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Marriage Markets with Externalities

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Marriage Markets with Externalities

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Abstract: This paper examines a marriage market with externality. We first develop an appropriate notion of stability for this market, called E-stability. We provide an example to show that an E-stable outcome need not exist. We then derive conditions under which an E-stable outcome exists.

Key words: Marriage model, stability, externalities.

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1 Introduction

In this paper we consider marriage markets with externalities.

Marriage models are an important subclass of two-sided matching models with one-to-one matchings. There are two disjoint class of agents, say men and women. An agent on one side of the market can be matched to another agent from the other side of the market, or he/she may remain single. In a classic paper, Gale and Shapley (1962) develop the standard solution concept for marriage markets, that of *stability*. An outcome is said to be stable if it cannot be blocked by individual or pairwise coalitions. They also establish that a stable outcome exists.

We generalize the standard marriage market model to allow for externalities, something that we believe are important in reality. Such externalities are likely to be exist whenever the agents, after the completion of the matching process, indulge in some activity where the outcome depends on the earlier assignment.

As an example, consider a labor market where every firm employs exactly one worker and the wage rate is exogenously given. In such a market the profit of a firm may depend on which worker is hired by a rival firm, or whether rival firms manage to fill their vacancies at all. Alternatively, consider a game of technology transfer from technologically advanced foreign firms to domestic firms. Interpreting the process of technology transfer as a matching process, the profits of the firms in the post-transfer game is clearly going to depend on the earlier matching, i.e. which domestic firm bought which technology etc. In fact, even in real marriage markets one can observe examples of externalities e.g. jealousy etc.

There is some work dealing with externalities in the context of assignment models, i.e. marriage markets with money. These include, among

others, Li (1993), Roy Chowdhury (1997) and Sonmez (1996).

To the best of our knowledge, the only other work on marriage markets with externalities is by Sasaki and Toda (1996). They consider a notion of stability where the agents are assumed to be very pessimistic. Existence is shown, as well as the fact that this notion of stability does not contradict Pareto optimality.

In this paper we consider an alternative notion of stability called *E-stability*, where agents are assumed to be not so pessimistic. Under this notion any coalition, either singleton or pairwise, block a matching if, by deviating, they are made better off under both the existing assignment, as well as the post-deviation assignment. We can think of this notion as an application of Cournot-Nash conjectures in that the deviating agents assume that the other, non-deviating, agents are not going to deviate from the proposed assignment. This is in contrast to Sasaki and Toda (1996) where a coalition blocks a matching if the coalition is made better off under all possible assignments.

Compared to the Sasaki-Toda (1996) definition, under our notion of stability deviations are more likely, and hence the existence of a 'stable' outcome is not guaranteed. In fact we provide an example to demonstrate that an E-stable outcome need not exist. We then derive sufficient conditions for the existence of an E-stable outcome.

2   The Model

We first develop the basic model without externalities and then go on to model the case with externalities.

2.1 No Externalities

There are two disjoint set of agents, denoted M and W, where M is the set of men and W is the set of women. Members of M are called the $m-$agents, and members of W are called the $w-$agents.

The preference of every man m is represented by an ordered list of preferences, $P(m)$, on the set $W \cup \{m\}$. Similarly, the preference of every woman w is represented by an ordered list of preferences, $P(w)$, on the set $M \cup \{w\}$. We assume that preferences are complete and transitive.

Let P denote the set of preferences for all agents in M and W. The triple $\{M, W, P\}$ denotes a marriage market. We write $w >_m w'$ when m strictly prefers w to w', and $w \geq_m w'$ when m weakly prefers w to w'. One can define $m >_w m'$ and $m \geq_w m'$ similarly.

We now introduce a series of definitions that we require for the analysis.

Definition. An *assignment* μ is an one-to-one correspondence from the set $M \cup W$ onto itself of order two such that if, for some $m \in M$, $\mu(m) \neq m$, then $\mu(m) \in W$ and if, for some $w \in W$, $\mu(w) \neq w$, then $\mu(w) \in M$.

