isid/ms/2002/32
December 18, 2002
http://www.isid.ac.in/~statmath/eprints

On Filtering with Ornstein-Uhlenbeck Process as Noise

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Abstract

We consider the nonlinear filtering model with Ornstein-Uhlenbeck process as noise and obtain an analogue of the Bayes' formula for the filter. For this we need to consider a modified model, where the instaneteneous effect $h(X_t)$ of the signal in the usual model is replaced by $\xi_t^\alpha = \alpha \int_{(t-\frac{1}{\alpha})\vee 0}^t h(X_u)\, du$, (where α is a large parameter). This means that there is a lingering effect of the signal for a time period $\frac{1}{\alpha}$.

Further, we also show the filter with Ornstein-Uhlenbeck converges to the usual filter in probability.

MSC 2000 subject classification: Primary 60G35 Secondary 60H10, 60G17, 60G44.

Key words and phrases: Nonlinear filtering, White noise, Ornstein-Uhlenbeck process, pathwise formulae for SDE

1 Introduction

Filtering thoery deals with the following situation: There is a process of interest, called the signal process (X_t) that is not directly observable. However, it is assumed that $h(X_t)$ is observable at time t in the presence of Noise. Here it is assumed that h is a known function. It is customary to assume that the noise is additive leading to a model

$$Y_t = \int_0^t h(X_u)\, du + N_t \tag{1.1}$$

where Y_t denotes the (accumulated) observation at time t and N_t denotes the noise over the time interval $[0, t]$.

The question of interest is to *estimate* the signal X_t having observed $\{Y_u : \ 0 \leq u \leq t\}$ This is known as filtering the noise (to recover the signal). In the classical approach to filtering theory, the noise (N_t) is modelled as a Brownian motion.

Kunita [5] had initiated study of filtering theory with general Gaussian noise processes. Filtering with Ornstein-Uhlenbeck noise was studied by Mandrekar and Mandal [6]. However, they had to require such a stringent condition on the paths of the signal process- namely that $h(X_t)$ is differentiable almost surely- that it rules out all the standard examples.

We will consider the filtering problem with noise (N_t) replaced by Orstein-Uhlenbeck process N_t^β (defined by (2.3))

$$Y_t = \int_0^t h(X_u)\, du + N_t^\beta.$$

Since N_t^β is differentiable (with derivative n_t^β given by equation (2.2)), it follows that the observation process Y_t is also differentable (with derivative y_t). The above model is then equivalent to

$$y_t = h(X_t) + n_t^\beta.$$

Here, $h(X_t)$ is the instantaneous effect of the signal and y_t is the instantaneous observation.

We will consider a modified model where the instantaneous effect $h(X_t)$ is replaced by a lingering efffect $\xi_t^\alpha = \alpha \int_{(t-\frac{1}{\alpha})\vee 0}^t h(X_u)\, du$ - where α is a large parameter. Note that as α tends to infinity, ξ_t^α converges to $h(X_t)$. This change allows us to get rid of the undesirable assumptions on X_t - such as differentiability of paths - made in a similar context in Mandrekar and Mandal [6].

Denoting the optimal filter for this model by $\pi^{\alpha,\beta}$, we first derive an analogue of the Kallianpur-Striebel Bayes' formula for $\pi^{\alpha,\beta}$.

It is well known that N_t^β converges to W_t as β converges to infinity and thus the model considered here can be thought of as a smooth approximation to the classical model of filtering with Wiener noise (for large α, β).

Thus we investigate the behaviour of the filter $\pi^{\alpha,\beta}$ as α, β tend to infinity. We show that $\pi^{\alpha,\beta}$ converges to the classical filter π_t in probability. This shows that the classical filter is robust under perturbation of the underlying noise process (namely Ornstein-Uhlenbeck process with a large parameter β). This result complements the (model) robustness results proved in Bhatt *et. al* [1] where it was shown that the filter is robust under perturbation of the law of the signal process as well as of the underlying function h.

For notational simplicity, we will consider the one dimensional case.

2 Bayes' formula for filter with OU noise

Fix a probability space $(\Omega, \mathcal{F}, P)$. Let E be a complete, separable metric space. We will assume that the signal process X takes values in E and that the paths of X are right continuous with left limits (r.c.l.l.). We will assume that the observation function h is continuous. Let W be a standard Brownian motion independent of X.

