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A Stepwise Planned Approach to the Solution of Hilbert's Sixth Problem. II : Supmech and Quantum Systems

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Abstract: Supmech, which is noncommutative Hamiltonian mechanics (NHM) (developed in paper I) with two extra ingredients : positive observable valued measures (PObVMs) [which serve to connect state-induced expectation values and classical probabilities] and the 'CC condition' [which stipulates that the sets of observables and pure states be mutually separating] is proposed as a universal mechanics aimed at covering all physical phenomena. Quantum systems, defined algebraically as supmech hamiltonian systems with non-supercommutative system algebras, are shown to inevitably have Hilbert space based realizations (so as to accommodate rigged Hilbet space based Dirac bra-ket formalism), generally admitting commutative superselection rules. Traditional features of quantum mechanics of finite particle systems appear naturally in an autonomous development. Treating massive particles as localizable elementary quantum systems, the Schrödinger wave functions with traditional Born interpretation appear as natural objects for their description and the Schrödinger equation for them is obtained without ever using a classical Hamiltonian or Lagrangian.

I. Introduction

This is the second of a series of papers aimed at obtaining a solution of Hilbert's sixth problem in the framework of a noncommutative geometry (NCG) based 'all-embracing' scheme of mechanics. In the first paper (Dass [14]; henceforth referred to as I), the 'bare skeleton' of that mechanics was presented in the form of noncommutative Hamiltonian mechanics (NHM) which combines elements of noncommutative symplectic geometry and noncommutative probability in an algebraic setting. Consideration of interaction between two systems in the NHM framework led to the division of physical systems into two 'worlds' — the 'commutative world' and the 'noncommutative world' [corresponding, respectively, to systems with (super-)commutative and non-(super-)commutative system algebras] — with no consistent description of interaction allowed between two systems belonging to different 'worlds'; in the 'noncommutative world', the system algebras are constrained by the formalism to have a 'quantum symplectic structure' characterized by a universal Planck type constant.

The formalism of NHM presented in paper I is deficient in that it does not connect smoothly to classical probability and, in the noncommutative case, to Hilbert space. A refined version of it, called Supmech, is presented in section 2 which has two extra ingredients aimed at overcoming these deficiencies.

The first ingredient is the introduction of classical probabilities as expectation values of 'supmech events' constituting 'positive observable-valued measures' (PObVMs) [a generalization of positive operator-valued measures]. All probabilities in the formalism — the transition probabilities between states as well as probabilities of outcomes in experimental situations — are stipulated to be of this type.

The second ingredient is the condition of 'compatible completeness' between observables and pure states (referred to as the 'CC condition') – the condition that the two sets be mutually separating. This condition is satisfied in classical Hamiltonian mechanics and in traditional Hilbert space quantum mechanics (QM). (It is, however, not generally satisfied in superclassical Hamiltonian systems with a finite number of fermionic generators; see section 2.3). It will be seen to play an important role in the whole development; in particular, it serves to smoothly connect — without making any extra assumptions — the algebraically defined quantum systems with the Hilbert space-based ones.

A general treatment of localizable systems (more general and simpler than that in the traditional approaches), which makes use of PObVMs, is given in section 2.4. In section 2.5, elementary systems are defined in supmech and the special case of nonrelativistic elementary systems is treated. The role of relativity groups in the identification of fundamental observables of elementary systems is emphasized. Particles are proposed to be treated as localizable elementary systems.

In section 3, quantum systems are treated as supmech Hamiltonian systems with non-(super-)commutative system algebras. As mentioned above, the CC condition ensures the existence of their Hilbert space based realizations. In the case of systems with finitely generated system algebras, one has an irreducible faithful representation of the system algebra; in the general case, one has a direct sum of such representations corresponding to situations with commutative superselection rules. Treating material particles as localizable elementary quantum systems, the Schrödinger wave functions are shown to appear naturally in the description of pure states; their traditional Born interpretation is obvious and the Schrödinger equation appears as a matter of course — without ever using the classical Hamiltonian or Lagrangian in the process of obtaining it. The Planck constant is introduced at the place dictated by the formalism (i.e. in the quantum symplectic form); its appearance everywhere else — canonical commutation relations, Heisenberg and Schrödinger equations, etc. — is automatic. In section 4, a transparent treatment of quantum - classical correspondence emphasizing some formal aspects is presented.

In section 5, a provisional set of axioms underlying the treatment of systems in the supmech framework is given. The last section contains some concluding remarks.

2. Augmented Noncommutative Hamiltonian Mechanics : Supmech

In this section we shall introduce two extra ingredients in NHM — the positive observable valued measures (PObVMs) and the CC condition — which serve to connect it smoothly to classical probability and to Hilbert space respectively; the resulting augmented NHM will be called 'Supmech'. The PObVMs will be used in section 2.4 in the treatment of localizable systems. The CC condition will be used in section 2.5 to allow the Hamiltonian action of a relativity group on the system algebra to be extended to a Poisson action which is an important simplification. Noncommutative Noether invariants of the Galilean group for a free massive spinless particle will be obtained in section 2.6.

We shall freely use the terminology and notation of I. We quickly recall here that, in NHM, a physical system is assumed to have associated with it a (topological) superalgebra $\mathcal{A}$ (with unit element I), the even hermitian elements of which are identified as the system observables. Observables of the form of finite sums $\sum A_i^* A_i$ ($A_i \in \mathcal{A}$) are called positive. A state ϕ of $\mathcal{A}$ is defined as a (continuous) positive linear functional on $\mathcal{A}$ satisfying the normalization condition $\phi(I) = 1$; the quantity $\phi(A)$ is to be interpreted as the expectation value of the observable A when the system is in the state ϕ. Sets of observables, states and pure states (those not expressible as nontrivial convex combinations of other states) of $\mathcal{A}$ are denoted as $\mathcal{O}(\mathcal{A}), \mathcal{S}(\mathcal{A})$ and $\mathcal{S}_1(\mathcal{A})$ respectively.

Note. In (I, section 2.1), the following convention about the *-operation in a superalgebra $\mathcal{A}$

[following (Dubois-Violette [20]; section 2)] was adopted :

$$(AB)^* = (-1)^{\epsilon_A \epsilon_B} B^* A^*$$

where ϵ_A is the parity of $A \in \mathcal{A}$. This convention, however, does not suit the needs of the work reported in this series (it was not used anywhere in I). We shall henceforth use the convention $(AB)^* = B^* A^*$. [Given two fermionic annihilation operators a, b, for example, we have $(ab)^* = b^* a^*$ and not $(ab)^* = -b^* a^*$. One can also check the appropriateness of the latter convention by taking $\mathcal{A}$ to be the superalgebra of linear operators on a superspace $V = V^{(0)} \oplus V^{(1)}$.]