Let μ_x denote an assignment where all matchings follow μ, except that agent x and agent $\mu(x)$ (in case $\mu(x) \neq x$) remain single.

Let μ_{mw} denote an assignment where all matchings follow μ, except that m and w are matched to each other. Moreover, if $\mu(m) \neq m$ then $\mu(m)$ remains single under μ_{mw}, and if $\mu(w) \neq w$ then $\mu(w)$ remains single under μ_{mw}.

We are now in a position to define the notion of stability.

Definition. A matching μ is *stable* if
(i) there exists no x such that x strictly prefers $\{x\}$ to $\mu(x)$, and
(ii) there exists no m and w such that, m strictly prefers w to $\mu(m)$, and

w strictly prefers m to $\mu(w)$.

For a marriage market without externality we have the following well known result by Gale and Shapley (1962).

Theorem 1. *For every marriage market* $\{M, W, P\}$, *a stable matching exists.*

We refer the readers to Roth and Sotomayor (1990) for a succinct discussion of the literature on marriage markets.

2.2   Introducing Externalities

We then examine the case where the preferences of the agents are a function of the assignment itself. Consider the following example.

Example 1. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Let the preferences of the agents be as follows.

(i) In case w_2 is single, m_1's preference is: $w_1 >_{m_1} \{m_1\} >_{m_1} w_2$, and w_1's preference is: $m_1 >_{w_1} \{w_1\} >_{w_1} m_2$.

(ii) In case w_2 is matched, m_1's preference is: $\{m_1\} >_{m_1} w_1 >_{m_1} w_2$, and w_1's preference is: $\{w_1\} >_{w_1} m_1 >_{w_1} m_2$.

(iii) In case w_1 is single, m_2's preference is: $\{m_2\} >_{m_2} w_2 >_{m_2} w_1$, and w_2's preference is: $\{w_2\} >_{w_2} m_2 >_{w_2} m_1$.

(iv) In case w_1 is matched, m_2's preference is: $w_2 >_{m_2} \{m_2\} >_{m_2} w_1$, and w_2's preference is: $m_2 >_{w_2} \{w_2\} >_{w_2} m_1$.

Let $P(m, \mu)$ denote the preference of m when the actual assignment is μ and let $P(w, \mu)$ denote the preference of w when the actual assignment is μ. As Example 1 demonstrates, it is possible that for two different μ and μ', $P(m, \mu) \neq P(m, \mu')$ and $P(w, \mu) \neq P(w, \mu')$.

We then formally define the notion of stability in this case, called *E-stability*. The definition of E-stability tries to capture the idea that for any individual or a group to deviate from an assignment μ, a minimal condition should be that the concerned deviation should be attractive under *both* μ, as well as the post-deviation assignment. Thus E-stability satisfies a minimal *no-regret* property so that post-deviation, no deviating coalition regrets the decision to deviate.

Definition. A matching μ is *E-stable* if and only if

(i) there exists no x such that x strictly prefers $\{x\}$ to $\mu(x)$ under both $P(x, \mu)$ and $P(x, \mu_x)$, and

(ii) there exists no m and w such that m strictly prefers w to $\mu(m)$ under both $P(m, \mu)$ and $P(m, \mu_{mw})$, and w strictly prefers m to $\mu(w)$ under both $P(w, \mu)$ and $P(w, \mu_{mw})$.

Clearly, in the absence of externalities, this definition reduces to the Gale and Shapley (1962) definition of stability.

3   Results

We first use Example 1 to show that in the presence of externalities an E-stable outcome need not exist.

Non-existence of E-stable outcomes. Consider Example 1. To begin with note that for m_1, being single is always strictly preferred to being matched to w_2. Hence there cannot be an E-stable outcome where m_1 is matched to w_2. Similarly, there cannot be an E-stable outcome where m_2 is matched to w_1. We then rule out the other possible candidate assignments one by one.

Case A. Consider μ such that all agents are single. Clearly, m_1 and w_1 prefer to be matched to each other under both μ and $\mu_{m_1 w_1}$.

Case B. Consider μ such that m_1 is matched to w_1 and m_2 is matched to w_2. Clearly, m_1 would prefer to remain single under both μ and μ_{m_1}.

