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Geometry of the Poisson Boolean model on a region of logarthmic width in the plane

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Geometry of the Poisson Boolean model on a region of logarthmic width in the plane

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Abstract

Consider the region $L := \{(x, y) : 0 \leq y \leq C \log(1 + x), x > 0\}$ for a constant $C > 0$. We study the percolation and coverage properties of this region.

For the percolation properties we place a Poisson point process of intensity λ on the region L. At each point of the process we centre a box of a random side length ρ. In case $\rho \leq R$ for some fixed $R > 0$ we study the critical intensity λ_c of percolation.

For the coverage properties we place a Poisson point process of intensity λ on the entire half space $\mathbb{R}_+ \times \mathbb{R}$ and associated with each Poisson point we place a box of a random side length ρ. Depending on the tail behaviour of the random variable ρ we exhibit a phase transition in the intensity for the eventual coverage of the region L.

Keywords: Boolean model, Poisson point process, percolation, coverage.
AMS Classification: 60K35.

1 Introduction

Let (X, λ, ρ) be a Poisson Boolean model on $\mathbb{H} := \mathbb{R}_+ \times \mathbb{R}$, i.e. $X := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots\}$ is a homogenous Poisson point process on $\mathbb{H}$ with intensity λ, and, at each point $\boldsymbol{x}_i$ we situate the box $\boldsymbol{x}_i + [0, \rho_i]^2$, where $\{\rho_i : i \geq 1\}$ is a collection of i.i.d. random variables, each ρ_i having the same distribution as the non-negative random variable ρ and is independent of the underlying Poisson process. The covered (or occupied) region of this Boolean model is defined as $C := \cup_{i \geq 1}(\boldsymbol{x}_i + [0, \rho_i]^2)$; while the vacant region is $V := \mathbb{H} \setminus C$. In general the shapes situated at points of the Poisson process are usually balls of random radius (see e.g. Stoyan [10], Hall [6]), Meester and Roy [7]); however for the convenience of writing we consider boxes instead of balls. It may be seen easily that all our results carry through for the standard case.

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For an unbounded connected region $L \subseteq \mathbb{H}$ we say that

- there is *occupied percolation in* L if $C \cap L$ admits an unbounded connected component with positive probability,
- there is *vacant percolation in* L if $V \cap L$ admits an unbounded connected component with positive probability, and,
- there is *eventual coverage of* L if, with positive probability, there exists $t > 0$ such that $L \cap \{(x, y) : x > t\} \subseteq C$.

It is easily seen that there is no equivalence of eventual coverage on the vacant region unless L has finite Lebesgue measure and the Boolean model (X, λ, ρ) is inhomogenous with side lengths of the squares decreasing rapidly with the distance of the Poisson points from the origin.

For a non-decreasing function $f : [0, \infty) \to [0, \infty)$ let

$$L_f := \{(x, y) : 0 \leq y \leq f(x)\}$$

and, the critical parameters are defined as

$$\lambda_c(L_f) := \inf\{\lambda : \text{occupied percolation occurs in } L_f\},$$

$$\lambda_c^\star(L_f) := \sup\{\lambda : \text{vacant percolation occurs in } L_f\},$$

$$\lambda_e(L_f) := \inf\{\lambda : \text{there is eventual coverage of } L_f\}.$$

Although, we have not specified it explicitly, in all the above three definitions there is an implicit dependence on the random variable ρ and the underlying space $\mathbb{H}$ on which the Poisson point process is defined.

In case of Bernoulli bond percolation on a region L of the 2-dimensional square latttice, the critical probability $p_c(L)$ of percolation is well studied. Grimmett [4] has shown that the critical probability $p_c(L)$ equals the critical probability 1/2 of percolation on the entire square lattice whenever the function f grows faster than $\log x$, and, $p_c(L)$ equals 1 whenever the function f grows slower than $\log x$. While, if f is such that $f(x) \sim a \log x$ for some constant $a > 0$, then $p_c(L_f)$ is obtained as the unique solution p of the equation $\xi(1-p) = a$, where $\xi(\cdot)$ is the correlation length of the 2-dimensional Bernoulli bond percolation process.

Our first result (Theorem 1.1) is similar to the above result of Grimmett. The method of proof is also similar, although the vacancy and the occupancy structures not being in a duality relation as in the case of Bernoulli bond percolation, we need to do some extra work.

Our second result (Theorem 1.2) studies $\lambda_e(L_f)$ for $f(x) = a\log(1 + x)$ for $a > 0$ and for ρ having a heavy tailed distribution as given by (1). It is to be noted that for any ρ having a tail either thicker or thinner than that given by (1) the critical intensity $\lambda_e(L_f)$ is trivial, i.e. $\lambda_e(L_f) = 0$ or ∞ respectively. The analogy of eventual coverage in L_f with that of the percolation result is also exhibited in Theorem 1.2, where we study the coverage of a 1-dimensional line $l_g := \{(x,y) : y = g(x)\}$ and show that $\lambda_e(L_f) = \lambda_e(l_g)$ for any non-decreasing $g : [0,\infty) \to [0,\infty)$ and f as given in the beginning of this paragraph.

To complement the above two theorems, we study the behaviour at criticality. It is shown that (i) the occurrence or otherwise of percolation in L_f at criticality for $f(x) = O(\log x)$ depends on the higher order terms of f, and (ii) the occurrence or otherwise of eventual coverage of L_f depends on the higher order terms of the tail distribution of ρ. These are discussed at the end of the proofs of the theorems in the appropriate sections.

Theorem 1.1 *Let ρ be such that $0 < \rho \leq R$ for some fixed $R > 0$. For $f : [0,\infty) \to [0,\infty)$ non-decreasing we have*

(i) if f is such that $f(x) = o(\log x)$ as $x \to \infty$ then $\lambda_c(L_f) = \infty$ and $\lambda_c^\star(L_f) = 0$,

(ii) if f is such that $\log x = o(f(x))$ as $x \to \infty$ then $\lambda_c(L_f) = \lambda_c^\star(L_f) = \lambda_c(\mathbb{R}^2) = \lambda_c^\star(\mathbb{R}^2)$, and

(iii) if f is such that $f(x)/\log x \to a$ as $x \to \infty$, for some $a > 0$ then $\lambda_c(L_f)$ is the unique $\lambda \in (\lambda_c(\mathbb{R}^2), \infty)$ satisfying $\xi^\star(\lambda) = a$, and $\lambda_c^\star(L_f)$ is the unique $\lambda \in (0, \lambda_c(\mathbb{R}^2))$ satisfying $\xi(\lambda) = a$, where the vacant and occupied correlation lengths $\xi^\star(\lambda)$ and $\xi(\lambda)$ are as defined in Propositions 2.1 and 2.2 respectively.

Remark: In (ii) above $\lambda_c(\mathbb{R}^2)$ and $\lambda_c^\star(\mathbb{R}^2)$ are the critical intensities for occupied and vacant percolation when the Boolean model is defined by a Poisson point process on the entire plane $\mathbb{R}^2$. Although the equality of $\lambda_c(\mathbb{R}^2)$ and $\lambda_c^\star(\mathbb{R}^2)$ is known only when the shapes are discs of bounded radius (see Meester and Roy [7]), the result can be easily extended to shapes which are squares of bounded side length.