Fix $\alpha > 0,\ \beta > 0$. Define processes ξ^α and n^β by

$$\xi_t^\alpha = \alpha \int_{(t-\frac{1}{\alpha})\vee 0}^t h(X_u)\, du, \tag{2.1}$$

$$n_t^\beta = e^{-\beta t} n_0^\beta + \beta \int_0^t e^{-\beta(t-u)} dW_u, \tag{2.2}$$

where n_0^β has a normal distribution with mean zero and variance $\beta/2$ and is independent of X and W. Then n^β is the stationary Ornstein - Uhlenbeck velocity process with covariance function

$$\rho(s,t) = \frac{\beta}{2} e^{-\beta|t-s|}.$$

Let N^β be defined by

$$N_t^\beta = \int_0^t n_u^\beta\, du = \frac{(1-e^{-\beta t})}{\beta} n_0^\beta + \int_0^t \left(1 - e^{-\beta(t-u)}\right) dW_u. \tag{2.3}$$

Remark 2.1. N^β is called the stationary Ornstein - Uhlenbeck displacement process. It is well-known that N^β converges to W in $L^2(P)$ as $\beta \to \infty$. (See Nelson [7]). This fact also follows easily from (2.3).

We will consider the filtering model with Ornstein - Uhlenbeck noise given by

$$Y_t^{\alpha,\beta} = \int_0^t \xi_s^\alpha ds + N_t^\beta, \quad 0 \le t \le T. \tag{2.4}$$

This can be equivalently written as

$$y_t^{\alpha,\beta} = \xi_t^\alpha + n_t^\beta, \quad 0 \le t \le T \tag{2.5}$$

where $y_t^{\alpha,\beta} = \frac{d}{dt} Y_t^{\alpha,\beta}$.

Remark 2.2. Note that using (2.1) the filtering model (2.4) can be rewritten as (for $t > \frac{1}{\alpha}$)

$$Y_t^{\alpha,\beta} = \int_0^{t-1/\alpha} h(X_u)\,du + \alpha \int_{t-\frac{1}{\alpha}}^{t} (t-u)h(X_u)\,du + N_t^{\beta}, \quad 0 \le t \le T. \tag{2.6}$$

In this form it can be more readily compared with the classical model (1.1). The above model can be thought of as corresponding to the system where there is a delay in registering the signal and hence the instantaneous signal is replaced by a lingering effect over a small, fixed time interval of length $1/\alpha$.

Our aim in this section is to get an expression for the filter $\pi^{\alpha,\beta}$ for the filtering model (2.4), where $\pi^{\alpha,\beta}$ is defined by

$$\pi_t^{\alpha,\beta}(f) = E\left[f(X_t)|Y_u^{\alpha,\beta} : u \le t\right], \quad \forall f \in C_b(E). \tag{2.7}$$

For this purpose we will recast (2.4) in a form that enables us to use the classical Kallianpur-Striebel Bayes' formula. Let

$$\widehat{y}_t^{\alpha,\beta} = e^{\beta t} y_t^{\alpha,\beta} - n_0^{\beta} \tag{2.8}$$

$$\widehat{\xi}_t^{\alpha,\beta} = e^{\beta t}\xi_t^{\alpha}, \tag{2.9}$$

$$\widehat{n}_t^{\beta} = e^{\beta t} n_t^{\beta} - n_0^{\beta} \tag{2.10}$$

Then it follows that

$$\widehat{y}_t^{\alpha,\beta} = \widehat{\xi}_t^{\alpha,\beta} + \widehat{n}_t^{\beta}, \quad 0 \le t \le T. \tag{2.11}$$

Using (2.2), it follows that

$$\widehat{n}_t^{\beta} = \beta \int_0^t e^{\beta u} dW_u,$$

so $\widehat{n}_t^{\beta}$ is a semimartingale and

$$\int_0^t \frac{e^{-\beta u}}{\beta} d\widehat{n}_u^{\beta} = W_t. \tag{2.12}$$

The relation (2.11) implies that $\widehat{y}_u^{\alpha,\beta}$ is also a semimartingale. Now define

$$\widetilde{Y}_t^{\alpha,\beta} = \int_0^t \frac{e^{-\beta u}}{\beta} d\widehat{y}_u^{\alpha,\beta} \tag{2.13}$$

$$\widetilde{\xi}_t^{\alpha,\beta} = \int_0^t \frac{e^{-\beta u}}{\beta} d\widehat{\xi}_u^{\alpha,\beta}. \tag{2.14}$$

It follows from (2.11)-(2.14) that

$$\widetilde{Y}_t^{\alpha,\beta} = \widetilde{\xi}_t^{\alpha,\beta} + W_t, \quad 0 \le t \le T.$$