Case C. Consider μ such that m_1 is matched to w_1 and the other agents are single. Clearly, m_2 and w_2 would prefer to be matched to each other under both μ and $\mu_{m_2 w_2}$.

Case D. Consider μ such that m_2 is matched to w_2 and the other agents are single. Clearly, m_2 would prefer to be single under both μ and μ_{m_2}.

The existence result in Gale and Shapley (1962) depends on the two-sidedness of the market, as well as the fact that matchings are one-to-one. For example, Gale and Shapley (1962) use the roommate problem to demonstrate that non-existence may occur if the market is not two-sided. Alkan (1986) use an example involving three-sided matching to make a similar point. Roth and Sotomayor (1990), on the other hand, use a many-to-one matching model to underline the crucial importance of the assumption that the matching process is one-to-one (see Example 2.7, pp. 25-26). Example 1 demonstrates that another critical assumption behind the Gale-Shapley result is the absence of externalities.

We then turn to the task of identifying conditions that ensure the existence of an E-stable outcome.

Consider Example 1. Note that with a change in the assignment, the ranking of the agents vis-a-vis the agents on the other side of the market are not changing, what is changing is the relative ranking between remaining single and the agents on the other side of the market. Assumption 1 seeks to rule out irregularities of such kind. In assignment models, for example, similar assumptions are quite standard (see Shapley and Shubik (1972)).

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Assumption 1. For all agents and for all possible assignments, remaining unmatched is the least preferred outcome.

Assumption 2 below imposes some additional regularity conditions on the structure of externalities. Consider, for example, the labor market discussed in the introduction. Clearly, if the firms and the workers are symmetric, then Assumption 2 holds.

Assumption 2. Let μ and μ' be two matchings such that the number of matchings under both equal $\min\{|M|,|W|\}$. Then $P(m,\mu) = P(m,\mu')$ for all m and $P(w,\mu) = P(w,\mu')$ for all w.

We are now in a position to write down the main result of this paper.

Theorem 2. *Suppose Assumptions 1 and 2 hold. Then an E-stable outcome exists.*

Proof. Let $A = \min\{|M|,|W|\}$. From Assumption 2, $P(m,\mu) = \tilde{P}(m)$ and $P(w,\mu) = \tilde{P}(w)$, for all μ involving A matchings. Now consider a standard matching model without externalities where the preference of an m-agent is given by $\tilde{P}(m)$ and that of a w-agent by $\tilde{P}(w)$. From Theorem 1, $\{M,W,\tilde{P}\}$ has a stable assignment, say $\tilde{\mu}$. Given Assumption 1, $\tilde{\mu}$ must involve $\min\{|M|,|W|\}$ matchings.

We then check if the matching is E-stable. Given Assumption 1, none of the agents who are matched under $\tilde{\mu}$ would prefer to be single rather than remain matched. Next consider the possibility of pairwise deviations. Since $\tilde{\mu}$ is stable (in the Gale-Shapley sense) for $\{M,W,\tilde{P}\}$, there exists no pair m and w such that m strictly prefers w to $\mu(m)$ under $P(m,\tilde{\mu})$ and w strictly prefers m to $\mu(w)$ under $P(w,\tilde{\mu})$. ∎

Corollary. Under Assumptions 1 and 2, an assignment is an E-stable outcome if and only if it is a stable outcome, in the Gale-Shapley sense, of the market $\{M, W, \tilde{P}\}$.

Proof. Clearly, given Assumption 1, any E-stable outcome must involve $\min\{|M|, |W|\}$ matchings. Hence, any outcome is E-stable if and only if it is a stable outcome of $\{M, W, \tilde{P}\}$. ∎

4   Conclusion

This paper examines a marriage market with externalities. We first provide an example to show that in the presence of externalities, a 'stable' (i.e. E-stable) outcome need not exist. We then identify conditions under which an E-stable outcome exists. *Inter alia*, we also relate the E-stable outcomes of this model to the stable outcomes of a related standard marriage market (without externalities).

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