2.1. Positive observable valued measures

We shall introduce classical probabilities in the formalism through a straightforward formalization of a measurement situation. To this end, we consider a measurable space $(\Omega, \mathcal{F})$ and associate, with every measurable set $E \in \mathcal{F}$, a positive observable $\nu(E)$ such that

$$(i)\ \nu(\emptyset) = 0,\ \ (ii)\ \nu(\Omega) = I,$$
$$(iii)\ \nu(\cup_i E_i) = \textstyle\sum_i \nu(E_i) \text{ (for disjoint unions).}$$

[The last equation means that, in the relevant topological algebra, the possibly infinite sum on the right hand side is well defined and equals the left hand side.] Then, given a state ϕ, we have a probability measure p_ϕ on $(\Omega, \mathcal{F})$ given by

$$p_\phi(E) = \phi(\nu(E)) \ \ \forall E \in \mathcal{F}. \tag{1}$$

The family $\{\nu(E), E \in \mathcal{F}\}$ will be called a *positive observable-valued measure* (PObVM) on $(\Omega, \mathcal{F})$. It is the abstract counterpart of the 'positive operator-valued measure' (POVM) employed in Hilbert space QM (Davies [16]; Holevo [25]; Busch, Grabowski, Lahti [11]). The objects $\nu(E)$ may be called *supmech events* (representing possible outcomes in a measurement situation); these are algebraic generalizations of the objects (projetion operators) called 'quantum events' (Parthasarathy [37]). A state assigns probabilities to these events. Eq.(1) represents the desired relationship between the supmech expectation values and classical probabilities.

In concrete applications, the space Ω represents the 'value space' (spectral space) of one or more observable quantities. The measurable subsets of Ω (elements of $\mathcal{F}$) represent idealised domains supposed to be experimentally accesible. In a classical probability space $(\Omega, \mathcal{F}, P^{cl})$, they are the 'events' to which probabilities are assigned by the probability measure P^{cl}; the classical probability of an event $E \in \mathcal{F}$ is

$$P^{cl}(E) = \int_\Omega \chi_E dP^{cl} \equiv \phi_{P^{cl}}(\chi_E) \tag{2}$$

where χ_E is the characteristic/indicator function of the subset E (the random variable which represents the classical observable distinguishing between the occurrence and non-occurrence of the event E. [These random variables are easily seen to constitute a PObVM on the commutative unital *-algebra $\tilde{\mathcal{A}}_{cl}$ of complex measurable functions on $(\Omega, \mathcal{F})$; the objects $\nu(E)$ described above are noncommutative generalizations of these.] The right hand side of (2) expresses the classical probability of occurrence of the event E as expectation value of the observable χ_E in the state $\phi_{P^{cl}}$ [represented by the probability measure P^{cl} on the measurable space $(\Omega, \mathcal{F})$] of the commutative algebra $\tilde{\mathcal{A}}_{cl}$.

We have here a more sophisticated scheme of probability theory which incorporates classical probability theory as a special case and is well equipped to take into consideration the influence of one measurement on probabilities of outcomes of other measurements. Moreover, this scheme appears embedded in an 'all-embracing' scheme of mechanics — in the true spirit of Hilbert's sixth problem.

Concrete examples of the objects $\nu(E)$ will appear in sections 2.4 and 3.4 (where observables related to localization are treated) and in section 4 on measurements on quantum systems.

2.2. The condition of compatible completeness on observables and pure states

In a sensible physical theory, the collection of pure states must be rich enough to distinguish between two different observables. (Mixed states represent averaging over ignorances over and above those implied by the irreducible probabilistic aspect of the theory; they, therefore, are not the proper objects for a statement of the above sort.) Similarly, there should be enough observables to distinguish between different pure states. These requirements are taken care of in supmech by stipulating that the pair $(\mathcal{O}(\mathcal{A}),\ \mathcal{S}_1(\mathcal{A}))$ be *compatibly complete* in the sense that

(i) given $A, B \in \mathcal{O}(\mathcal{A}), A \neq B$, there should be a state $\phi \in \mathcal{S}_1(\mathcal{A})$ such that $\phi(A) \neq \phi(B)$;

(ii) given two different states ϕ_1 and ϕ_2 in $\mathcal{S}_1(\mathcal{A})$, there should be an $A \in \mathcal{O}(\mathcal{A})$ such that $\phi_1(A) \neq \phi_2(A)$.

We shall refer to this condition as the 'CC condition' for the pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}))$.

Proposition 2.1 *The CC condition holds for (i) a classical Hamiltonian system* (M, ω_{cl}, H_{cl}) *[where* (M, ω_{cl}) *is a finite dimensional symplectic manifold and the Hamiltonian* H_{cl} *is a smooth real valued function on M] and (ii) a traditional quantum system represented by a quantum triple* $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ *where* $\mathcal{H}$ *is a complex separable Hilbert space,* $\mathcal{D}$ *a dense linear subset of* $\mathcal{H}$ *and* $\mathcal{A}$ *is an Op*-algebra based on the pair* $(\mathcal{H}, \mathcal{D})$ *acting irreducibly [i.e. such that there does not exist a smaller quantum triple* $(\mathcal{H}', \mathcal{D}', \mathcal{A})$ *with* $\mathcal{D}' \subset \mathcal{D}, \mathcal{A}\mathcal{D}' \subset \mathcal{D}'$ *and* $\mathcal{H}'$ *is a proper subspace of* $\mathcal{H}$*].*

[Note. Op*- algebras (Horuzhy [27]) and quantum triples were defined in section 3.4 of I.]

Proof. (i) For a classical hamiltonian system (M, ω_{cl}, H_{cl}), observables are smooth real valued functions on M and pure states are Dirac measures (or, equivalently, points of M) $\mu_{\xi_0}(\xi_0 \in M)$; the expectation value of the observable f in the pure state ϕ_{ξ_0} corresponding to the Dirac measure μ_{ξ_0} is given by $\phi_{\xi_0}(f) = \int f d\mu_{\xi_0} = f(\xi_0)$. Given two different real-valued smooth functions on M, there is a point of M at which they take different values; conversely, given two different points ξ_1 and ξ_2 of M, there is a real-valued smooth function on M which takes different values at those points. [To show the existence of such a function, let U be an open neighborhood of ξ_1 not containing ξ_2; now appeal to lemma (2) on page 92 of (Matsushima [34]) which guarantees the existence of a smooth function non-vanishing at ξ_1 and vanishing outside U.]

(ii) The observables are the Hermitian elements of $\mathcal{A}$ and pure states are unit rays represented by normalized elements of $\mathcal{D}$.