Let $\widetilde{\psi}_t^{\alpha,\beta} = \frac{d}{dt}\widetilde{\xi}_t^{\alpha,\beta}$. Then

$$\widetilde{\psi}_t^{\alpha,\beta} = \xi_t^{\alpha} + \frac{\alpha}{\beta}\left(h(X_t) - h(X_{(t-\frac{1}{\alpha})\vee 0})\right). \tag{2.15}$$

The filtering model now becomes

$$\widetilde{Y}_t^{\alpha,\beta} = \int_0^t \widetilde{\psi}_u^{\alpha,\beta}\, du + W_t, \quad 0 \le t \le T. \tag{2.16}$$

Also, it follows from (2.8) and (2.13) that

$$\begin{aligned}\sigma\left(\widetilde{Y}_u^{\alpha,\beta} : u \le t; n_0^\beta\right) &= \sigma\left(\widehat{y}_u^{\alpha,\beta} : u \le t; n_0^\beta\right) = \sigma\left(y_u^{\alpha,\beta} : u \le t; n_0^\beta\right) \\ &= \sigma\left(y_u^{\alpha,\beta} : u \le t\right) \end{aligned} \tag{2.17}$$

The last equality follows from the fact that $y_0^{\alpha,\beta} = n_0^\beta$.

We need to introduce an independent copy of X. For this purpose let $\bar{X}$ be a process defined on some $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ such that

$$\bar{P} \circ \bar{X}^{-1} = P \circ X^{-1}.$$

Following (2.1) and (2.15) define

$$\bar{\xi}_t^\alpha = \alpha \int_{(t-\frac{1}{\alpha})\vee 0}^t h\left(\bar{X}_u\right)\, du \tag{2.18}$$

and

$$\bar{\psi}_t^{\alpha,\beta} = \bar{\xi}_t^\alpha + \frac{\alpha}{\beta}\left(h\left(\bar{X}_t\right) - h\left(\bar{X}_{(t-\frac{1}{\alpha})\vee 0}\right)\right). \tag{2.19}$$

With an abuse of notation, we will consider the processes $\bar{X}$, W and $\widetilde{Y}^{\alpha,\beta}$ to be defined on the product space $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}\right) \otimes (\Omega, \mathcal{F}, P)$.

Theorem 2.1. *Consider the filtering model (2.4). The optimal non-linear filter $\pi_t^{\alpha,\beta}$ admits the representation*

$$\pi_t^{\alpha,\beta}(f)(\omega) = \frac{\int f\left(\bar{X}_t(\bar{\omega})\right) q_t^{\alpha,\beta}(\bar{\omega},\omega) d\bar{P}(\bar{\omega})}{\int q_t^{\alpha,\beta}(\bar{\omega},\omega) d\bar{P}(\bar{\omega})}, \forall f \in C_b(E) \tag{2.20}$$

where

$$q_t^{\alpha,\beta}(\bar{\omega},\omega) = \exp\left\{\int_0^t \bar{\psi}_u^{\alpha,\beta}(\bar{\omega}) d\widetilde{Y}_u^{\alpha,\beta}(\omega) - \frac{1}{2}\int_0^t (\bar{\psi}_u^{\alpha,\beta}(\bar{\omega}))^2\, du\right\}. \tag{2.21}$$

Proof : Let $\widetilde{\pi}^{\alpha,\beta}$ denote the optimal non-linear filter for the model (2.16). i.e.

$$\widetilde{\pi}_t^{\alpha,\beta}(f) = E\left[f(X_t) | \widetilde{Y}_u^{\alpha,\beta} : u \le t\right], \quad \forall f \in C_b(E).$$

The filtering model (2.16) is the classical nonlinear filtering model with the Brownian motion noise being independent of the signal. It is well-known that $\widetilde{\pi}^{\alpha,\beta}$ admits the representation

$$\widetilde{\pi}_t^{\alpha,\beta}(f)(\omega) = \frac{\int f(\bar{X}_t(\bar{\omega})) q_t^{\alpha,\beta}(\bar{\omega},\omega) d\bar{P}(\bar{\omega})}{\int q_t^{\alpha,\beta}(\bar{\omega},\omega) d\bar{P}(\bar{\omega})} \quad \forall f \in C_b(E), \text{ for a.a.}\omega,$$

where $q^{\alpha,\beta}$ is as in (2.21). (See Kallianpur and Karandikar [3, p. 575]).

