $f(R_N) = y$. This means for all $C \in g(R_N)$ there is $j \in C; j \neq i$ such that $r_1(R_j) = y$. If $r_1(R'_i) = y$ then using the fact that $g(R_N) = g(R'_N)$ we have $f(R_N) = f(R'_N)$. Now suppose $r_1(R'_i) = \{x, y\}$. Then by RNDA it follows that for all $C \in g(R'_N)$ there is $j \in C$ such that $r_1(R_j) = y$. So $f(R_N) = f(R'_N)$. This completes the proof that f is strategy-proof.

(Only if part) Consider a unanimous and strategy-proof social choice function f on $\mathcal{D}^n$. Take $C \subsetneq N$ and let $R_C \in \bar{\mathcal{D}}^C$ be a collection of preferences of the players in C. Let $\mathcal{D}^n(R_C) = \{R'_N \in \hat{\mathcal{D}}^n : R'_C = R_C \text{ and } r_1(R'_i) \in \{x, y\} \text{ for all } i \notin C\}$. Take $R_N, R'_N \in \mathcal{D}^n(R_C)$. In view Lemma 3.6, it follows by standard monotonicity argument that for all $R_N, R'_N \in \mathcal{D}^n(R_C)$, $f(R_N) = x$ and $N^x(R'_N) \supseteq N^x(R_N)$ imply $f(R'_N) = x$. Moreover by unanimity, $f(R_N) = x$ implies $N^x(R_N) \neq \emptyset$. This means there exists $\mathcal{C}(R_C) \in \mathcal{P}_0(\mathcal{P}_0(N \setminus C))$ such that for all $R_N \in \mathcal{D}^n(R_C)$, $f(R_N) = x$ if and only if there is $C' \in \mathcal{C}(R_C)$ such that $N^x(R_N) \supseteq C'$. Define the function $g : \hat{\mathcal{D}}^n \to \mathcal{P}_0(\mathcal{P}_0(N))$ such that for all $R_N \in \hat{\mathcal{D}}^n$, $g(R_N) = \mathcal{C}(R_C)$ where $C = C(R_N)$. Now we show g satisfies IDA and RNDA.

First we show that g satisfies IDA. Suppose not. Then either there is $C' \in g(R_N)$ such that $C' \notin g(R'_N)$, or there is $C' \in g(R'_N)$ such that $C' \notin g(R_N)$ where $R'_N = (R'_i, R_{N\setminus i})$. Suppose there is $C' \in g(R_N)$ such that $C' \notin g(R'_N)$. First we show that $C'' \notin g(R'_N)$ for all $C'' \subsetneq C'$. Consider $\bar{R}_N \in \mathcal{D}^n(R_C)$ where $C = C(R_N)$ and $r_1(\bar{R}_j) = x$ if and only if $j \in C'' \cup i$. Then by the definition of f we have $f(R_N) = y$ and $f(R'_N) = x$, and hence player i manipulates at R_N via R'_i. Finally, consider the profile $\bar{R}_N \in \mathcal{D}^n(R_C)$ such that $r_1(\bar{R}_j) = x$ if and only if $j \in C'$. This means $f(R_N) = x$ and $f(R'_N) = y$, and hence player i manipulates at R_N via R'_i.

Now suppose there is $C' \in g(R'_N)$ such that $C' \notin g(R_N)$. First we show that $C'' \notin g(R_N)$ for all $C'' \subsetneq C'$. Consider $\bar{R}_N \in \mathcal{D}^n(R_C)$ where $C = C(R_N)$ and $r_1(\bar{R}_j) = x$ if and only if $j \in C'' \cup i$. Then by the definition of f we have $f(R_N) = x$ and $f(R'_N) = y$, and hence player i manipulates at R_N via R'_i. Finally, consider the profile $\bar{R}_N \in \mathcal{D}^n$ such that $r_1(R_j) = x$ if and only if $j \in C' \cup i$. This means $f(R_N) = y$ and $f(R'_N) = x$, and hence player i manipulates at R_N via R'_i. This proves that g satisfies IDA.