(a) Given $A, B \in \mathcal{O}(\mathcal{A})$, and $(\psi, A\psi) = (\psi, B\psi)$ for all normalized ψ in $\mathcal{D}$ (hence for all ψ in $\mathcal{D}$), we have $(\chi, A\psi) = (\chi, B\psi)$ for all $\chi, \psi \in \mathcal{D}$, implying A = B. [Hint : Consider the given equality with the state vectors $(\chi + \psi)/\sqrt{2}$ and $(\chi + i\psi)/\sqrt{2}$.]

(b) Given normalized vectors ψ_1, ψ_2 in $\mathcal{D}$ and $(\psi_1, A\psi_1) = (\psi_2, A\psi_2)$ for all $A \in \mathcal{O}(\mathcal{A})$, we must prove that $\psi_1 = \psi_2$ (up to a multiplicative phase factor). Considering the 2-dimensional subspace V of $\mathcal{H}$ spanned by ψ_1 and ψ_2 and choosing an appropriate orthonormal basis in V, we can write

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{with } |a|^2 + |b|^2 = 1.$$

It is easily seen that $\psi_2 = U\psi_1$ where (writing $a = |a|e^{i\alpha},\ b = |b|e^{i\beta}$) U is the unitary matrix

$$U = \begin{pmatrix} a & be^{i(\alpha-\beta)} \\ b & -ae^{i(\beta-\alpha)} \end{pmatrix}.$$

Extending U trivially to a unitary operator on $\mathcal{H}$ (and denoting the extended operator by U) we again have $\psi_2 = U\psi_1$ (in $\mathcal{H}$). The given equality and denseness of $\mathcal{D}$ then give $U^*AU = A$ (for all $A \in \mathcal{O}(\mathcal{A})$, hence all $A \in \mathcal{A}$). The irreducibility of $\mathcal{A}$-action now implies U = I up to a multiplicative phase factor. □

Note. The irreducibility of $\mathcal{A}$-action assumed above implies that all elements of $\mathcal{D}$ represent pure states. This excludes the situations when $\mathcal{H}$ is a direct sum of more than one coherent subspaces in the presence of superselection rules.

The noncommutative Hamiltonian mechanics (NHM) described in I augmented by the two inclusions — PObVMs and the CC condition — is being hereby projected as the 'all-embracing' mechanics covering (in the sense of providing a common framework for the description of) all motion in nature; we shall henceforth refer to it as *Supmech.*

We have seen in section 3.4 of I that both — classical Hamiltonian mechanics and traditional Hilbert space quantum mechanics — are subdisciplines of NHM. Since the two new ingredients — PObVMs and the CC condition — are present in both of them, both are subdisciplines of supmech as well.

2.3. Superclassical systems; Violation of the CC condition

Superclassical mechanics is an extension of classical mechanics which employs, besides the traditional phase space variables, Grassmann variables θ^α ($\alpha = 1, ..n$, say) satisfying the relations

$$\theta^\alpha\theta^\beta + \theta^\beta\theta^\alpha = 0 \quad \text{for all } \alpha, \beta;$$

[in particular $(\theta^\alpha)^2 = 0$ for all α]. These objects generate the so -called Grassmann algebra (with n generators) $\mathcal{G}_n$ whose elements are functions of the form

$$f(\theta) = a_0 + a_\alpha\theta^\alpha + a_{\alpha\beta}\theta^\beta\theta^\alpha + ...$$

where the coefficients $a_{..}$ are complex numbers; the right hand side is obviously a finite sum. If the coefficients $a_{..}$ are taken to be smooth functions on, say, $\mathbb{R}^m$, the resulting functions $f(x,\theta)$ are referred to as smooth functions on the superspace $\mathbb{R}^{m|n}$; the algebra of these functions is denoted as $C^\infty(\mathbb{R}^{m|n})$. With parity zero assigned to the variables x^a (a = 1,..,m) and one to the $\theta^\alpha, C^\infty(R^{m|n})$ is a supercommutative superalgebra. Restricting the variables x^a to an open subset U of R^m, one obtains the superdomain $U^{m|n}$ and the superalgebra $C^\infty(U^{m|n})$ in the above-mentioned sense. Gluing such superdomains appropriately, one obtains the objects called supermanifolds. These are the objects serving as phase spaces in superclassical mechanics. We shall, for simplicity, restrict ourselves to the simplest supermanifolds $\mathbb{R}^{m|n}$ and take, as system algebra, $\mathcal{A} = C^\infty(\mathbb{R}^{m|n})$. A *-operation is assumed to be defined on $\mathcal{A}$ with respect to which the 'coordinate variables' x^a and θ^α are assumed to be hermitian.

States in superclassical mechanics are normalized positive linear functionals on $\mathcal{A} = C^\infty(\mathbb{R}^{m|n})$; they are generalizations of the states in classical statistical mechanics given by

$$\phi(f) = \int_{R^{m|n}} f(x,\theta)d\mu(x,\theta)$$

where the measure μ satisfies the normalization and positivity conditions

$$1 = \phi(1) = \int d\mu(x,\theta); \tag{3}$$

$$0 \leq \int ff^* d\mu \quad \text{for all } f \in \mathcal{A}. \tag{4}$$

For states admitting a density function, we have

$$d\mu(x,\theta) = \rho(x,\theta)d\theta^1...d\theta^n d^m x.$$

To ensure real expectation values for observables, $\rho(.,.)$ must be even (odd) for n even (odd). The condition (3) implies that

$$\rho(x,\theta) = \rho_0(x)\theta^n...\theta^1 + \text{terms of lower order in } \theta \tag{5}$$

where ρ_0 is a probability density on $\mathbb{R}^m$. The inequality (4) implies inequalities involving the coefficient functions on the right in Eq.(5). They eventually determine a convex domain $\mathcal{D}$ in a real vector space. Pure states correspond to points on the boundary of $\mathcal{D}$ (which is generally not a manifold).

The CC condition is generally not satisfied by the pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}))$ in super-classical mechanics. To show this, it is adequate to give an example (Berezin [7]). Taking $\mathcal{A} = C^\infty(\mathbb{R}^{0|3}) \equiv \mathcal{G}_3$, we have a general state represented by a density function of the form

$$\rho(\theta) = \theta^3\theta^2\theta^1 + c_\alpha\theta^\alpha.$$

The inequality (4) with $f = a\theta^1 + b\theta^2$ (with a and b arbitrary complex numbers) implies $c_3 = 0$; similarly, $c_1 = c_2 = 0$, giving, finally

$$\rho(\theta) = \theta^3\theta^2\theta^1.$$

There is only one possible state which must be pure. This state does not distinguish, for example, observables $f = a + b\theta^1\theta^2$ with the same 'a' but different 'b', thus verifying the assertion made above.