Now, since n_0^β is independent of X and W, we get for all $f \in C_b(E)$

$$E_P\left[f(X_t)|\widetilde{Y}_u^{\alpha,\beta} : u \le t\right] = E_P\left[f(X_t)|\widetilde{Y}_u^{\alpha,\beta} : u \le t; n_0^\beta\right].$$

It now follows from the observation (2.17) and the definitions of $\pi_t^{\alpha,\beta}$ and $\widetilde{\pi}^{\alpha,\beta}$ that

$$\pi_t^{\alpha,\beta}(f) = \widetilde{\pi}_t^{\alpha,\beta}(f) \quad \forall f \in C_b(E), \text{ a.s. } [P]$$

This now completes the proof. ■

Remark 2.3. We will use the pathwise formula for stochastic integral (see Karandikar [4]) while defining $q_t^{\alpha,\beta}(\bar{\omega},\omega)$ in (2.21). The same will be the case in the sequel wherever stochastic integrals are used. This will allow us, for example, to deduce (2.25).

Remark 2.4. In Kallianpur and Karandikar [3], (See also [1]), when the noise is an independent Brownian motion the nonlinear filter is expressed as a Wiener functional evaluated at the observation path. The same interpretation can be given to representation (2.20). We will briefly describe it and use it in the next section.

Consider the model (2.16). Let $P_0^{\alpha,\beta}$ be the probability measure defined by

$$\frac{dP_0^{\alpha,\beta}}{dP} = \exp\left\{-\int_0^T \widetilde{\psi}_u^{\alpha,\beta} dW_u - \frac{1}{2}\int_0^T \left(\widetilde{\psi}_u^{\alpha,\beta}\right)^2 du\right\}.$$

Since X and W are independent, $P_0^{\alpha,\beta}$ is indeed a probability measure on $(\Omega,\mathcal{F})$. Moreover, under $P_0^{\alpha,\beta}$, $\widetilde{Y}^{\alpha,\beta}$ is a Brownian motion independent of X and $P_0^{\alpha,\beta} \circ X^{-1} = P \circ X^{-1}$.

Let $\Omega^0 = C([0,T], I\!R)$, $\mathcal{F}^0$ be the Borel σ-field on Ω^0 and Q be the Wiener measure on $(\Omega^0,\mathcal{F}^0)$. Let W^0 be the co-ordinate process on Ω^0. Consider the product space $(\bar{\Omega},\bar{\mathcal{F}},\bar{P}) \otimes (\Omega^0,\mathcal{F}^0,Q)$. Note that

$$P_0^{\alpha,\beta} \circ \left(X, \widetilde{Y}^{\alpha,\beta}\right)^{-1} = \left(\bar{P} \otimes Q\right) \circ \left(\bar{X}, W^0\right)^{-1}.$$

Define

$$p_t^{\alpha,\beta}(\bar{\omega},\omega^0) = \exp\left\{\int_0^t \bar{\psi}_u^{\alpha,\beta}(\bar{\omega}) dW^0(\omega^0) - \frac{1}{2}\int_0^t \left(\bar{\psi}_u^{\alpha,\beta}(\bar{\omega})\right)^2 du\right\}, \tag{2.22}$$

$$F_t^{\alpha,\beta}(\omega^0)(f) = \int f\left(\bar{X}_t(\bar{\omega})\right) p_t^{\alpha,\beta}(\bar{\omega},\omega^0) d\bar{P}(\bar{\omega}), \quad \forall f \in C_b(E) \tag{2.23}$$

and

$$H_t^{\alpha,\beta}(\omega^0)(f) = \frac{F_t^{\alpha,\beta}(\omega^0)(f)}{F_t^{\alpha,\beta}(\omega^0)(\mathbf{1})}. \tag{2.24}$$

Then (see Remark 2.3)

$$q_t^{\alpha,\beta}(\bar{\omega},\omega) = p_t^{\alpha,\beta}\left(\bar{\omega}, \widetilde{Y}^{\alpha,\beta}(\omega)\right) \quad \text{a.s.}[P]. \tag{2.25}$$

and

$$\pi_t^{\alpha,\beta}(\omega) = H_t^{\alpha,\beta}(\widetilde{Y}^{\alpha,\beta}(\omega)) \quad \text{a.s.}[P]. \tag{2.26}$$

This expresses the filter for the OU noise model as a Wiener functional. For later use, let us define the so-called unnormalised filter σ_t by

$$\sigma_t^{\alpha,\beta}(\omega) = F_t^{\alpha,\beta}(\widetilde{Y}^{\alpha,\beta}(\omega)). \tag{2.27}$$

3 Approximation of the classical filter by filter with OU noise

The classical nonlinear filtering model with signal X and independent noise W is given by

$$Y_t = \int_0^t h(X_s)ds + W_t \quad 0 \le t \le T. \tag{3.1}$$

We first note that the model (2.4) approximates (3.1) as $\alpha \to \infty$, $\beta \to \infty$. We had already noted (Remark 2.1) that $N^\beta \to W$ in $L^2(P)$ as processes as $\beta \to \infty$. The rest follows from the following Lemma.