Theorem 1.2 *Let $f(x) = a\log(1+x)$ for some $a > 0$. We have for any $g : [0,\infty) \to [0,\infty)$ nondecreasing*

(i) if ρ is such that, for all x large

$$P(\rho \geq x) = \frac{K_\rho + \eta(x)}{x^2} \tag{1}$$

for some $K_\rho > 0$ and $\eta(x) \to 0$ as $x \to \infty$, then

$$\lambda_e(L_f) = \lambda_e(l_g) = \frac{1}{2K_\rho}; \tag{2}$$

(ii) if ρ is such that,

$$x^2 P(\rho \geq x) \to 0 \text{ as } x \to \infty, \tag{3}$$

then

$$\lambda_e(L_f) = \lambda_e(l_g) = \infty; \tag{4}$$

(iii) if ρ is such that,

$$x^2 P(\rho \geq x) \to \infty \text{ as } x \to \infty, \tag{5}$$

then

$$\lambda_e(L_f) = \lambda_e(l_g) = 0. \tag{6}$$

The result on percolation extends the work of Tanemura [11]. Tanemura uses a Grimmett and Marstrand method to study continuum percolation on slabs, half spaces and other regions of space. In particular Tanemura [11] shows that $\lambda_c(\mathbb{R}^d) = \lambda_c(\mathbb{R}^d_+)$ and $\lambda^\star_c(\mathbb{R}^d) = \lambda^\star_c(\mathbb{R}^d_+)$. We extend these results to further subsets of $\mathbb{R}^2$. Towards this we need to develop correlation lengths for both occupied and vacant connectivity functions. Tanemura [12] uses lace expansion techniques to study such connectivity functions in high dimensions.

Eventual coverage has been studied for quadrants and octants by Athreya, Roy and Sarkar [1]. It is the natural analogue of complete coverage of space in Boolean models. Hall [6] shows that for the Boolean model defined on $\mathbb{R}^d$ complete coverage occurs, i.e. $C = \mathbb{R}^d$ almost surely, if and only if $E\rho^d = \infty$. Molchanov and Scherbakov [8] study the question of complete coverage for an inhomogenous Poisson Boolean model. Here we study eventual coverage of the region under the log function.

The rest of the article is organized as follows. In section 2, for each of the cases of infinite occupied and infinite vacant components, we first derive in Propositions 2.1 and 2.2 the properties of the connectivity functions or correlation lengths by vacant or occupied paths respectively. These are then used to prove Theorem 1.1 for infinite occupied and infinite vacant components respectively. In section 3, the Boolean model is first compared to two discrete models; then eventual coverage or otherwise of the Boolean model follows from the same for the discrete models. Finally the discrete model is studied.

2 Proof of Theorem 1.1

2.1 The case of infinite occupied component

We begin by noting that for a Poisson point process X of intensity λ and an independent collection of i.i.d. random variables $\{\rho_i : i \geq 1\}$ the process

$$Y := \{\boldsymbol{x}_i + \frac{1}{2}(\rho_i, \cdots, \rho_i) : \boldsymbol{x}_i \in X\}$$

is again a Poisson point process of intensity λ. Hence it suffices to prove Theorem 1.1 for the Boolean model obtained by centering the Poisson point in its associated box, i.e. $\boldsymbol{x}_i + [-\rho_i/2, \rho_i/2]^d$. We also assume $R = 1$, i.e. $\rho \leq 1$. First we derive properties of the connectivity function.

The box of side length $2n$ centered at the origin will be denoted by B_n. We are interested in the probability of the event that there is a vacant path from the origin to ∂B_n, an event that we write as $\mathbf{0} \overset{\star}{\longleftrightarrow} \partial B_n$. Consider a box of side length one centered at the origin denoted by $D(\mathbf{0})$ and consider the event of a vacant path from this box to ∂B_n, an event we write as $D(\mathbf{0}) \overset{\star}{\longleftrightarrow} \partial B_n$, whose probability we denote by $\beta^\star(n)$. We first prove

Proposition 2.1 *There are positive constants ρ and σ so that*

$$\rho n^{1-d} e^{-n\phi^\star(\lambda)} \leq P_\lambda(D(\mathbf{0}) \overset{\star}{\longleftrightarrow} \partial B_n) \leq \sigma n^{d-1} e^{-n\phi^\star(\lambda)}. \tag{7}$$

The limit of $-\frac{1}{m}\log\beta^\star(m) = \phi^\star(\lambda)$ is a continuous function of λ. Moreover $\phi^\star(\lambda) = 0$ for $\lambda \leq \lambda_c$, $\phi^\star(\lambda)$ is increasing on (λ_c, ∞) and $\phi^\star(\lambda) \uparrow \infty$ as $\lambda \uparrow \infty$. As customary, the vacant correlation length is defined by $\xi^\star(\lambda) = 1/\phi^\star(\lambda)$.

Remark: If there is a vacant path from the origin to ∂B_n that implies $D(\mathbf{0}) \overset{\star}{\longleftrightarrow} \partial B_n$. On the other hand if there is a box of side length three around the origin which is devoid of Poisson points from X, and a vacant path from $D(\mathbf{0})$ to ∂B_n then that implies $\mathbf{0} \overset{\star}{\longleftrightarrow} \partial B_n$. Using the FKG inequality it now follows easily that $P_\lambda(\mathbf{0} \overset{\star}{\longleftrightarrow} \partial B_n)$ is bounded as in (7) above but with different positive constants σ and ρ.

Proof: We put boxes of side lengths one centered at the integer points of ∂B_{m+2}. We notice that if there is a vacant path from $D(\mathbf{0})$ to ∂B_{m+n+2} then there is a vacant path from $D(\mathbf{0})$ to ∂B_m and a vacant path from $D(\boldsymbol{x})$, where $D(\boldsymbol{x})$ is a box of side length one around some $\boldsymbol{x} \in \partial B_{m+2}$ with integer coordinates, to the boundary of a box of side $2n$ around this box $D(\boldsymbol{x})$. In addition these two events happen disjointly as the maximum of ρ is assumed to be one. By the extension of the BK inequality for the Poisson Boolean model proved in Gupta and Rao [5] we then have considering all such $D(\boldsymbol{x})$ with $\boldsymbol{x}$ having integer coordinates situated on ∂B_{m+2},

$$\begin{aligned}
\beta^\star(m+n+2) &\leq \beta^\star(m). \sum_{\boldsymbol{x} \in \partial B_{m+2}} \beta^\star(n) \\
\beta^\star(m+n+2) &\leq \beta^\star(m).2d((2(m+2)+1)^{d-1}.\beta^\star(n) \\
\text{i.e. } \log\beta^\star(m+n+2) &\leq \log\beta^\star(m) + \log\beta^\star(n) \\
&\quad +(d-1)\log(2(m+2)+1) + \log 2d,
\end{aligned}$$

which we write for convenience as

$$\log\beta^\star(m+n+2) \leq \log\beta^\star(m) + \log\beta^\star(n) + g_1(m) + c_1, \tag{8}$$

where $g_1(m) \sim \log m$ and c_1 is a constant.