Note. It would not do to stipulate exclusion of θ-dependence in observables. Treatments in superclassical mechanics, of particles with spin, for example, employ θ-dependent observables (Berezin [7], Dass [13]).

The fermionic extension of classical mechanics, therefore, appears to have a fundamental inadequacy; no wonder, therefore, that it does not appear to be realized by systems in nature.

The argument presented above, however, does not apply to the $n = \infty$ case.

2.4. Systems with configuration space; lacalizability

We shall now consider the class of systems each of which has a configuration space (say, M) associated with it and it is meaningful to ask questions about the localization of the system in subsets of M. To start with, we shall take M to be a topological space and take the permitted domains of localization to belong to B(M), the family of Borel subsets of M.

Some good references containing detailed treatment of localization in conventional approaches are (Newton and Wigner [36], Wightman [45], Varadarajan [43], Bacry [3]). We shall follow a relatively more economical path exploiting some of the constructions described above and in I.

We shall say that a system S [with associated symplectic superalgebra $(\mathcal{A}, \omega)$] is *localizable* in M if we have a positive observable-valued measure (as defined in section 2.1 above) on the measurable space (M,B(M)), which means that, corresponding to every subset $D \in B(M)$, there is a positive observable P(D) in $\mathcal{A}$ satisfying the three conditions
(i) $P(\emptyset) = 0$; (ii) P(M) = I;
(iii) for any countable family of mutually disjoint sets $D_i \in B(M)$,

$$P(\cup_i D_i) = \sum_i P(D_i). \tag{6}$$

For such a system, we can associate, with any state ϕ, a probability measure μ_ϕ on the measurable space $(M, B(M))$ defined by [see Eq.(1)]

$$\mu_\phi(D) = \phi(P(D)), \tag{7}$$

making the triple $(M, B(M), \mu_\phi)$ a probability space. The quantity $\mu_\phi(D)$ is to be interpreted as the probability of the system, given in the state ϕ, being found (on observation/measurement) in the domain D.

Generally it is of interest to consider localizations having suitable invariance properties under a transformation group G. Typically G is a topological group with continuous action on M assigning, to each $g \in G$, a bijection $T_g : M \to M$ such that, in obvious notation, $T_g T_{g'} = T_{gg'}$ and $T_e = id_M$; it also has a symplectic action on $\mathcal{A}$ and $\mathcal{S}(\mathcal{A})$ given by the mappings $\Phi_1(g)$ and $\Phi_2(g)$ introduced in section 3.5 of I [$\Phi_1(g)$, for every $g \in G$, is a canonical transformation of $\mathcal{A}$ and $\Phi_2(g) = ([\Phi_1(g)]^{-1})^T$ acts on states].

. The localization in M described above will be called *G-covariant* (or, loosely, G-invariant) if

$$\Phi_1(g)(P(D)) = P(T_g(D)) \ \ \forall g \in G \text{ and } D \in B(M). \tag{8}$$

Proposition 2.2 *In a G-covariant localization as described above, the localization probabilities (7) satisfy the covariance condition*

$$\mu_{\Phi_2(g)(\phi)}(D) = \mu_\phi(T_{g^{-1}}(D)) \text{for all } \phi \in \mathcal{S}(\mathcal{A}) \text{ and } D \in B(M). \tag{9}$$

Proof. We have

$$\begin{aligned} \mu_{\Phi_2(g)(\phi)}(D) &= < \Phi_2(g)(\phi), P(D) >=< \phi, \Phi_1(g^{-1})(P(D)) > \\ &= < \phi, P(T_{g^{-1}}(D)) >= \mu_\phi(T_{g^{-1}}(D)). \ \ \square \end{aligned}$$

In most practical applications, M is a manifold and G a Lie group with smooth action on M and a Poisson action on the symplectic superalgebra $(\mathcal{A}, \omega)$. In this case, the 'hamiltonian' h_ξ corresponding to an element ξ of the Lie algebra $\mathcal{G}$ of G is an observables which serves, through Poisson brackets, as the infinitesimal generator of the one-parameter group of canonical transformations induced by the action of the one-parameter group generated by ξ on the system algebra $\mathcal{A}$ (I, section 3.5). The Poisson brackets between these hamiltonins correspond to the commutation relations in $\mathcal{G}$ [se Eq.(59) in I and Eq.(13) below].

In Hilbert space QM, the problem of G-covariant localization is traditionally formulated in terms of the so-called 'systems of imprimitivity' (Mackey [33], Varadarajan [43], Wightman [45]). We are operating in the more general algebraic setting trying to exploit the machinery of noncommutative symplectics developed in I. Clearly, there is considerable scope for mathematical developments in this context parallel to those relating to systems of imprimitivity. We shall, however, restrict ourselves to some essential developments relevant to the treatment of localizable elementary systems (massive particles) later.

We shall be mostly concerned with $M = \mathbb{R}^n$ (equipped with the Euclidean metric). In this case, one can consider averages of the form (denoting the natural coordinates on $\mathbb{R}^n$ by x_j)

$$\int_{\mathbb{R}^n} x_j d\mu_\phi(x), \quad j = 1, ..., n. \tag{10}$$

It is natural to introduce *position/configuration observables* X_j such that the quantity (10) is $\phi(X_j)$. Let E_n denote the (identity component of) Euclidean group in n dimensions and let $p_j, m_{jk}(= -m_{kj})$ be its generators satisfying the commutation relations

$$\begin{aligned} [p_j, p_k] = 0, \quad [m_{jk}, p_l] = \delta_{jl} p_k - \delta_{kl} p_j \\ [m_{jk}, m_{pq}] = \delta_{jp} m_{kq} - \delta_{kp} m_{jq} - \delta_{jq} m_{kp} + \delta_{kq} m_{jp}. \end{aligned} \tag{11}$$

We shall say that a system S with configuration space $\mathbb{R}^n$ has *concrete Euclidean-covariant localization* if it is localizable as above in $\mathbb{R}^n$ and
(i) it has position observables $X_j \in \mathcal{A}$ such that, in any state ϕ,

$$\phi(X_j) \quad = \quad \int_{R^n} x_j d\mu_\phi(x); \tag{12}$$

(The term 'concrete' is understood to imply this condition.)
(ii) the group E_n has a Poisson action on $\mathcal{A}$ so that we have the hamiltonians P_j, M_{jk} associated with the generators p_j, m_{jk} such that

$$\begin{aligned} \{P_j, P_k\} = 0, \quad \{M_{jk}, P_l\} = \delta_{jl} P_k - \delta_{kl} P_j \\ \{M_{jk}, M_{pq}\} = \delta_{jp} M_{kq} - \delta_{kp} M_{jq} - \delta_{jq} M_{kp} + \delta_{kq} M_{jp}; \end{aligned} \tag{13}$$