Lemma 3.1.

$$\int_0^T |\xi_s^\alpha - h(X_s)|^2 ds \to 0 \quad a.s.\ [P]$$

as $\alpha \to \infty$.

Proof. Recall that the signal process X is r.c.l.l. a.s.[P]. Let $N \subset \Omega$ be such that for $\omega \notin N$, $X.(\omega)$ is r.c.l.l. and such that $P(N) = 0$. Fix $\omega \notin N$. Since h is continuous, $h(X_t(\omega))$ is r.c.l.l. Hence $\sup_{0 \le t \le T} |h(X_t(\omega))| < \infty$.

Further, it is clear from (2.1) that $\lim_{\alpha \to \infty} \xi_t^\alpha(\omega) = h(X_t(\omega))$ for all continuity points t of $h(X.(\omega))$. Since the discontinuity points of this function are at most countable in number, a simple application of the bounded convergence theorem proves the lemma. ■

Our aim in this section is to show that the nonlinear filter $\pi^{\alpha,\beta}$ for the approximating model (2.4) converges to the filter π corresponding to the model (3.1) as $\alpha, \beta \to \infty$. Here the filter π is defined by

$$\pi_t(f) = E_P\left[f(X_t)|Y_u : u \le t\right] \quad \forall f \in C_b(E). \tag{3.2}$$

Let $\bar{X}$ defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ and W^0 defined on $(\Omega^0, \mathcal{F}^0, Q)$ be as in the previous section. We define processes p_t, F_t and H_t as follows.

$$p_t(\bar{\omega}, \omega^0) = \exp\left\{\int_0^t h(\bar{X}_s(\bar{\omega}))dW^0(\omega^0) - \frac{1}{2}\int_0^t (h(\bar{X}_s(\bar{\omega}))^2 ds)\right\}, \tag{3.3}$$

$$F_t(\omega^0)(f) = \int f(\bar{X}_t(\bar{\omega}))p_t(\bar{\omega}, \omega^0)d\bar{P}(\bar{\omega}), \quad \forall f \in C_b(E) \tag{3.4}$$

and

$$H_t(\omega^0)(f) = \frac{F_t(\omega^0)(f)}{F_t(\omega^0)(\mathbf{1})}. \tag{3.5}$$

(See Remark 2.3). Then we have (See Kallianpur and Karandikar [3])

$$\pi_t(f)(\omega) = H_t(Y(\omega))(f) \quad \text{a.s. } P \tag{3.6}$$

Also, let

$$\sigma_t(f)(\omega) = F_t(Y(\omega))(f) \tag{3.7}$$

Let $\mathcal{M}_+(E)$ denote the set of positive finite measures on E. Let d denote the Prohorov metric on $\mathcal{M}_+(E)$. We will also denote by $\mathcal{P}(E)$, the set of probability measures on E. We are now ready to prove our first result on robustness. As usual, $\Rightarrow$ denotes weak convergence.

Theorem 3.2. *Let the signal process X be r.c.l.l. and continuous in probability. Let $\alpha \to \infty$, $\beta \to \infty$ such that $\frac{\alpha}{\beta} \leq M$ for some $M < \infty$. Then*

(i) $F^{\alpha,\beta} \to F$ *in Q- probability as $C\left([0,T], \mathcal{M}_+(E)\right)$- valued random variables.*

(ii) $H^{\alpha,\beta} \to H$ *in Q- probability as $C\left([0,T], \mathcal{P}(E)\right)$- valued random variables.*

(iii) $\pi^{\alpha,\beta} \Rightarrow \pi$.

(iv) $\sigma^{\alpha,\beta} \Rightarrow \sigma$.