Conversely we consider events which imply a vacant path from $D(\mathbf{0})$ to ∂B_{m+n}. Writing $D(\boldsymbol{x}) = [x_1 - 1/2, x_1 + 1/2] \times [x_2 - 1/2, x_2 + 1/2] \times \cdots \times [x_d - 1/2, x_d + 1/2]$, consider the collection $\mathcal{D}_m = \{D(\boldsymbol{x}) : \boldsymbol{x}$ is such that $\boldsymbol{x}$ has integer coordinates and $x_i = \pm m$ for some $i\}$. Let $U_{\boldsymbol{x}} = \{D(\mathbf{0}) \overset{\star}{\longleftrightarrow} D(\boldsymbol{x})\}$ for some $\boldsymbol{x}$ with $D(\boldsymbol{x}) \in \mathcal{D}_m$. Without loss of generality let $\boldsymbol{x}$ be such that $x_1 = m$. Consider also the event $V_{\boldsymbol{x}}$ of a vacant path from $D(\boldsymbol{x})$ to $\partial B_{m+n} \cap \{y_1 = m+n\}$. If in addition we consider a box of side length three around $\boldsymbol{x}$ and require it to be empty then these three decreasing events imply a vacant path from $D(\mathbf{0})$ to ∂B_{m+n}. By the FKG inequality then

$$\beta^\star(m+n) \geq P_\lambda(U_{\boldsymbol{x}}) e^{-\lambda 3^d} P_\lambda(V_{\boldsymbol{x}}). \tag{9}$$

Now using translation invariance for $V_{\boldsymbol{x}}$, a vacant path from $D(\mathbf{0})$ to ∂B_n implies the union of vacant paths in fixed directions, and these probabilities in fixed directions are same by symmetry. Let us define $\gamma^\star(n) = P_\lambda(D(\mathbf{0}) \overset{\star}{\longleftrightarrow} \partial B_n$ in a given direction). So $P_\lambda(V_{\boldsymbol{x}}) = \gamma^\star(n) \leq \beta^\star(n) \leq 2d.\gamma^\star(n)$. On the other hand

$$\beta^\star(m) \leq P_\lambda(\cup_{\boldsymbol{x} \in \partial B_m} U_{\boldsymbol{x}}) \leq \sum_{\boldsymbol{x} \in \partial B_m} P_\lambda(U_{\boldsymbol{x}}),$$

where the union and the sum are over the integer points on ∂B_m. Considering the maximum over integer points $\boldsymbol{x}$ in fixed directions we see that there is an $\boldsymbol{x}$ so that $P_\lambda(U_{\boldsymbol{x}}) \geq \beta^\star(m)/|\partial B_m|$, where $|\partial B_m|$ counts integer points on ∂B_m. Using this and $P_\lambda(V_{\boldsymbol{x}}) = \gamma^\star(n) \geq \beta^\star(n)/2d$, (9) gives

$$\beta^\star(m+n) \geq \frac{e^{-\lambda 3^d} \beta^\star(m) \beta^\star(n)}{2d|\partial B_m|},$$

which implies

$$-\log \beta^\star(m+n) \leq -\log \beta^\star(m) - \log \beta^\star(n) + 3^d \lambda + g_2(m) + c_2, \tag{10}$$

where $g_2(m) \sim \log m$ and c_2 is a constant. From appendix II of Grimmett the subadditive inequalities (8) and (10) imply the existence of a limit as $m \to \infty$ for $-(\log \beta^\star(m))/m$, say $\phi^\star(\lambda)$, and after adjusting for the largest of the constants in absolute value as well as the largest of $g_1(m)$ and $g_2(m)$, we also get the inequalities

$$-const. - g(m) \leq m\phi^\star(\lambda) - \log \beta^\star(m) - const.\lambda \leq const. + g(m), \tag{11}$$

for λ lying in a compact set where $g(m) = (d-1)\log(2(m+2)+1) = O(\log m)$. We have established the bounds (7).

We now remember that $\beta^\star(m)$ is also a function of λ, so that if we can argue $\beta^\star(m)$ is a continuous function of λ for fixed m, then the inequalities (7) after making $m \to \infty$ would imply the continuity of $\phi^\star(\lambda)$. As λ increases the probability $P_\lambda(D(\mathbf{0}) \overset{\star}{\longleftrightarrow} \partial B_m)$ decreases.

Thus considering the superposition of two independent Poisson processes with intensities $\lambda - \epsilon$ and 2ϵ respectively, a simple coupling argument gives

$$\begin{aligned} P_{\lambda-\epsilon}(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_m) e^{-2\epsilon . Vol(B_m)} &\leq P_{\lambda+\epsilon}(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_m) \\ &\leq P_{\lambda-\epsilon}(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_m). \end{aligned} \tag{12}$$

Thus

$$\begin{aligned} & P_{\lambda-\epsilon}(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_m) - P_{\lambda-\epsilon}(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_m) e^{-2\epsilon Vol(B_m)} \\ & \geq P_{\lambda-\epsilon}(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_m) - P_{\lambda+\epsilon}(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_m) \\ & \geq 0, \end{aligned} \tag{13}$$

leading to continuity of $P_\lambda(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_m)$. Hence $\phi^\star(\lambda)$ is a continuous function of λ.

Now

$$\begin{aligned} \theta^\star(\lambda) &:= P_\lambda(\text{there is an unbounded vacant} \\ & \qquad \text{component containing the origin}) \\ &= \lim_{n\to\infty} P_\lambda(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_n), \end{aligned}$$

and here $\lambda < \lambda_c^\star$ implies $\theta^\star(\lambda) > 0$. From this and the right side of the inequality (7) we get $\phi^\star(\lambda) = 0$ for $\lambda < \lambda_c^\star$. Continuity gives $\phi^\star(\lambda_c) = 0$. On the other hand if $\lambda > \lambda_c^\star$ then by Lemma 4.1 of Meester and Roy [7] $P_\lambda(d(V) \geq a) \leq C_1 e^{-aC_2}$, for positive constants C_1, C_2 where $d(V)$ denotes the diameter of the vacant component of the origin. This along with the left side of the inequality (7) implies $\phi^\star(\lambda) > 0$ if $\lambda > \lambda_c^\star$.

Finally we want to show that for $\lambda > \lambda_c$, $\phi^\star(\lambda)$ is increasing and goes to infinity as $\lambda \to \infty$. For this we approximate the Poisson process by independent Bernoulli in small squares of volume $1/m^d$ with failure probability $q_m = e^{-(\lambda/m^d)}$. Our event $D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_n$, under such a discretized setting, is decreasing and imitating the proof of Theorem 2.38 in Grimmett [3] we get that for any decreasing event depending on finitely many of these boxes the probability $h(q_m)$ satisfies $h(q_m^\gamma) \leq h(q_m)^\gamma$ for $\gamma > 1$. As $m \to \infty$ the Bernoulli probability converges to the Poisson probability and we get $P_{\gamma\lambda}(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_n) \leq P_\lambda(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_n)^\gamma$, which after taking logarithm gives

$$-\gamma \frac{1}{n} \log P_\lambda(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_n) \leq -\frac{1}{n} \log P_{\gamma\lambda}(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B_n).$$

As we make $n \to \infty$ we get $\phi^\star(\gamma\lambda) \geq \gamma\phi^\star(\lambda) > \phi^\star(\lambda)$ since $\gamma > 1$. This shows that $\phi^\star(\lambda)$ is increasing for λ above $\lambda_c^\star$ and goes to infinity as $\lambda \to \infty$. This completes the proof of the proposition. $\square$

Remember, $\phi^\star(\lambda) = 1/\xi^\star(\lambda)$, and note that the results in Theorem 1.1 (iii) are stated in this notation. Note also that as stated in the remark after the statement of Theorem 1.1, with $d = 2$ we have $\lambda_c = \lambda_c^\star$.