(iii) the covariance condition (9) holds with the Euclidean group action on $\mathbb{R}^n$ given by

$$T_{(R,a)}x = Rx + a, \quad R \in SO(n),\ a \in \mathbb{R}^n. \tag{14}$$

Proposition 2.3 *For supmech systems with concrete Euclidean - covariant localization, the infinitesimal Euclidean transformations of the localization observables X_j are given by the PB relations*

$$\{P_j, X_k\} = \delta_{jk} I, \quad \{M_{jk}, X_l\} = \delta_{jl} X_k - \delta_{kl} X_j. \tag{15}$$

Proof. Using Eq.(12) with ϕ replaced by $\phi' = \Phi_2(g)(\phi)$, we have

$$\phi'(X_j) = \int x_j d\mu_{\phi'}(x) = \int x_j d\mu_\phi(x') = \int (x'_j - \delta x_j) d\mu_\phi(x')$$

where $x' \equiv T_{g^{-1}}(x) \equiv x + \delta x$ and we have used Eq.(9) to write $d\mu_{\phi'}(x) = d\mu_\phi(x')$. [Application of the transformation rule for integration over a measure (DeWitt-Morette and Elworthy [18]; p.130) gives the same result.] Writing $\phi' = \phi + \delta\phi$ and taking T_g to be a general infinitesimal transformation generated by $\epsilon\xi = \epsilon^a \xi_a$, we have [recalling Eq.(53) of I]

$$-(\delta\phi)(X_j) = \epsilon\phi(\{h_\xi, X_j\}) = \int_{R^n} \delta x_j d\mu_\phi(x). \tag{16}$$

For translations, with $\xi = p_k,\ h_{p_k} = P_k,\ \delta x_j = \epsilon\delta_{jk}$, Eq.(16) gives

$$\phi(\{P_k, X_j\}) = \delta_{jk} = \delta_{jk}\phi(I).$$

Since this holds for all $\phi \in \mathcal{S}(\mathcal{A})$, we have the first of the equations (15). The second equation is similarly obtained by taking, in obvious notation, $\epsilon\xi = \frac{1}{2}\epsilon_{jk} m_{jk}$ and

$$\delta x_l = \epsilon_{lk} x_k = \epsilon_{jk}\delta_{jl} x_k = \frac{1}{2}\epsilon_{jk}(\delta_{jl} x_k - \delta_{kl} x_j). \ \square$$

The hamiltonians P_j and M_{jk} will be referred to as the *momentum* and *angular momentum* observables of the system S. It should be noted that the PBs obtained above do not include the expected relations $\{X_j, X_k\} = 0$; these relations, as we shall see in the following subsection, come from the relativity group. [Recall that, in the treatments of localalization based on systems of imprimitivity, the commutators $[X_j, X_k] = 0$ appear because there the analogues of the objects P(D) are assumed to be projection operators satisfying the relation $P(D)P(D') = P(D \cap D')(= P(D')P(D))$. In our more general approach, we do not have such a relation.]

2.5. Elementary systems; Particles

We shall now obtain, in the framework of supmech, the fundamental observables relating to the characterization/labelling and kinematics of a particle. Relativity group will be seen to play an important role in this context.

Particles are irreducible entities localized in 'space' and their dynamics involves 'time'. Their description, therefore, belongs to the subdomain of supmech admitting space-time descriptions of systems. The space-time M will be assumed here to be a (3+1)- dimensional differentiable manifold equipped with a suitable metric to define spatial distances and time-intervals. A *reference frame* is an atlas on M providing a coordinatization of its points. *Observers* are supposedly intelligent beings employing reference frames for doing concrete physics; they will be understood to be in one-to-one correspondence with reference frames.

To take into consideration observer-dependence of observables, we adopt the *principle of relativity* formalized as follows :

(i) There is a preferred class of reference frames whose space-time coordinatisations are related through the action of a connected Lie group G_0 (the *relativity group*).
(ii) The relativity group G_0 has a hamiltonian action on the symplectic superalgebra $(\mathcal{A}, \omega)$ [or the generalized symplectic superalgebra $(\mathcal{A}, \mathcal{X}, \omega)$ (se section 3.7 of I) in appropriate situations] associated with a system.
(iii) All reference frames in the chosen class are physically equivalent in the sense that the fundamental equations of the theory are covariant with respect to the G_0-transformations of the relevant variables.

We shall call such a scheme G_0-*relativity* and systems covered by it G_0-*relativistic*. In the present work, G_0 will be assumed to have the one-parameter group $\mathcal{T}$ of time translations as a subgroup. This allows us to relate the Heisenberg and Schrödinger pictures of dynamics corresponding to two observers O and O$'$ through the symplectic action of G_0 by following the strategy adopted in (Sudarshan and Mukunda [42]; referred to as SM below). Showing the observer dependence of the algebra elements explicitly, the two Heisenberg picture descriptions A(O,t) and A(O$'$,t$'$) of an element A of $\mathcal{A}$ can be related through the sequence (assuming a common zero of time for the two observers)

$$A(O,t) \longrightarrow A(O,0) \longrightarrow A(O',0) \longrightarrow A(O',t')$$

where the first and the last steps involve the operations of time translations in the two frames. We shall be concerned only with the symplectic action of G_0 on $\mathcal{A}$ involved in the middle step.

To formalize the notion of a (relativistic, quantum) particle as an irreducible entity, Wigner [47] introduced the concept of an 'elementary system' as a quantum system whose Hilbert space carries a projective unitary irreducible representation of the Poincaré group. The basic idea is that the state space of an elementary system should not admit a decomposition into more than one invariant (under the action of the relevant relativity group) subspaces. Following this idea, elementary systems in classical mechanics (SM; Alonso [1]) have been defined in terms of a transitive action of the relativity group on the phase space of the system. Our treatment of elementary systems in supmech will cover classical and quantum elementary systems as special cases.

A system S having associated with it the symplectic triple $(\mathcal{A}, \mathcal{S}_1, \omega)$ will be called an *elementary system* in G_0-relativity if it is a G_0-relativistic system such that the action of G_0 on the space $\mathcal{S}_1$ of its pure states is transitive. Formally, an elementary system may be represented as a collection $\mathcal{E} = (G_0, \mathcal{A}, \mathcal{S}_1, \omega, \Phi)$ where $\Phi = (\Phi_1, \Phi_2)$ are mappings as in section 3.5 of I implementing the G_0-actions — Φ_1 describing a hamiltonian action on $(\mathcal{A}, \omega)$ and $\Phi_2[= (\tilde{\Phi}^{-1})]$ a transitive action on $\mathcal{S}_1$.