Proof : Let $\alpha_n, \beta_n \to \infty$ with $\alpha_n \leq M\beta_n \ \ \forall n$. We will denote F^{α_n,β_n} by F^n and p^{α_n,β_n} by p^n. Similarly, denote $\bar{\xi}^{\alpha_n}$ by $\bar{\xi}^n$ and $\bar{\psi}^{\alpha_n,\beta_n}$ by $\bar{\psi}^n$. Since $\bar{P}\circ\bar{X}^{-1} = P\circ X^{-1}$, we get

$$\bar{P}\circ\left(\bar{\xi}^n\right)^{-1} = P\circ\left(\xi^{\alpha_n}\right)^{-1}; \bar{P}\circ\left(\bar{\psi}^n\right)^{-1} = P\circ\left(\widetilde{\psi}^{\alpha_n,\beta_n}\right)^{-1}.$$

Thus using Lemma 3.1, we get

$$\lim_{n\to\infty}\int_0^T \left|\bar{\xi}^n_s - h(\bar{X}_s)\right|^2 ds = 0, \qquad \text{a.s } [\bar{P}].$$

Also,

$$\begin{aligned}\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\int_0^T & \left[h(\bar{X}_s) - h(\bar{X}_{(s-\frac{1}{\alpha})\vee 0})\right]^2 ds \\ & \leq \ M\lim_{n\to\infty}\int_0^T\left[h(\bar{X}_s) - h(\bar{X}_{(s-\frac{1}{\alpha})\vee 0})\right]^2 ds \\ & = \ 0 \quad \text{a.s. } [\bar{P}].\end{aligned}$$

The last equality follows since for almost all s, $h(\bar{X}_s) - h(\bar{X}_{(s-\frac{1}{\alpha})V0}) \to 0$ a.s.$[\bar{P}]$, and for a fixed $\bar{\omega}$ for which $h(\bar{X})$ is r.c.l.l., the integrand is bounded. Now, we get from (2.19) that

$$\lim_{n\to\infty}\int_0^T\left[\bar{\psi}^n_s - h(\bar{X}_s)\right]^2 ds = 0, \quad \text{a.s.}[\bar{P}] \tag{3.8}$$

Let $t_n \to t$. (3.8) now implies that $p^n_{t_n} \to p_t$ in $\bar{P}\otimes Q$ probability where p_t is defined by (3.3). Further note that

$$\int p^n_{t_n} d\bar{P}\otimes Q = \int p_t d\bar{P}\otimes Q = 1.$$

Hence by Scheffe's Lemma we get

$$p^n_{t_n} \to p_t \text{ in } L^1(\bar{P}\otimes Q).$$

Since $\bar{X}$ is continuous in probability we get $f(\bar{X}_{t_n}) \to f(\bar{X}_t)$ in probability for all $f \in C_b(E)$. Hence

$$f(\bar{X}_{t_n})p^n_{t_n} \to f(\bar{X}_t)p_t \text{ in } L^1(\bar{P}\otimes Q)$$

This implies

$$\int f(\bar{X}_{t_n})p^n_{t_n} d\bar{P} \to \int f(\bar{X}_t)p_t d\bar{P} \text{ in } L^1(Q).$$

In particular, for $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} Q\left(|F^n_{t_n}(f) - F_t(f)| > \varepsilon\right) = 0, \quad \forall f \in C_b(E) \\
&\Rightarrow \lim_{n\to\infty} Q\left(d(F^n_{t_n}, F_t) > \varepsilon\right) = 0. \\
&\Rightarrow \lim_{n\to\infty} Q\left(\sup_{t\in[0,T]} d(F^n_t, F_t) > \varepsilon\right) = 0.
\end{aligned}$$

This completes the proof of part (i).

Part (ii) follows immediately from Part (i) and the fact that $F^n_t(1)$ and $F_t(1)$ are exponential martingales and hence satisfy

$$\inf_t F^n_t(1) > 0 \quad \text{a.s. } [Q], \tag{3.9}$$

$$\inf_t F_t(1) > 0 \quad \text{a.s. } [Q] \tag{3.10}$$

For (iii), note that as in (2.26), the filter π can be written as

$$\pi_t(\omega) = H_t(Y(\omega)) \quad \text{a.s. } [P] \tag{3.11}$$

Let $G \in C_b(C[0,T], \mathcal{P}(E))$. Then

$$\begin{aligned}
E_P\left[G(\pi^n)\right] &= E_P\left[G\left(H^n(\widetilde{Y}^n)\right)\right] \\
&= E_{\bar{P}\otimes Q}\left[G(H^n)p^n_T\right] \\
&\to E_{\bar{P}\otimes Q}\left[G(H)p_T\right] \\
&= E_P\left[G(H)(Y)\right] \\
&= E_P\left[G(\pi)\right].
\end{aligned}$$

Part (iv) follows similarly using (i). ■

We will now show that the convergence of $\pi^{\alpha,\beta}$ to π is in a much stronger sense- in probabiity. For this, we need the following technical Lemma which also appears in Bhatt and Karandikar [2] in a similar context. It is included here for the sake of completeness.