Proof of Theorem 1.1 (iii) for infinite occupied component: Suppose $\lambda > \lambda_c$ and $a < \xi^\star(\lambda)$. Then we want to show L_f almost surely contains no unbounded occupied cluster. Let λ_a be the solution of $\xi^\star(\lambda) = a$. Note that over $(\lambda_c^\star, \infty)$, $\xi^\star(\lambda)$ is decreasing, thus $a < \xi^\star(\lambda)$ implies $\lambda < \lambda_a$. Choosing $\delta > 0$ such that $(1+\delta)a < \xi^\star(\lambda)$, we define $\boldsymbol{w}_k = (k^{1+\delta}, 0)$. Let B_k be the smallest square with $\boldsymbol{w}_k$ in the middle of the lower side with the upper side just above the curve $v = f(u)$. By our assumption $f(u)/\log u \to a$, hence the side length l_k of B_k satisfies

$$l_k = a(1+o(1))\log k^{1+\delta} \text{ as } k \to \infty.$$

Now B_k has side length l_k and center $\boldsymbol{w}_k + (0, l_k/2)$, and let A_k be the event that there is a vacant path from the top edge to the bottom edge of B_k. By the FKG inequality

$$P_\lambda(A_k) \geq \{\frac{1}{4}P_\lambda(D(\mathbf{0}) \overset{\star}{\longleftrightarrow} \partial B_k(\frac{1}{2}l_k))\}^2 e^{-3^2\lambda}.$$

However,

$$P_\lambda(D(\mathbf{0}) \overset{\star}{\longleftrightarrow} \partial B_k(\frac{1}{2}l_k)) \approx e^{-l_k(2\xi^\star(\lambda))^{-1}} \text{ as } k \to \infty,$$

by (7) where the $\approx$ sign means equality in the limit after taking logarithm and dividing by l_k. Hence $P_\lambda(A_k) \geq (1/16)e^{-3^2\lambda}k^{-(1+o(1))(1+\delta)a/\xi^\star(\lambda)}$ as $k \to \infty$, i.e. $\sum P_\lambda(A_k) = \infty$ since $(1+\delta)a < \xi^\star(\lambda)$. On the other hand for large k the squares B_k are separated by more than twice the maximum of the sides of the Boolean squares (assumed $R = 1$ here), hence the configurations inside B_k's are independent for large k. Thus A_k occurs infinitely often almost surely.

Secondly, suppose $\lambda > \lambda_c$ and $a > \xi^\star(\lambda)$. We want to show L_f contains almost surely an infinite occupied cluster. Choose α such that $a > \alpha > \xi^\star(\lambda)$ and D_k be the box with center $(k, 0)$ and side length $2\alpha \log k$. For large values of k, D_k lies strictly beneath the curve $v = f(u)$. Let E_k be the event that $(k, 0)$ is joined by a vacant path to ∂D_k. From (7) we have

$$P_\lambda(E_k) = P_\lambda(\mathbf{0} \overset{\star}{\longleftrightarrow} \partial B(\alpha\log k)) \leq P_\lambda(D(\mathbf{0}) \overset{\star}{\longleftrightarrow} \partial B(\alpha \log k)) \approx k^{-(\alpha/\xi^\star(\lambda))}, \tag{14}$$

as $k \to \infty$. This gives $\sum P_\lambda(E_k) < \infty$ from the assumption that $\alpha > \xi^\star(\lambda)$. Therefore there exists M such that

$$P_\lambda\Big(\cup_{k\geq M} E_k\Big) < 1/2. \tag{15}$$

However if none of the events $\{E_k, k \geq M\}$ occurs, then a vacant path cannot join $f(u)$ and the x-axis, and L_f contains almost surely an infinite open cluster.

Combining the two steps this proves that $\lambda_c(L_f)$ is the unique solution in $(\lambda_c(\mathbb{R}^2), \infty)$ of $\xi^\star(\lambda) = a$. □

Proof of Theorem 1.1 (i), (ii) for infinite occupied component: In (i), $\lambda_c(L_f) = \infty$ follows from the fact that the solution of $\xi^\star(\lambda) = a$ goes to infinity as $a \downarrow 0$. Similarly in (ii), $\lambda_c(L_f) = \lambda_c(\mathbb{R}^2)$ follows from the fact that the solution of $\xi^\star(\lambda) = a$ goes to $\lambda_c(\mathbb{R}^2)$ as $a \uparrow \infty$. □

2.2 The case of infinite vacant component

Again we first derive properties of the connectivity function. When we consider the event that there is an occupied path from the origin to ∂B_n, much of the argument remains the same, except that now some of the events become increasing events as opposed to earlier.

First of all, there is a $t > 0$ such that $P(\rho > 2t) > 0$ (as before $\rho \leq 1$). We divide the square of side three around $\mathbf{0}$ into $[3^d/t^d]$ many small squares of side t and consider the event that in each smaller square there is at least one Poisson point with associated $\rho > 2t$. If an occupied path from $D(\mathbf{0})$ to ∂B_m also exists then these two events imply $\mathbf{0} \longleftrightarrow \partial B_m$, which is our notation for an occupied path from one to the other. Thus using the FKG inequality it is enough to work with $P(D(\mathbf{0}) \longleftrightarrow \partial B_m) = \beta(m)$ and parallel arguments lead to the existence of the limit

$$\phi(\lambda) = \lim -\frac{1}{m} \log P_\lambda(D(\mathbf{0}) \longleftrightarrow \partial B_m). \tag{16}$$

which satisfies the inequalities

$$-const. - g(m) \leq m\phi(\lambda) - \log \beta(m) - const.\lambda \leq const. + g(m), \tag{17}$$

for λ lying in a compact set where $g(m) = (d-1)\log(2(m+2)+1) = O(\log m)$. The precise statement can be written as (for covenience we use the same notation for constants ρ and σ)

Proposition 2.2 *There are positive constants ρ and σ so that*

$$\rho n^{1-d} e^{-n\phi(\lambda)} \leq P_\lambda(D(\mathbf{0}) \longleftrightarrow \partial B_n) \leq \sigma n^{d-1} e^{-n\phi(\lambda)}. \tag{18}$$

The limit of $-\frac{1}{m}\log \beta(m) = \phi(\lambda)$ is a continuous function of λ. Moreover $\phi(\lambda) = 0$ for $\lambda \geq \lambda_c$, $\phi(\lambda)$ is decreasing on $(0, \lambda_c)$ and $\phi(\lambda) \uparrow \infty$ as $\lambda \downarrow 0$. As customary, the occupied correlation length is defined by $\xi(\lambda) = 1/\phi(\lambda)$.

Proof: Continuity of $\phi(\lambda)$ follows as before from the continuity of $P_\lambda(D(\mathbf{0}) \longleftrightarrow \partial B_n)$ in λ for each fixed n, which can be proved noting that in the inequalities (12) and (13) we can substitute $P_{\lambda\pm\epsilon}\{(D(\mathbf{0}) \longleftrightarrow \partial B_n)^c\}$ for $P_{\lambda\pm\epsilon}(D(\mathbf{0}) \overset{\star}{\longleftrightarrow} \partial B_n)$ keeping the inequality signs unchanged, using the previous argument on vacancy. Using continuity and taking limit in (17) we have proved there are positive constants ρ and σ so that (18) holds where $\phi(\lambda)$ is a continuous function of λ.

It then follows that $\phi(\lambda) > 0$ for $0 < \lambda < \lambda_c$, is zero for $\lambda \geq \lambda_c$. Since $P_\lambda(D(\mathbf{0}) \longleftrightarrow \partial B_n) \leq P_\lambda(|W| \geq n)$ where $|W|$ is the number of Poisson points in the occupied component of the origin, from Theorem 10.1 of Penrose [9] it follows that $\phi(\lambda) \geq \lim -(1/n)\log P_\lambda(|W| \geq n) = \zeta(\lambda)$ and since $\zeta(\lambda) \uparrow \infty$ as $\lambda \downarrow 0$, we then have $\phi(\lambda) \uparrow \infty$ as $\lambda \downarrow 0$.