Proposition 2.4 *In the G_0-relativity scheme, a G_0-invariant observable must be a multiple of the unit element.*

Proof. Let Q be such an observable and ϕ_1, ϕ_2 two pure states. The transitive action of G_0 on $\mathcal{S}_1$ implies that $\phi_2 = \Phi_2(g)(\phi_1)$ for some $g \in G_0$. We have

$$< \phi_2, Q > = < \Phi_2(g)(\phi_1), Q > = < \phi_1, \Phi_1(g^{-1})(Q) >=< \phi_1, Q >$$

showing that the expectation value of Q is the same in every pure state. Denoting this common expectation value of Q by q (we shall call it the *value* of Q for the system), we have, by the CC condition, Q = qI. □

This has the important implication that, for an elementary system, a Poisson action [of G_0 or of its projective group $\hat{G}_0$ (see section 3.5 of I)] is always available; this is because, if G_0 does not admit Poisson action, the values $\alpha(\xi, \eta)$ of the cocycle α of section 3.5 of I (where ξ, η are elements of the Lie algebra $\mathcal{G}_0$ of G_0), since they have vanishing Poisson brackets (PBs) with all elements of $\mathcal{A}$ (hence with the hamiltonians corresponding to G_0), are multiples of the unit element and the hamiltonian action of G_0 can be extended to a Poisson action of $\hat{G}_0$. [See the discussion following Eq.(62) of I.] In the remainder of this subsection, $\hat{G}_0$ will stand for the effective relativity group which will be G_0 or its projective group depending on whether or not G_0 admits Poisson action on $\mathcal{A}$.

Let ξ_a (a = 1,..,r) be a basis in the Lie algebra $\hat{\mathcal{G}}_0$ of $\hat{G}_0$ satisfying the commutation relations $[\xi_a, \xi_b] = C_{ab}^c \xi_c$. Corresponding to the generators ξ_a, we have the hamiltonians $h_a \equiv h_{\xi_a}$ in $\mathcal{A}$ satisfying the PB relations

$$\{h_a, h_b\} = C_{ab}^c \; h_c. \tag{17}$$

These relations are the same for all elementary systems in G_0-relativity.

In classical mechanics, one has an isomorphism between the symplectic structure on the symplectic manifold of an elementary system and that on a coadjoint orbit in $\hat{\mathcal{G}}_0^*$ (the conjugate space of the Lie algebra $\hat{\mathcal{G}}_0$). In our case, the state spaces of elementary systems and coadjoint orbits of relativity groups are generally spaces of different types and the question of an isomorphism does not arise. The appropriate relation in supmech corresponding to the above mentioned relation in classical Hamiltonian mechanics is given by proposition 2.5 below. Adopting/(adapting from) the notation of section 3.6 of I, we have the mapping $h : \hat{\mathcal{G}}_0 \rightarrow \mathcal{A}$ given by $h(\xi) = h_\xi$; the noncommutative momentum map is the restriction to $\mathcal{S}_1$ of the transposed map $\tilde{h} : \mathcal{A}^* \rightarrow \hat{\mathcal{G}}_0^*$:

$$< \tilde{h}(\phi), \xi > = < \phi, h(\xi) > = < \phi, h_\xi > \quad \text{for all } \phi \in \mathcal{S}_1. \tag{18}$$

The equivariance condition for the noncommutative momentum map $\tilde{h}$ [(68) of I] is

$$\tilde{h}(\Phi_2(g)\phi) = Cad_g(\tilde{h}(\phi)) \tag{19}$$

where Cad stands for the co-adjoint action of $\hat{G}_0$ on $\hat{\mathcal{G}}_0^*$.

Proposition 2.5 *Adopting the notations introduced above in the context of elementary systems in G_0-relativity, we have*
(a) the $\tilde{h}$-images of pure states of an elementary system in supmech are co-adjoint orbits;
(b) the coordinates $u_a(g)$ of a general point of the co-adjoint orbit corresponding to the pure state ϕ [defined by $Cad_g[\tilde{h}(\phi)] = u_a(g)\lambda^a$ where $\{\lambda^a\}$ is the dual basis in $\hat{\mathcal{G}}_0^*$ corresponding to the basis $\{\xi_a\}$ in $\hat{\mathcal{G}}_0$] are given by

$$u_a(g) =< \phi, \Phi_1(g^{-1})h_a > . \tag{20}$$

Proof. Part (a) follows immediately from Eq.(19) and the transitivity of the $\hat{G}_0$-action on the pure states.
Part(b). We have

$$u_a(g) = < Cad_g[\tilde{h}(\phi)], \xi_a > = < \tilde{h}[\Phi_2(g)(\phi)], \xi_a > = < \phi, \Phi_1(g^{-1})h_a > . \ \square$$

Eq.(20) shows that the transformation properties of the hamiltonians h_a are directly related to those of the corresponding coordinates (with respect to the dual basis) of points on the relevant co-adjoint orbit. This is adequate to enable us to to use the descriptions of the relevant co-adjoint actions in (Alonso [1]) and draw parallel conclusions.

For the treatment of elementary systems in a given relativity scheme, we shall adopt the following strategy :

(i) Obtain the PBs (17).

(ii) Use these PBs to identify some *fundamental observables* [i.e. those which cannot be obtained from other observables (through algebraic relations or PBs)]. These include observables (like mass) that Poisson-commute with all h_as and the momentum observables (if the group of space translations is a subgroup of the relativity group considered).

(iii) Determine the transformation laws of h_as under finite transformations of G_0 following the relevant developments in (SM; Alonso [1]). Use these transformation laws to identify the G_0-invariants and some other fundamental observables (the latter are configuration and spin observables in the schemes of Galilean and special relativity). The values of the invariant observables serve to characterize/label an elementary system.

(iv) The system algebra $\mathcal{A}$ for an elementary system is to be taken as the one generated by the fundamental observables and the identity element.

(v) Obtain (to the extent possible) the general form of the Hamiltonian as a function of the fundamental observables as dictated by the PB relations (17).

For illustration, we consider the scheme of Galilean relativity.