Now we show that g satisfies RNDA. Suppose not. Then, either there is $C'' \in g(R_N) \setminus g(R'_N)$ such that $i \notin C''$ and $r_1(R'_i) = \{x, y\}$, or for some $C'' \in g(R'_N) \setminus g(R_N)$ there is no $\bar{C} \in g(R_N)$ such that $\bar{C} \setminus i \subseteq C''$ and $i \in \bar{C}$ where $R'_N = (R'_i, R_{N \setminus i})$. Suppose there is $C'' \in g(R_N) \setminus g(R'_N)$ such that $i \notin C''$. Since $C'' \notin g(R'_N)$, using similar argument as before it follows that that $\hat{C} \notin g(R'_N)$ for all $\hat{C} \subseteq C''$. Consider $\bar{R}_N \in \mathcal{D}^n(R_C)$ where $C = C(R_N)$ and $r_1(\bar{R}_j) = x$ if and only if $j \in C''$. This means $f(R_N) = x$ and $f(R'_N) = y$ and hence player i manipulates at R_N via R'_i.

Now suppose for some $C'' \in g(R'_N) \setminus g(R_N)$ there is no $\bar{C} \in g(R_N)$ such that $\bar{C} \setminus i \subseteq C''$ and $i \in \bar{C}$. Consider $\bar{R}_N \in \mathcal{D}^n(R_C)$ where $C = C(R_N)$ and $r_1(\bar{R}_j) = x$ if and only if $j \in C'' \cup i$. This means $f(R_N) = y$ and $f(R'_N) = x$ and hence player i manipulates at R_N via R'_i. Hence, g satisfies RNDA. This completes the proof of the theorem. ■

3.2 Necessary Condition for Binary Support

In this section we provide conditions on a domain that are necessary to ensure that every strategy-proof and unanimous social choice function will have binary support. It follows that the necessary condition is very close to the sufficiency one meaning that our sufficient condition is very weak, or in some sense *almost* necessary. We leave the problem of finding the necessary and sufficient condition for future research.

Definition 3.5. A domain of weak preferences is called a *almost binary restricted domain* over $\{x, y\}$, where $x, y \in A$, if

1. for all $R \in \mathcal{D}$, $r_1(R) \in \{\{x\}, \{y\}, \{x, y\}\}$,
2. there does not exist $z_R \in R[x, y]$ for all $R \in \mathcal{D}^x$, with the property that $z_R R z_{R'}$ $\forall R, R' \in \mathcal{D}^x$, such that
   (a) for each $R' \in \mathcal{D}^y$, either $z_R \in R'[y, x]$ for all $R \in \mathcal{D}^x$, or $z_R \notin R'[y, x]$ for all $R \in \mathcal{D}^x$,
   (b) there exists $R' \in \mathcal{D}^y$ such that $z_R \in R'[y, x]$ for all $R \in \mathcal{D}^x$,
3. there does not exist $z_R \in R[y, x]$ for all $R \in \mathcal{D}^y$, with the property that $z_R R z_{R'}$ $\forall R, R' \in \mathcal{D}^y$, such that

(a) for each $R' \in \mathcal{D}^x$, either $z_R \in R'[x,y]$ for all $R \in \mathcal{D}^y$, or $z_R \notin R'[x,y]$ for all $R \in \mathcal{D}^y$,

(b) there exists $R' \in \mathcal{D}^x$ such that $z_R \in R'[x,y]$ for all $R \in \mathcal{D}^y$.

Theorem 3.6. *Let $\mathcal{D}$ be a domain and $n \geq 2$ such that every strategy-proof and unanimous PSCF defined on $\mathcal{D}^n$ has binary support. Then $\mathcal{D}$ is almost binary restricted domain.*

Proof. Let $\mathcal{D}$ be a domain and $n \geq 2$ such that every strategy-proof and unanimous PSCF defined on $\mathcal{D}^n$ has binary support. We show that it is a almost binary restricted domain. Assume for contradiction that the domain does not satisfy the condition 1 of the definition of almost binary restricted domain. We construct a strategy-proof and unanimous PSCF on $\mathcal{D}^n$ that does not have binary support.