To show that $\phi(\lambda)$ is strictly decreasing on $(-\infty, \lambda_c)$ we adapt the argument of Grimmett [4] in our continuum setting. Let $N(k)$ be the number of Poisson(λ) points in the component of the origin which fall in the annulus $B_k \cap B_{k-1}^c$ and $\mathbf{N}(n) = (N(1), N(2), \cdots, N(n))$. Consider another intensity λ', $0 < \lambda' < \lambda < \lambda_c$. The points with intensity λ are called 'Light' and each of them can be 'White' with conditional probability λ'/λ. Let A_n be the event that the origin is joined to ∂B_n by a 'White' path. If $\mathbf{m}(n) = (m(1), m(2), \cdots, m(n))$, then writing $\epsilon = 1 - (\lambda'/\lambda)$, we have

$$P(A_n|\mathbf{N}(n)) = \mathbf{m}(n)) \leq \Pi_{i=1}^n (1 - \epsilon^{m(i)}) \leq e^{-\sum_{i=1}^n \epsilon^{m(i)}},$$

since each 'Light' point is not 'White' with probability ϵ. Now

$$\beta_{\lambda'}(n) = P(A_n) = \sum P(A_n|\mathbf{N}(n))P(\mathbf{N}(n) = \mathbf{m}(n)),$$

where the sum is over all vectors $\mathbf{m}(n)$ such that the conditional probability is nonzero. Dividing the sum into two parts depending on $\sum_{i=1}^n m(i) \leq Mn$ or $\sum_{i=1}^n m(i) > Mn$, where M will be specified later, we get

$$\beta_{\lambda'}(n) \leq \sum_{\mathbf{m}:\sum_i m(i) \leq Mn} e^{-\sum_{i=1}^n \epsilon^{m(i)}} P(\mathbf{N}(n) = \mathbf{m}(n)) + P(|L| > Mn), \tag{19}$$

where $|L|$ denotes the number of Poisson points in the 'Light' cluster containing the origin. By the inequality between the arithmetic and geometric means and Theorem 10.1 of Penrose [9] which says that under intensity λ the probability that the number of Poisson points in the occupied component containing the origin is greater than k behaves like $e^{-k\zeta(\lambda)}$ where $\zeta(\lambda) > 0$ for $0 < \lambda < \lambda_c$, we get

$$\begin{aligned}\beta_{\lambda'}(n) &\leq \sum_{\mathbf{m}:\sum_i m(i) \leq Mn} e^{-n\epsilon^M} P(\mathbf{N}(n) = \mathbf{m}(n)) + P(|L| > Mn) \\ &\leq e^{-n\epsilon^M} \beta_\lambda(n) + c_1 e^{-\zeta(\lambda)Mn}.\end{aligned}$$

Choosing M large enough so that $\zeta(\lambda)M > \epsilon^M + 2\phi(\lambda)$ we get from (16)

$$\beta_{\lambda'}(n) \leq 2\beta_\lambda(n) e^{-n\epsilon^M}, \text{ for all large } n.$$

Combining this with (16) we see that $\phi(\lambda') \geq \phi(\lambda) + \epsilon^M$, completing the proof of the decreasing nature of ϕ. The proof of the proposition is complete. □

Proof of Theorem 1.1 (iii) for infinite vacant component: Suppose $\lambda < \lambda_c$ and remember $\phi(\lambda) = 1/\xi(\lambda)$. When $a < \xi(\lambda)$, then we want to show L_f almost surely contains no unbounded vacant cluster. Let $\lambda_a^\star$ be the unique solution of $\xi(\lambda) = a$. Note that over $(0, \lambda_c)$, $\xi(\lambda)$ is increasing, thus $a < \xi(\lambda)$ implies $\lambda > \lambda_a^\star$. Fixing $\delta > 0$ such that $(1+\delta)a < \xi(\lambda)$, we define $\boldsymbol{w}_k = (k^{1+\delta}, 0)$. Let B_k be the smallest square with $\boldsymbol{w}_k$ in the middle of the lower side with the upper side just above the curve $v = f(u)$. $f(u)/\log u \to a$, hence the side length l_k of B_k satisfies

$$l_k = a(1 + o(1)) \log k^{1+\delta} \text{ as } k \to \infty.$$

Now B_k has side length l_k and center $\boldsymbol{w}_k + (0, l_k/2)$, and let A_k be the event that there is an occupied path from the top edge to the bottom edge of B_k. By the FKG inequality

$$P_\lambda(A_k) \geq \{\frac{1}{4} P_\lambda(D(\mathbf{0}) \longleftrightarrow \partial B_k(\frac{1}{2} l_k))\}^2 (1 - e^{-\lambda t^2 P(\rho > 2t)})^{[3^2/t^2]},$$

where t was defined at the beginning of this subsection satisfying $P(\rho > 2t) > 0$. However,

$$P_\lambda(D(\mathbf{0}) \longleftrightarrow \partial B_k(\frac{1}{2} l_k)) \approx e^{-l_k (2\xi(\lambda))^{-1}} \text{ as } k \to \infty,$$

by (18). Hence $P_\lambda(A_k) \geq (1/16) k^{-(1+o(1))(1+\delta)a/\xi(\lambda)} (1 - e^{-\lambda t^2 P(\rho > 2t)})^{[3^2/t^2]}$ as $k \to \infty$, i.e. $\sum P_\lambda(A_k) = \infty$ since $(1+\delta)a < \xi(\lambda)$. On the other hand the squares B_k are separated by more than twice the maximum of the sides of the Boolean squares (assumed $R = 1$ here), hence the configurations inside B_k's are independent for large k. Thus A_k occurs infinitely often almost surely.

Secondly, suppose $\lambda < \lambda_c$ and $a > \xi(\lambda)$. We want to show L_f contains almost surely an infinite vacant cluster. Choose α such that $a > \alpha > \xi(\lambda)$ and D_k be the box with center $(k, 0)$ and side length $2\alpha \log k$. For large values of k, D_k lies strictly beneath the curve $v = f(u)$. Let E_k be the event that $(k, 0)$ is joined by an occupied path to ∂D_k. From (18) we have

$$P_\lambda(E_k) = P_\lambda(\mathbf{0} \longleftrightarrow \partial B(\alpha \log k)) \leq P_\lambda(D(\mathbf{0}) \longleftrightarrow \partial B(\alpha \log k)) \approx k^{-(\alpha/\xi(\lambda))},$$

as $k \to \infty$. This gives $\sum P_\lambda(E_k) < \infty$ from the assumption that $\alpha > \xi(\lambda)$. Therefore there exists M such that

$$P_\lambda\Big(\cup_{k \geq M} E_k \Big) < 1/2. \tag{20}$$

However if none of the events $\{E_k, k \geq M\}$ occurs, then an occupied path cannot join $f(u)$ and $\mathbb{R}^+$, and L_f contains almost surely an infinite vacant cluster.