Nonrelativistic elementary systems

In the nonrel;ativistic domain, the relativity group G_0 is the Galilean group of transformations of the Newtonian space-time $\mathbb{R}^3 \times \mathbb{R}$ given by

$$g = (b, a, v, R) : (x, t) \mapsto (Rx + tv + a, t + b) \tag{21}$$

where $R \in SO(3),\ v \in \mathbb{R}^3,\ a \in \mathbb{R}^3$ and $b \in \mathbb{R}$. Choosing a basis of the 10-dimensional Lie algebra $\mathcal{G}_0$ of G_0 in accordance with the representation

$$g = exp(b\mathcal{H})\ exp(a.\mathcal{P})\ exp(v.\mathcal{K})\ exp(w.\mathcal{J})$$

the commutators $[\mathcal{J}_j, \mathcal{J}_k], [[\mathcal{J}_j, \mathcal{K}_j]$ are standard; the nontrivial commutators are

$$[\mathcal{K}_j, \mathcal{H}] = \mathcal{P}_j, \quad [\mathcal{K}_j, \mathcal{P}_k] = 0. \tag{22}$$

[The last one should be obvious from Eq.(21).]

Recalling the discussion relating to Poisson action of Lie groups on symplectic superalgebras in section 3.5 of I, the cohomology group $H_0^2(\mathcal{G}_0, \mathbb{R})$ does not vanish (implying non-implementability of a Poisson action of G_0) and has dimension one (Cariñena, Santander [12]; Alonso [1]; Guillemin, Sternberg [24]; SM). Choosing the representative cocycle in $Z_0^2(\mathcal{G}_0, \mathbb{R})$ as $\eta(\mathcal{K}_j, \mathcal{P}_k) = -\delta_{jk}\mathcal{M}$, Eq. (63) of I implies the replacement of the second equation in (22) by

$$[\mathcal{K}_j, \mathcal{P}_k] = -\delta_{jk}\mathcal{M}. \tag{23}$$

Supplementing the so modified commutation relations of $\mathcal{G}_0$ with the vanishing commutators of $\mathcal{M}$ with the ten generators of $\mathcal{G}_0$, we obtain the commutation relations of the 11-dimensional Lie algebra $\hat{\mathcal{G}}_0$ of the projective group $\hat{G}_0$ of the Galilean group G_0.

The hamiltonians J_i, K_i, P_i, H, M corresponding to the generators $\mathcal{J}_i, \mathcal{K}_i, \mathcal{P}_i (i = 1, 2, 3), \mathcal{H}, \mathcal{M}$ of $\hat{G}_0$ [so that $h_{\mathcal{P}_i} = P_i$ etc] satisfy the Poisson bracket relations (SM)

$$\{J_i, J_j\} = -\epsilon_{ijk} J_k, \quad \{J_i, K_j\} = -\epsilon_{ijk} K_k, \quad \{J_i, P_j\} = -\epsilon_{ijk} P_k$$
$$\{K_i, H\} = -P_i, \quad \{K_i, P_j\} = -\delta_{ij} M; \tag{24}$$

all other PBs vanish. By the argument presented above, we must have M= mI, $m \in \mathbb{R}$. We shall identify m as the mass of the elementary system. The condition $m \geq 0$ will follow later from an appropriate physical requirement. The objects P_i and J_i, being generators of the Euclidean subgroup E_3 of G_0, are the momentum and angular momentum observables of subsection 2.4 above.

The transformation laws of the hamiltonians of $\hat{G}_0$ under its adjoint action (SM; Alonso [1]) yield the following three independent invariants

$$M, \quad C_1 \equiv 2MH - \mathbf{P}^2, \quad C_2 \equiv (M\mathbf{J} - \mathbf{K} \times \mathbf{P})^2. \tag{25}$$

Of these, the first one is obvious; the vanishing of PBs of C_1 with all the hamiltonians is also easily checked. Writing $C_2 = B_j B_j$ where

$$B_j = MJ_j - \epsilon_{jkl} K_k P_l,$$

it is easily verified that

$$\{J_j, B_k\} = -\epsilon_{jkl} B_l, \quad \{K_j, B_k\} = \{P_j, B_k\} = \{H, B_k\} = 0$$

which finally leads to the vanishing of PBs of C_2 with all the hamiltonians. The values of these three invariants characterize a Galilean elementary system in supmech.

We henceforth restrict ourselves to elementary systems with $m \neq 0$. Defining $X_i = m^{-1} K_i$, we have

$$\{X_j, X_k\} = 0, \quad \{P_j, X_k\} = \delta_{jk} I, \qquad \{J_j, X_k\} = -\epsilon_{jkl} X_l. \tag{26}$$

Comparing the last two equations above with the equations (15)(for n=3), we identify X_j with the position observables of section 2.4. Note that the fact that the X_js mutually Poisson-commute comes from the relativity group.

Writing $\mathbf{S} = \mathbf{J} - \mathbf{X} \times \mathbf{P}$, we have $C_2 = m^2 \mathbf{S}^2$. We have the PB relations

$$\{S_i, S_j\} = -\epsilon_{ijk} S_k, \quad \{S_i, X_j\} = 0 = \{S_i, P_j\}. \tag{27}$$

We identify **S** with the internal angular momentum or spin of the elementary system.

The invariant quantity

$$U \equiv \frac{C_1}{2m} = H - \frac{\mathbf{P}^2}{2m} \tag{28}$$

is interpreted as the *internal energy* of the elementary system; its appearance as one of the invariant observables of a Galilean elementary system reflects the possibility that such an elementary system may have an internal dynamics involving dynamical variables which are invariant under the action of the Galilean group. It is the appearance of this quantity (which plays no role in Newtonian mechanics) which is responsible for energy being defined in Newtonian mechanics only up to an additive constant.

Writing $\mathbf{S}^2 = \sigma I$ and U = u I, we see that Galilean elementary systems with $m \neq 0$ can be taken to be characterized/labelled by the parameters m, σ and u. The fundamental kinematical observables are X_j, P_j and S_j (j=1,2,3). The system algebra $\mathcal{A}$ of a nonrelativistic elementary system is assumed to be the one generated by the fundamental observables and the identity element.

Particles are defined as the elementary systems with u = 0. Eq.(28) now gives

$$H = \frac{\mathbf{P}^2}{2m} \tag{29}$$

which is the Hamiltonian for a free Galilean particle in supmech.

Note. (i) Full Galilean invariance (more generally, full invariance under a relativity group) applies only to an isolated system. Interactions/(external influences) are usually described with (explicit or implicit) reference to a fixed reference frame or a restricted class of frames. For example, the interaction described by a central potential implicitly assumes that the center of force is at the origin of axes of the chosen reference frame.
(ii) In the presence of external influences, invariance under space translations is lost and the PB $\{H, P_i\} = 0$ must be dropped. For a spinless particle, the Hamiltonian, being an element of the system algebra generated by the fundamental observables **X** and **P**, has the general form

$$H = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X}, \mathbf{P}). \tag{30}$$

In most practical situations, V is a function of **X** only.