Lemma 3.3. *Let U be a random variable and $\{U_n\}$ be a sequence of random variables on a probability space $(\Omega^*, \mathcal{F}^*, P^*)$ such that*

(a) $P^* \circ (U_n)^{-1} \to P^* \circ U^{-1}$

(b) $\liminf U_n \geq U$ *a.s P^**

Then $U_n \to U$ in P^*-probability.

Proof. Let $V_n = \tan^{-1}(U_n)$ and $V = \tan^{-1}(U)$. Then V_n, V are bounded, $P^* \circ (V_n)^{-1} \Rightarrow P^* \circ (V)^{-1}$ and

$$\liminf V_n \geq V \quad \text{a.s. } [P^*] \tag{3.12}$$

Since $\{V_n\}$ are bounded, we get $E(V_n) \to E(V)$

On the other hand, using boundedness of $\{U_n\}$, we get using Fatou's lemma

$$E(\liminf_{n\to\infty} V_n) \le \liminf_{n\to\infty} E(V_n) = E(V). \tag{3.13}$$

Now (3.12) and (3.13) imply

$$\liminf_{n\to\infty} V_n = V \quad \text{a.s. } [P^*]. \tag{3.14}$$

Let $\widetilde{V}_m = \inf_{n\ge m} V_n$. Then $\widetilde{V}_m \to \liminf V_n = V$ a.s.

We thus have $\widetilde{V}_n \le V_n$, $\widetilde{V}_n \to V$ a.s. and $V_n \Rightarrow V$. Since $\{\widetilde{V}_n\}$ and $\{V_n\}$ are converging in law, the sequence $\left\{\left(\widetilde{V}_n, V_n\right)\right\}$ is tight as $I\!R^2$-valued random variables. If $\left(\widetilde{V}_{n_k}, V_{n_k}\right)$ is a convergent subsequence, with $\left(\widetilde{V}_0, V_0\right)$ as a weak limit, then $\widetilde{V}_{n_k} \le V_{n_k}$ implies that $\widetilde{V}_0 \le V_0$. On the other hand, $\widetilde{V}_0, V_0$ both have the same law as V. Hence, $\widetilde{V}_0 = V_0$. We then conclude,

$$\left(\widetilde{V}_n, V_n\right) \to (V, V).$$

Thus $P(|\widetilde{V}_n - V_n| \ge \varepsilon) \to P(|V - V| \ge \varepsilon) = 0$ for any $\varepsilon > 0$. Since $\widetilde{V}_n \to V$ a.s., it follows that $V_n \to V$ in probability. ■

Theorem 3.4. *Let the signal process X be r.c.l.l. and continuous in probability. Let $\alpha \to \infty$, $\beta \to \infty$ such that $\frac{\alpha}{\beta} \le M$ for some $M < \infty$. Then $\sigma^{\alpha,\beta} \to \sigma$ as $C([0,T], \mathcal{M}_+(E))$ valued processes and $\pi^{\alpha,\beta} \to \pi$ as $C([0,T], \mathcal{P}(E))$ valued processes in P- probability.*

Proof. As in the previous theorem, it suffices to prove that if $\alpha_n \to \infty$, $\beta_n \to \infty$, $\frac{\alpha_n}{\beta_n} \le M$, $t_n \to t$ and $f \in C_b(E)$, then

$$\pi_{t_n}^{\alpha_n,\beta_n}(f) \to \pi_t(f) \text{ in } P-\text{probability} \tag{3.15}$$

This in turn would follow if we show

$$\sigma_{t_n}^{\alpha_n\beta_n}(f) \to \sigma_t(f) \text{ in } P-\text{probability.} \tag{3.16}$$

Here, (3.10) implies

$$\inf_t \sigma_t(1) > 0 \quad \text{a.s. } [P]$$

and then (3.15) would follow from (3.16).

Fix α_n, β_n, t_n, f as above. Let us write $q_s^n = q_s^{\alpha_n,\beta_n}$, $\bar{\psi}_s^n = \bar{\psi}_s^{\alpha_n,\beta_n}$, $\widetilde{\psi}_s^n = \widetilde{\psi}_s^{\alpha_n,\beta_n}$, $\widetilde{Y}_s^n = \widetilde{Y}_s^{\alpha_n,\beta_n}$, $\theta^n = \sigma_{t_n}^{\alpha_n,\beta_n}(f)$, $\theta = \sigma_t(f)$.