Combining the above this proves that $\lambda_c^\star(L_f)$ is the unique solution in $(0, \lambda_c(\mathbb{R}^2))$ of $\xi(\lambda) = a$. $\square$

Proof of Theorem 1.1 (i), (ii) for infinite vacant component: In (i), $\lambda_c^\star(L_f) = 0$ follows from the fact that the solution of $\xi(\lambda) = a$ goes to zero as $a \downarrow 0$. Similarly in (ii), $\lambda_c^\star(L_f) = \lambda_c(\mathbb{R}^2)$ follows from the fact that the solution of $\xi(\lambda) = a$ goes to $\lambda_c(\mathbb{R}^2)$ as $a \uparrow \infty$. $\square$

Remark: At criticality, i.e. when $f(x) \sim a \log x$ for some $a > 0$ and $\lambda = \lambda_c(L_f)$ as obtained in Theorem 1.1 (iii), infinite occupied component is possible and similarly at $\lambda_c^\star(L_f)$ infinite vacant component is possible. Let us consider the case of $\lambda_c(L_f)$ and show that there exists a function f such that $f(x)/\log x \to a$ as $a \to \infty$ and with $\xi^\star(\lambda_a) = a$ we have

$$P_{\lambda_a}(L_f \text{ contains an infinite occupied cluster}) = 1. \tag{21}$$

Consider the function f satisfying $f(u) = a \log u + b \log\log u$ for all large u where $b > 2a$. Let D_k be the largest box having center at $(k, 0)$ and lying strictly beneath the curve $v = f(u)$.

Then D_k has side length $2f(k)+O(1)$ as $k \to \infty$ and let E_k be the event that $(k,0)$ is joined by a vacant path to ∂D_k. Instead of (14), $P_\lambda(E_k)$ can be bounded more precisely by the inequality on the right side of (7) as

$$\begin{aligned} P_\lambda(E_k) &\leq P_\lambda(D(\mathbf{0}) \stackrel{\star}{\longleftrightarrow} \partial B(f(k)+O(1))) \\ &\leq \sigma(\log k)\exp\{-\frac{a\log k + b\log\log k}{\xi^\star(\lambda)}\} \end{aligned}$$

for all large k. At λ_a we have $\xi^\star(\lambda_a) = a$ and then $P_{\lambda_a}(E_k) \leq \sigma/k(\log k)^\alpha$ where $\alpha = (b/a) - 1 > 1$. Summability of $P_{\lambda_a}(E_k)$ as before gives (21) following the argument after (15). The calculations for $\lambda_a^\star$ are similar. $\square$

3 Proof of Theorem 1.2

In this section we work with the original Boolean model $\{\boldsymbol{x}_i + [0,\rho_i]^2 : \boldsymbol{x}_i \in X\}$. Note, in case we center the boxes at the Poisson points then we would get results on the complete coverage of space rather than eventual coverage.

Proof of Theorem 1.2 (i): Without loss of generality we take $a = 1$, i.e. $f(x)/\log x \to 1$ as $x \to \infty$. For eventual coverage, following Athreya, Roy and Sarkar [1] the Boolean model is compared to two discrete models as follows. Under the red model if a square of the $\mathbb{Z}^+ \times \mathbb{Z}$ lattice has at least one Poisson point, then the lower left hand corner of that square is declared open and a square of side $\max\{\rho_1, \cdots, \rho_N\}+1$ is placed there where N is the number of Poisson points in the previous square. It can be checked that the side of the new square follows the distribution

$$\begin{aligned} F_{red}(m) &= P(\max\{\rho_1, \cdots, \rho_N\} < m-1 | N \geq 1) \\ &= \sum_{j=1}^{\infty} \frac{e^{-\lambda}\lambda^j}{(1-e^{-\lambda})j!} P(\rho < m-1)^j \\ &= e^{-\lambda}\frac{e^{\lambda P(\rho < m-1)} - 1}{1-e^{-\lambda}} \\ &= \frac{e^{-\lambda P(\rho \geq m-1)} - e^{-\lambda}}{1-e^{-\lambda}}, \end{aligned}$$

and that each lower left corner of the new square is open with probability $1-e^{-\lambda}$. As in Lemma 3.1 of Athreya, Roy and Sarkar [1] it is clear that eventual coverage of L_f under the Boolean model ensures the same under the above discrete model in which each point of the lattice is open with probability $p = 1-e^{-\lambda}$ and at each lattice a square with integer sides following the distribution F_{red} is placed. We write $G_{red}(m) = 1 - F_{red}(m-1)$, and assume that the tail $G(m) = 1 - F(m-1)$ of the distribution of ρ satisfies

$$G(m) = \frac{K_\rho}{m^2} + \frac{\eta(m)}{m^2} \text{ for all large } m, \tag{22}$$

where $K_\rho > 0$ and $\eta(m) \to 0$ as $m \to \infty$. Under this assumption $G_{red}(m)$ satisfies

$$G_{red}(m) = \frac{\lambda K_\rho}{1-e^{-\lambda}} \frac{1}{m^2} + \frac{\eta_{red}(m)}{m^2}, \text{ for all large } m,$$

where $\eta_{red}(m) \to 0$ as $m \to \infty$. In Corollary 3.1 we shall prove that in a discrete model with G as in (22) if $2pK_\rho < 1$, then L_f is not eventually covered almost surely. Thus under the red model L_f is not eventually covered almost surely if

$$2(1-e^{-\lambda}) \frac{\lambda K_\rho}{1-e^{-\lambda}} < 1,$$

hence under the above condition under the Boolean model L_f is not eventually covered almost surely

Similarly we consider a green model in which if a square of the $\mathbb{Z}^+ \times \mathbb{Z}$ lattice has at least one Poisson point with $\rho \geq 3$, then the upper right hand corner of that square is declared open and a square of side $\max\{\rho_1, \cdots, \rho_{N'}\} - 3$ is placed there where N' is the number of Poisson points with $\rho \geq 3$ in the previous square. It can be checked that the side of the new square follows the distribution (here ρ' has the same distribution as ρ conditioned on the fact that $\rho \geq 3$)

$$\begin{aligned} F_{green}(m) &= P(\max\{\rho'_1, \cdots, \rho'_{N'}\} < m+3 | N' \geq 1) \\ &= \frac{e^{-\lambda P(\rho \geq 3)}}{1-e^{-\lambda P(\rho\geq 3)}} \sum_{j=1}^{\infty} \frac{(\lambda P(\rho \geq 3))^j}{j!} P(\rho' < m+3)^j \\ &= \frac{e^{-\lambda P(\rho \geq m+3)} - e^{-\lambda P(\rho\geq 3)}}{1-e^{-\lambda P(\rho \geq 3)}}, \end{aligned}$$

and that each lower left corner of the new square is open with probability $1 - e^{-\lambda P(\rho \geq 3)}$.

As in Lemma 3.2 of Athreya, Roy and Sarkar [1] we see that eventual coverage of L_f under the Boolean model is ensured by the same for the green model which is a discrete model in which each point of the lattice is open with probability $p = 1 - e^{-\lambda P(\rho \geq 3)} = 1 - e^{-\lambda G(4)}$ and at each lattice a square with integer sides following the distribution F_{green} is placed. From our assumed form of G we get the behavior of

$$G_{green}(m) = 1 - F_{green}(m-1) = \frac{\lambda K_\rho}{1-e^{-\lambda G(4)}} \frac{1}{m^2} + \frac{\eta_{green}(m)}{m^2},$$

for all large m, where $\eta_{green}(m) \to 0$ as $m \to \infty$. In Corollary 3.1 we shall show that in a discrete model with G as in (22) if $2pK_\rho > 1$ then L_f is eventually covered almost surely. Thus under the green model L_f is eventually covered almost surely if

$$2(1-e^{-\lambda G(4)}) \frac{\lambda K_\rho}{1-e^{-\lambda G(4)}} < 1,$$

hence under the above condition under the Boolean model L_f is covered eventually almost surely.