Note that if there exists $R \in \mathcal{D}$ with $r_1(R) \cap \{x,y\} = \emptyset$ then we are done by unanimity since unanimity implies $\Phi_x(R_N) = \Phi_y(R_N) = 0$ where $R_i = R$ for all $i \in N$. So, assume $r_1(R) \cap \{x,y\} \neq \emptyset$ for all $R \in \mathcal{D}$ and $\cup_{R \in \mathcal{D}} r_1(R) \supseteq \{x,y,z\}$ for some $z \in A$. For $R \in \mathcal{D}$ and $X \subseteq A$, we denote the maximal set of alternatives amongst X at R by $B_R(X) = \{z : zRx, \forall x \in X\}$. Note that $B_R(A) = r_1(R)$. Consider the PSCF

$$\Phi(R_N) = U\left(B_{R_n}(B_{R_{n-1}}(\ldots(B_{R_2}(B_{R_1}(A)))\ldots))\right)$$

where $U(S)$ denotes uniform probability distribution over the elements in the set S. We call such type of PSCF a serial random dictatorial rule with the dictatorial order of players $1 \succ 2 \succ \ldots \succ n$.

The PSCF Φ is unanimous by definition. To see that $\{x,y,z\} \subseteq Supp(\Phi)$ consider the unanimous profile R_N where $R_i = R_j$ for all $i,j \in N$ and $z \in r_1(R_1)$. By the definition of Φ, z gets positive probability at this profile. It remains to show that Φ is strategy-proof. Agent 1 cannot manipulate Φ since $\Phi_{r_1(R_1)}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$. Suppose the agents $1,2,\ldots,k$ cannot manipulate Φ. We show agent $k+1$ cannot manipulate Φ. Note that $B_{R_{k+1}}(X) R_{k+1} B_{R'_{k+1}}(X)$ for all $X \subseteq A$, R_{k+1} and R'_{k+1}. This, together with the fact that $\Phi_{B_{R_{k+1}}(X)}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ where $X = B_{R_k}(B_{R_{k-1}}(\ldots(B_{R_2}(B_{R_1}(A)))\ldots))$, implies that agent $k+1$ cannot manipulate Φ. This proves Φ is strategy-proof. Hence, $\mathcal{D}$ must satisfy condition 1 of the definition of almost binary restricted domain.

Now we assume that a domain satisfies condition 1 of almost binary restricted domain, and show that both condition 2 and condition 3 are necessary to ensure binary support for every strategy-proof and uannimous rule. We show the necessity of the condition 2, the proof of the necessity of condition 3 is similar. Consider a domain $\mathcal{D}$ that satisfies condition 1 but does not satisfy condition 2. We construct a strategy-proof and unanimous PSCF $\Phi : \mathcal{D}^n \to \triangle A$ that does not have binary support. Since condition 2 is not satisfied there exist $z_R \in R[x,y]$ for all $R \in \mathcal{D}^x$, with the property that $z_R R z_{R'} \; \forall R, R' \in \mathcal{D}^x$, and $\mathcal{D}^y$ can be partitioned into two subsets $\mathcal{D}^y_1$ and $\mathcal{D}^y_2$ such that $\mathcal{D}^y_1 = \{R \in \mathcal{D}^y : z_{R'} \notin R[y,x] \text{ for all } R' \in \mathcal{D}^x\}$, $\mathcal{D}^y_2 = \{R \in \mathcal{D}^y : z_{R'} \in R[y,x] \text{ for all } R' \in \mathcal{D}^x\}$ where $\mathcal{D}^y_2$ is non-empty. Consider the rule Φ as follows:

$$\Phi(R_N) = \begin{cases} U\left(r_1(R_1) \cap r_1(R_2) \cap \ldots \cap r_1(R_n)\right) \text{ if } r_1(R_1) \cap r_1(R_2) \cap \ldots \cap r_1(R_n) \neq \emptyset \\ U\left(r_1(R_1) \cap r_1(R_2)\right) \text{ if } r_1(R_1) \cap r_1(R_2) \cap \ldots \cap r_1(R_n) = \emptyset \text{ and } r_1(R_1) \cap r_1(R_2) \neq \emptyset \\ \Phi_{(x,z_{R_1},y)}(R_N) = (\frac{1}{3}, \frac{1}{6}, \frac{1}{2}) \text{ if } R_1 \in \mathcal{D}^x, \;\; R_2 \in \mathcal{D}^y_2 \\ \Phi_{(x,y)}(R_N) = (\frac{1}{2}, \frac{1}{2}) \text{ if } R_1 \in \mathcal{D}^x, \;\; R_2 \in \mathcal{D}^y_1 \\ \Phi_{(x,y)}(R_N) = (\frac{1}{2}, \frac{1}{2}) \text{ if } R_1 \in \mathcal{D}^y \text{ and } R_2 \in \mathcal{D}^x \end{cases}$$

where by $\Phi_{(x,z_{R_1},y)}(R_N) = (\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$ we mean $\Phi_x(R_N) = \frac{1}{3}, \Phi_{z_{R_1}}(R_N) = \frac{1}{6}$, and $\Phi_{z_{R_1}}(R_N) = \frac{1}{3}$.