Now,

$$\begin{aligned} q_s^n &= \exp\left\{\int_0^s \bar{\psi}_u^n d\widetilde{Y}_u^n - \frac{1}{2}\int_0^s (\bar{\psi}_u^n)^2\, du\right\} \\ &= \exp\left\{\int_0^s \bar{\psi}_u^n dW_u + \int_0^s \bar{\psi}_u^n \widetilde{\psi}_u^n\, du - \frac{1}{2}\int_0^s (\bar{\psi}_u^n)^2\, du\right\} \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} q_s &= \exp\left\{\int_0^s h(\bar{X}_u) dY_u - \frac{1}{2}\int_0^s (h(\bar{X}_u))^2\, du\right\} \\ &= \exp\left\{\int_0^s h(\bar{X}_u) dW_u + \int_0^s h(\bar{X}_u)h(X_u)\, du - \frac{1}{2}\int_0^s (h(\bar{X}_u))^2\, du\right\} \end{aligned} \tag{3.18}$$

As seen earlier

$$\int_0^T \left(\bar{\psi}_u^n - h(\bar{X}_u)\right)^2 du \to 0 \quad \text{in } \bar{P} - \text{probability.}$$

Similarly

$$\int_0^T \left(\widetilde{\psi}_u^n - h(X_u)\right)^2 du \to 0 \quad \text{in } \; P - \text{probability.}$$

As a consequence

$$q_{t_n}^n \to q_t \quad \text{in } \bar{P} \otimes P - \text{probability.}$$

Now

$$\theta^n = \int f(\bar{X}_{t_n}) q_{t_n}^n d\bar{P}$$

and

$$\theta = \int f(\bar{X}_t) q_t d\bar{P}.$$

Here $f(\bar{X}_{t_n}) \to f(\bar{X}_t)$ in $\bar{P}$- probability. It was shown in Theorem 3.2 that $\theta^n \to \theta$ in law. Let us write $g_n(\bar{\omega}, \omega) = f(\bar{X}_{t_n}(\bar{\omega})) q_{t_n}^n(\bar{\omega}, \omega))$ and $g(\bar{\omega}, \omega) = f(X_t(\bar{\omega})) q_t(\bar{\omega}, \omega)$. Then we have

$$g_n \to g \text{ in } \bar{P} \otimes P \text{ probability} \tag{3.19}$$

$$\theta_n(\omega) = \int g_n(\bar{\omega}, \omega) d\bar{P}(\bar{\omega}) \tag{3.20}$$

and

$$\theta(\omega) = \int g(\bar{\omega}, \omega) d\bar{P}(\bar{\omega}) \tag{3.21}$$

$$P \circ (\theta^n)^{-1} \Rightarrow P \circ (\theta)^{-1} \tag{3.22}$$

To prove $\theta^n \to \theta$ in probability, suffices to show that given a subsequence $\{n_k\}$, there exists a further subsequence $\{n_{k_j}\}$ such that $\theta^{n_{k_j}} \to \theta$ in probability. Thus, given $\{n_k\}$, choose $\{n_{k_j}\}$ such that $\bar{g}_j = g_{n_{k_j}}$ converges to g a.s. P. Let $\bar{\theta}_j = \theta_{n_{k_j}}$

Applying Fatou's lemma, it follows that

$$\begin{aligned} \theta = \int g d\bar{P} &= \int \liminf_{j\to\infty} \bar{g}_j d\bar{P} \\ &\leq \liminf_{j\to\infty} \int \bar{g}_j dP = \liminf_{j\to\infty} \bar{\theta}_j \;\; a.s. \, P \end{aligned}$$

So,

$$\liminf_{j\to\infty} \bar{\theta}_j \geq \theta \;\; a.s. \, P$$

and of course

$$\bar{\theta}_j \Rightarrow \theta.$$

Using Lemma 3.3 it follows that $\bar{\theta}_j \to \theta$ in P-probability.

The subsequence argument given above implies $\theta_n \to \theta$ in P-probability. This completes the proof. ∎

Remark 3.1. From the proof given above, it can be seen that the distribution of n_0^β does not play any role. Thus even if the OU noise n_t^β is not stationary (with some initial distribution, possibly degenerate), the results given above on the expression for the filter as well as the approximation results continue to be true verbatim.

Remark 3.2. Here we have only considered the one dimensional case for notational convienience. The results carry over to the multi-dimensional case without any difficulty whatsoever.

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