Combining the two cases we see that using Corollary 3.1 and the above domination of the Boolean model by two discrete models, the critical intensity for the eventual coverage of L_f under the Boolean model is given by $\lambda_e = 1/(2K_\rho)$. It will also follow similarly from Corollary 3.2 that the critical intensity for eventual coverage of the line $l_g = \{(x, g(x)), x \geq 0\}$ where $g : [0, \infty) \to [0, \infty)$ is nondecreasing, is also $\lambda_e = 1/(2K_\rho)$. This completes the proof of Theorem 1.2 (i). $\square$

Proof of Theorem 1.2 (ii), (iii): It will be proved in Corollary 3.3 that in the discrete case there is no eventual coverage for any $p < 1$ in case of tail of ρ thinner than $1/x^2$, and there is eventual coverage for any $p > 0$ in the case of tail of ρ thicker than $1/x^2$. Then by a similar domination of the Boolean model by two discrete models in case of thinner tails there is no eventual coverage for any $\lambda > 0$, i.e. $\lambda_e = \infty$, and in the case of thicker tails there is eventual coverage for any $\lambda > 0$, i.e. $\lambda_e = 0$. $\square$

It remains to discuss the discrete model and derive the conditions for eventual coverage in that model.

3.1 Eventual coverage in the discrete model

In the discrete model, at each $(i, j) \in \mathbb{N} \times \mathbb{Z}$, we put iid Bernoulli random variables with probability of success $p, 0 < p < 1$. Let $\{\rho_{(i,j)} : (i, j) \in \mathbb{N} \times \mathbb{Z}\}$ be a collection of nonnegative integer valued random variables having the same distribution as ρ. The probability measure will be denoted by P_p. Now define the covered region

$$C := \bigcup_{\{(i,j)\in\mathbb{N}\times\mathbb{Z}, X_{i,j}=1\}} \Big\{(i, j) + [0, \rho_{(i,j)}]^2\Big\}.$$

Let us consider a nondecreasing nonnegative function $f(i), i \in \mathbb{Z}^+$. Let $L_f = \{(i, j) : 0 \leq j \leq f(i)\}$ be the discrete version of the region L_f defined earlier. We say that L_f is eventually covered with probability one if with probability one for every realization of the $X_{i,j}$'s and the $\rho_{(i,j)}$'s there is some $N < \infty$, such that $\{(i, j) \in L_f : i \geq N\} \subset C$. We now have

Proposition 3.1 *Define for $(i, j) \in \mathbb{N} \times \mathbb{Z}$, the event $A_{(i,j)} = \{(i, j) \notin C\}$. Notice that by invariance $P_p(A_{(i,j)}) = P_p(A_{(i,0)})$ for any $j \in \mathbb{Z}$. We have that (a) for $g(i) : \mathbb{Z}^+ \to \mathbb{Z}^+$ a nondecreasing function, the line $l_g = \{(i, g(i)) : i \geq 1\}$ is covered or not covered eventually with probability one if and only if $\sum_{i=1}^{\infty} P_p(A_{(i,0)})$ is finite or infinite, (b) L_f is eventually covered with probability one $\sum_{i=1}^{\infty}(f(i) + 1)P_p(A_{(i,0)}) < \infty$ and L_f is not eventually covered with probability one if $\sum_{i=1}^{\infty} P_p(A_{(i,0)}) = \infty$.*

Proof: (a) Note that $(i, g(i))$ can only be covered by squares situated on the lattice points of the region $[0, i] \times (-\infty, g(i)]$. Since the ρ random variables are nonnegative, as soon as we observe the first uncovered point say $(i_1, g(i_1))$, the coverage by the points in the region

$[0, i_1] \times (-\infty, g(i_1)]$ and their ρ's is below the line $\mathbb{Z}^+ \times \{g(i_1)\}$ from $(i_1, g(i_1))$ onwards to the right. Since g is nondecreasing, the coverage of the next point $(i_2, g(i_2))$ is then determined by the points in $[i_1+1, i_2] \times (-\infty, g(i_2)]$ and their ρ's. Thus $A_{(i,g(i))}$ is a renewal event (see Feller [2]) satisfying

$$P_p(A_{(i,g(i))} \cap A_{(k,g(k))}) = P_p(A_{(i,g(i))}) P_p(A_{(k-i,g(k)-g(i))}), i < k,$$

and by $P_p(A_{(i,g(i))}) = P_p(A_{(i,0)})$, if $\sum_{i=1}^{\infty} P_p(A_{(i,0)}) = \infty$ then on the line $l_g = \{(i, g(i)) : i \geq 1\}$ there are infinitely many uncovered points with probability one. On the other hand if $\sum_{i=1}^{\infty} P_p(A_{(i,0)}) < \infty$ the the Borel-Cantelli lemma says that with probability one only finitely many of the events $A_{(i,g(i))}$ can happen.

(b) Since f is assumed to be nondecreasing, using part (a) under the assumption $\sum_{i=1}^{\infty} P_p(A_{(i,0)}) = \infty$ we have every horizontal line is not eventually covered with probability one, and the same holds for L_f. On the other hand when $\sum_{i=1}^{\infty}(f(i)+1)P_p(A_{(i,0)}) < \infty$, then for each i, we consider the points $0 \leq j \leq f(i)$ and remember that by invariance $P_p(A_{(i,j)}) = P_p(A_{(i,0)})$. By considering union over i, $P_p(\cup_{i=1}^{\infty} \cup_{j=0}^{f(i)} A_{(i,j)}) \leq \sum_{i=1}^{\infty}(1+f(i))P_p(A_{(i,0)}) < \infty$, and the Borel-Cantelli lemma says that with probability one only finitely many of $\cup_{j=0}^{f(i)} A_{(i,j)}, i \geq 1$ can happen. $\square$.

Under assumption (1) on the tail of ρ, the proposition gives a simple criterion for deciding the threshold between eventual coverage and not eventual coverage for L_f in terms of p. At the end in a remark we shall also see that for thinner tails there is no eventual coverage for any $p < 1$, and for fatter tails there is eventual coverage for any $p > 0$. We shall also discuss what can happen for p at the threshold.

Let F be the distribution function of ρ, i. e. $F(i) := P_p(\rho \leq i)$ and let $G(i) = 1 - F(i-1)$. The following formula will be used

$$\begin{aligned} & P_p(A_{(i,0)}) \\ = \; & (1-p) \prod_{t=1}^{i-1} \Big((1-p) + pF(t-1)\Big)^{2t+1} \prod_{k=i}^{\infty} \Big((1-p) + pF(k-1)\Big)^{i} \\ = \; & \prod_{t=0}^{i-1} (1 - pG(t))^{2t+1} \prod_{k=i}^{\infty} (1 - pG(k))^{i}. \end{aligned} \tag{23}$$

Now we need to find conditions on ρ so that $\sum_{i=1}^{\infty} P_p(A_{(i,0)})$ is infinite and $\sum_{i=1}^{\infty}(f(i)+1)P_p(A_{(i,0)})$ is finite. Under assumption (1) on the tail of ρ we first derive bounds on the behavior of $P_p(A_{(i,0)})$, and show that

Lemma 3.1 *Given any positive ϵ we can find constants $0 < C_1(\epsilon) \leq C_2(\epsilon) < \infty$ so that*

$$C_1(\epsilon) i^{-2pK_\rho} e^{-\epsilon \log i} \leq P_p(A_{(i,0)}) \leq C_2(\epsilon) i^{-2pK_\rho} e^{\epsilon \log i} \tag{24}$$

for all large i.