By definition Φ is unanimous and does not have binary support. We show that Φ is strategy-proof. It is easy to see that no agent other than 1 and 2 can manipulate Φ. Moreover, it is clear that agents 1 and 2 will not manipulate if $r_1(R_1) \cap r_1(R_2) \neq \emptyset$. Consider a profile R_N where $R_1 \in \mathcal{D}^x$ and $R_2 \in \mathcal{D}^y$. Agent 1 will not manipulate since $z_{R_1} R_1 z_{R'_1} \forall R_1, R'_1 \in \mathcal{D}^x$. Moreover, agent 2 cannot manipulate since either $z_{R_1} \in R_2[y,x]$ for all $R_1 \in \mathcal{D}^x$, or $z_{R_1} \notin R_2[y,x]$ for all $R_1 \in \mathcal{D}^x$. Now consider a profile where $R_1 \in \mathcal{D}^y$ and $R_2 \in \mathcal{D}^x$. Note that the outcome does not change if an agent changes his/her preference keeping the top same, on the other hand if an agent changes his top then by unanimity the top of the other agent gets probability 1. This proves that Φ is strategy-proof. ■

4 Applications

In this section we provide a few applications of our result. We give a few examples of domains that are binary restricted. One such well-known example is the domain of single dipped preference relations relative to a given order of the alternatives. Another such example is the domain of single-peaked preferences where the peaks are restricted to be one of two adjacent alternatives.

4.1 Single-dipped Domain

In this section we apply our results to single-dipped domain and characterize all strategy-proof and unanimous PSCFs on this domain.

Definition 4.1. A preference relation of individual $i \in N$, R_i is single dipped on A relative to a linear order $\succ$ of the set of alternatives if

1. R_i has a unique minimal element $d(R_i)$, called the dip of R_i and

2. for all $y, z \in X$, $[d(R_i) \succeq y \succ z$ or $z \succ y \succeq d(R_i)] \Rightarrow zP_iy$

Let $\mathcal{D}_{\succ}$ denote the set of all single dipped preference relative to the order $\succ$ while $\mathcal{R}_{\succ}$ denote a subset of the single dipped preference relative to $\succ$. Clearly $\mathcal{D}_{\succ}$ is a binary restricted domain. Moreover, every restricted single dipped preference domain $\mathcal{R}_{\succ}$ is binary restricted if it satisfies condition 2 in the definition of binary restricted domain. So we deduce the following theorems for the single-dipped domain.

Theorem 4.1. *Single dipped domain is a deterministic extreme point (DEP) domain.*

Proof. This follows from Theorem 3.4. ∎

We now consider a restricted single-dipped domain where the alternatives are assumed to be equidistant from each other and the preference is derived from the distance from the dip. More formally, whenever the distance of an alternative from the dip of an agent is higher than that of another alternative, the agent prefers the former alternative to the latter one. We call such a domain a distance single-dipped domain. If ties are broken in both ways. Note that such a restricted single-dipped domain is binary restricted, and hence our results apply. However, if the ties are broken in favour of left side (or right side) only, then the domain is no more a binary restricted domain. In Example 4.1 we show that there exists strategy-proof and unanimous probabilistic rule that does not have binary support.

Example 4.1. Consider the domain presented in the table below. This domain is distance single-dipped domain where ties are always broken in favour of the left alternative. The PSCF given in the table is strategy-proof and unanimous, however it does not have binary support.

$1\backslash 2$	$x_1x_2x_3x_4$	$x_4x_3x_2x_1$	$x_4x_1x_3x_2$	$x_1x_2x_4x_3$
$x_1x_2x_3x_4$	$(1,0,0,0)$	$(\alpha-\beta,\beta,0,1-\alpha)$	$(\alpha,0,0,1-\alpha)$	$(1,0,0,0)$
$x_4x_3x_2x_1$	$(\epsilon-\gamma,\gamma,0,1-\epsilon)$	$(0,0,0,1)$	$(0,0,0,1)$	$(\epsilon-\gamma,\gamma,0,1-\epsilon)$
$x_4x_1x_3x_2$	$(\epsilon,0,0,1-\epsilon)$	$(0,0,0,1)$	$(0,0,0,1)$	$(\epsilon,0,0,1-\epsilon)$
$x_1x_2x_4x_3$	$(1,0,0,0)$	$(\alpha-\beta,\beta,0,1-\alpha)$	$(\alpha,0,0,1-\alpha)$	$(1,0,0,0)$

4.2 Adjacent Single-peaked Domain

Another well known domain of preferences is single peaked domain which is defined below.

Definition 4.2. A complete, reflexive, transitive and anti-symmetric preference relation of individual $i \in N$, P_i is single peaked on A relative to a linear order $\succ$ of the set of alternatives if

1. there exists $\tau(P_i) \in A$, called the peak of P_i and
2. for all $y, z \in A$, $[\tau(P_i) \succeq y \succ z$ or $z \succ y \succeq \tau(P_i)] \Rightarrow yP_iz$.

Let $\mathcal{D}^{\succ}$ denote the set of all single-peaked preferences relative to the order $\succ$ while $\mathcal{R}^{\succ}$ denote a subset of the single dipped preference relative to $\succ$. Clearly, when we have at least three alternatives the single peaked domain $\mathcal{D}^{\succ}$ relative to a linear order $\succ$ is not a binary restricted domain. However, a subset of single peaked domain where peaks are restricted to a set of two adjacent alternatives is a binary restricted domain over those two adjacent alternatives. We call such a domain adjacent single-peaked domain. We define this formally.

Definition 4.3. A restricted single-peaked domain $\mathcal{R}^{\succ}$ over a set $A = \{a_1, a_2, \ldots, a_k\}$ relative to the linear order $\succ$ over A where $a_1 \succ a_2 \succ \ldots \succ a_k$ is called adjacent single-peaked if there exists $l \in \{1, 2, \ldots, k-1\}$ such that $\tau(P_i) \in \{x_l, x_{l+1}\}$ for all $P_i \in \mathcal{R}^{\succ}$.

It is can be easily seen that $\mathcal{R}^{\succ}$ is a binary restricted domain, and hence every strategy-proof and unanimous PSCF defined on $\mathcal{R}^{\succ}$ has binary support and is a mixture of strategy-proof and unanimous DSCFs defined on $\mathcal{R}^{\succ}$.

5 Conclusion

In this paper we show that every strategy-proof and unanimous PSCF defined over a binary restricted domain has a binary support. We further show that, every such PSCF is a convex combination of strategy-proof and unanimous deterministic rules defined on that domain. We apply our results on some well-known binary restricted domains like single-dipped domain and adjacent single-peaked domain, and characterize all strategy-proof and unanimous PSCFs on those domains.

References

[1] Salvador Barberà, Dolors Berga, and Bernardo Moreno. Domains, ranges and strategy-proofness: the case of single-dipped preferences. *Social Choice and Welfare*, 39(2-3):335–352, 2012.

[2] Shurojit Chatterji, Souvik Roy, and Arunava Sen. The structure of strategy-proof random social choice functions over product domains and lexicographically separable preferences. *Journal of Mathematical Economics*, 48(6):353–366, 2012.

[3] Shurojit Chatterji, Arunava Sen, and Huaxia Zeng. Random dictatorship domains. *Games and Economic Behavior*, 86:212–236, 2014.

[4] Allan Gibbard. Manipulation of schemes that mix voting with chance. *Econometrica: Journal of the Econometric Society*, pages 665–681, 1977.

[5] Vikram Manjunath. Efficient and strategy-proof social choice when preferences are single-dipped. *International Journal of Game Theory*, 43(3):579–597, 2014.

[6] Hans Peters, Souvik Roy, Arunava Sen, and Ton Storcken. Probabilistic strategy-proof rules over single-peaked domains. *Journal of Mathematical Economics*, 52:123–127, 2014.

[7] Arunava Sen. The gibbard random dictatorship theorem: a generalization and a new proof. *SERIEs*, 2(4):515–527, 2011.