Proof: After taking logarithm of $P_p(A_{(i,0)})$ given by (23) we first consider the term

$$\begin{aligned} i\sum_{k=i}^{\infty}\log(1-pG(k)) &= i\sum_{k=i}^{\infty}\{-pG(k)+O(1/k^4)\} \\ &= -pi\sum_{k=i}^{\infty}(\frac{K_\rho}{k^2}+\frac{\eta(k)}{k^2})+O(i\sum_{k=i}^{\infty}\frac{1}{k^4}) \\ &\sim -pi\{(K_\rho/i)+O(1/i)\} \\ &\sim -pK_\rho+O(1), \end{aligned} \tag{25}$$

where we remember that $\eta(i)\to 0$ as $i\to\infty$. Now consider the crucial term

$$\begin{aligned} &\sum_{t=1}^{i-1}(2t+1)\log(1-pG(t)) \\ =& \sum_{t=1}^{i-1}(2t+1)\{-pG(t)+O(1/t^4)\} \\ =& -pK_\rho\sum_{t=1}^{i-1}\frac{2t+1}{t^2}-p\sum_{t=1}^{i-1}\frac{(2t+1)\eta(t)}{t^2}+O(\sum_{t=1}^{i-1}\frac{1}{t^3}) \\ =& -(2pK_\rho)\log i+O(1/i)-p\sum_{t=1}^{N}\frac{(2t+1)\eta(t)}{t^2} \\ & -p\sum_{t=N}^{i-1}\frac{(2t+1)\eta(t)}{t^2}+O(1), \end{aligned} \tag{26}$$

where N is a large fixed integer so that $|\eta(t)|<\epsilon/2$ for $t>N$. We use

$$|-p\sum_{t=N}^{i-1}\frac{(2t+1)\eta(t)}{t^2}|\le(\epsilon/2)\sum_{t=1}^{i-1}\frac{(2t+1)}{t^2}\le\epsilon\log i+O(1),$$

and then (25) and (26) give (24). □

Using part (b) of Proposition 3.1, we now have the following result.

Corollary 3.1 *Under the assumption (1), we have that for $f=a\log(1+x)$, for some $a>0$,*

(a) L_f is eventually covered almost surely-P_p if $2pK_\rho>1$.

(b) L_f is not eventually covered almost surely-P_p if $2pK_\rho<1$.

Proof: For part (a) we need to prove the finiteness of $\sum_{i=1}^{\infty}(\log(1+i))P_p(A_{(i,0)})$. From (24), $(\log(1+i))P_p(A_{(i,0)})\le C_2(\epsilon)(i^\epsilon\log(1+i))/i^{2pK_\rho}$ for all large i. Choosing ϵ so that $2pK_\rho>1+\epsilon$, gives the finiteness of the required sum.

For part (b) using (24) again we have $P_p(A_{(i,0)}\ge C_1(\epsilon)1/i^{2pK_\rho+\epsilon}$ for all large i, and choosing ϵ such that $2pK_\rho+\epsilon<1$ get $\sum_{i=1}^{\infty}P_p(A_{(i,0)}=\infty$, completing the proof. □

Let us define,

$$p_e(L_f) := \inf\{p > 0 : P_p\{\text{eventual coverage occurs}\} = 1\}.$$

Then, by Corollary 3.1 under the assumption (1), we have $p_e = \frac{1}{2K_\rho}$. Similar calculations for part (a) of Proposition 3.1 prove

Corollary 3.2 *For $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ nondecreasing, the line $l_g = \{(i, g(i)) : i \geq 1\}$ is eventually covered almost surely if $2pK_\rho > 1$ and is not eventually covered almost surely if $2pK_\rho < 1$,*

which shows that the critical probabilities for the wedge L_f with f as above and any line $\{(i, g(i) : i \geq 1\}$ for g as described are the same.

Finally we discuss what happens if the tail of ρ is thinner or fatter in the following

Corollary 3.3 *If $t^2 G(t) \to 0$ as $t \to \infty$ then $p_e(L_f) = 1$. On the other hand if $t^2 G(t) \to \infty$ as $t \to \infty$ then $p_e(L_f) = 0$.*

Proof: In the first case we can take $K_\rho = 0$ and with the same assumption on η as before, see from the order determining terms in the last lines of (25) and (26) that $P_p(A_{(i,0)}) \geq C_1(\epsilon)/i^\epsilon$ for any $\epsilon > 0$ for sufficiently large i. Thus $\sum P_p(A_{(i,0)}) = \infty$ establishing no eventual coverage for any $p > 0$, i.e. $p_e = 1$.

In the second case we can take $G(t) = K_\rho(t)/t^2$ where $K_\rho(t) \to \infty$ as $t \to \infty$. Now for any $p > 0$, from a certain i onwards $2pK_\rho(i) > \beta > 1$, for some constant β and from the order determining terms in the last line of (25) and the third line of (26) we get $P_p(A_{(i,0)}) \leq const.i^{-\beta}$ for all large i. It follows that for any $p > 0$, we have $\sum \log(1 + i) P_p(A_{(i,0)}) < \infty$, showing that for any $p > 0$ there is eventual coverage, i.e. $p_e = 0$. $\square$

Similar arguments hold for an extension of Corollary 3.2 related to the eventual coverage of a line.

Eventual coverage at criticality in the Boolean model: If $K_\rho > 0$ and $G(i) = \frac{K_\rho}{i^2} + \frac{\gamma}{i^2 \log i}$ where $\gamma > 0$, then at λ_e depending on $\gamma > 2K_\rho$ or $\gamma \leq K_\rho$ respectively, eventual coverage may or may not occur. The proof of this will follow by comparing the Boolean model to the Red and Green discrete models if we prove the corresponding statement in the discrete case.

Thus we need to show that in the discrete case at the critical point both the scenarios, i.e., eventual coverage and no eventual coverage, can happen. For example, let us take the following special case:

$$G(i) = \frac{K_\rho}{i^2} + \frac{\gamma}{i^2 \log i} \tag{27}$$

where $\gamma > 0$. In other words, we have $\eta(i) = \gamma/\log i$ for all sufficiently large i. Then it can be checked that the new estimate for the sum in the third line of (26)

$$-p\sum_{t=2}^{i-1}\frac{(2t+1)\eta(t)}{t^2} = -p\sum_{t=2}^{i-1}\frac{(2t+1)\gamma}{t^2\log t} \sim -2p\gamma\log\log i + O(1)$$

gives the asymptotic behavior

$$\frac{D_1}{i^{2pK_\rho}(\log i)^{2p\gamma}} \leq P_p(A_{(i,0)}) \leq \frac{D_2}{i^{2pK_\rho}(\log i)^{2p\gamma}} \tag{28}$$

for all large i where $0 < D_1 \leq D_2 < \infty$ are constants. Hence, at the critical point $p = 1/(2K_\rho)$, the sum $\sum_{i=1}^{\infty} P_p(A_{(i,0)}) < \infty$ if and only if $\gamma > K_\rho$ whereas the sum $\sum_{i=1}^{\infty}(\log(1+i))P_p(A_{(i,0)}) < \infty$ if $\gamma > 2K_\rho$. Thus, at the critical point $p = p_e = 1/(2K_\rho)$ for $\gamma \leq K_\rho$, no eventual coverage occurs almost surely, while for $\gamma > 2K_\rho$, eventual coverage occurs almost surely. $\square$

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