The Hamiltonian was assumed in section 3.4 of I to be bounded below (in the sense that its expectation values in all states are bounded below); this rules out the case $m < 0$ because, by Eq.(29), this will allow arbitrarily large negative expectation values for energy. (Expectation values of the observable $\mathbf{P}^2$ are expected to have no upper bound.)

Recalling the demonstration of the classical Hamiltonian mechanics as a subdiscipline of NHM in section 3.4 of I, the classical Hamiltonian system for a massive spinless Galilean particle is easily seen to be the special case of the corresponding supmech Hamiltonian system with $\mathcal{A} = C^\infty(\mathbb{R}^6)$. The corresponding quantum system is also (recalling the example in section 3.3 of I) a special case of a supmech Hamiltonian system with the system algebra generated by the position and momentum observables in Schrödinger theory. More detailed treatment (with justification of the Schrödinger theory) will appear in section 3.4.

2.6. Noncommutative Noether invariants of the Galilean group for a free massive spinless particle

In section 3.8 of I, the noncommutative analog of the symplectic version of Noether's theorem was proved. Given a noncommutative Hamiltonian system $(\mathcal{A}, \omega, H)$, one constructs a presymplectic algebra $(\mathcal{A}^e, \Omega)$ where

$$\mathcal{A}^e = C^\infty(\mathbb{R}) \otimes \mathcal{A} \quad \text{and} \quad \Omega = 1 \otimes \omega + d_1 t \otimes d_2 H.$$

Here the real line $\mathbb{R}$ is the carrier space of the evolution parameter ('time') t and d_1 and d_2 are the exterior derivatives in the differential calculi based on the algebras $C^\infty(\mathbb{R})$ and $\mathcal{A}$ respectively. Considering t and H as elements of $\mathcal{A}^e$ and employing the d operator for $\mathcal{A}^e$ [defined in terms of d_1 and d_2 through Eq.(28) of I], we have [see Eq.(72) of I]

$$d_1 t \otimes d_2 H = dt \wedge dH = -dH \wedge dt.$$

When a Lie group G with Lie algebra $\mathcal{G}$ has a hamiltonian action on the presymplectic algebra $(\mathcal{A}^e, \Omega)$, the noncommutative Noether's theorem predicts the constancy in 'time' of the hamiltonians $\hat{h}_\xi$ corresponding to the elements $\xi \in \mathcal{G}$ defined by [see Eq.(76) of I]

$$i_{\hat{Z}_\xi} \Omega = -d\hat{h}_\xi. \tag{31}$$

where $\hat{Z}_\xi$ is the superderivation of $\mathcal{A}^e$ serving as the infinitesimal generator of the 1-parameter subgroup of the automorphisms of $(\mathcal{A}^e, \Omega)$ generated by ξ.

Here we are interested in the explicit construction of the Noether invariants $\hat{h}_\xi$ when G is the projective group $\hat{G}_0$ of the Galilean group G_0 and $\mathcal{A}$ the algebra generated by the fundamental observables X_j and P_j (j=1,2,3) of a free nonrelativistic spinless particle and the identity element I and H is given by Eq.(29). Construction of these objects involves consideration of the transformation of the time variable which was bypassed in the previous subsection. The formalism of section 3.8 of I has obvious limitations in this regard because time was treated as an external parameter in the Poisson brackets employed there. This, however, is no problem for the Galilean group where the only admitted transformations of the time variable are translations.

We shall now obtain an equation that will be useful for the identification of the objects $\hat{h}_\xi$. Let $\xi \in \hat{\mathcal{G}}_0$ generate an infinitesimal transformation giving $\delta t = \epsilon f(t)$ (and possibly some changes in other quantities). [In view of the limitations of the formalism mentioned above, arguments other than t for the function f have been excluded.] The relation between the induced derivations Z_ξ on $\mathcal{A}$ and $\hat{Z}_\xi$ on $\mathcal{A}^e$ is given by

$$\hat{Z}_\xi = Z_\xi + f(t)\frac{\partial}{\partial t}. \tag{32}$$

We have $Z_\xi = Y_{h_\xi}$ where h_ξ is the hamiltonian corresponding to ξ in in the Poisson action of $\hat{G}_0$ on $\mathcal{A}$ (see section 3.5 of I). We look for the quantity $\hat{h}_\xi$ (the prospective Noether invariant) such that Eq.(31) holds. (Finding such a quantity will establish invariance of Ω under the relevant group action and also determine the corresponding Noether invariant.) Equation (32) above and the equation defining Ω above now give the desired relation (here $\tilde{\omega} \equiv 1 \otimes \omega$)

$$\begin{aligned} i_{\hat{Z}_\xi}\Omega &= i_{Z_\xi}\tilde{\omega} - i_{Z_\xi}(dH)dt + f(t)dH \\ &= -dh_\xi - \{h_\xi, H\}dt + f(t)dH. \end{aligned} \tag{33}$$

To obtain, for each ξ in the chosen basis of $\hat{\mathcal{G}}_0$, a 'hamiltonian' $\hat{h}_\xi$ such that Eq.(31) holds, we must show the exactness of the form on the right hand side of Eq.(33). We have

(i)for rotations ($\xi = \mathcal{J}_i, h_\xi = J_i$) f(t) =0, $\{h_\xi, H\} = 0$, giving $\hat{h}_\xi = h_\xi = J_i$;

(ii) for space translations ($\xi = \mathcal{P}_i, h_\xi = P_i$) f(t) =0, $\{h_\xi, H\} = 0$, giving $\hat{h}_\xi = h_\xi = P_i$;

(iii) for Galilean boosts ($\xi = \mathcal{K}_i, h_\xi = K_i = mX_i$) f(t) = 0, $\{K_i, H\} = -P_i$ giving $\hat{h}_\xi = mX_i - P_i t$;

(iv) for time translations ($\xi = \mathcal{H}, h_\xi = H$) f(t)=1, $\{h_\xi, H\} = 0$, giving $\hat{h}_\xi = H$;

(v) for the one-parameter group generated by $\mathcal{M}$ ($\xi = \mathcal{M},\ h_\xi = M = mI$), f(t) = 0, $\{h_\xi, H\} = 0$, giving $\hat{h}_\xi = M = mI$.

Finally, we have

Proposition 2.6 *The noncommutative Noether invariants of projective group $\hat{G}_0$ of the Galilean group G_0 for a free nonrelativistic spinless particle of mass m are*

$$\mathbf{J},\ \mathbf{P},\ m\mathbf{X} - \mathbf{P}t,\ \ H,\ \ M = mI. \tag{34}$$

Note that the first four of these are (up-to signs) the supmech avatars of those in (Souriau [41]; p.162).

Note. If, instead of taking $X_j = m^{-1}K_j$ in a treatment bypassing the involvement of time in the symplectic transformations as above, we had proceeded to identify observables through Noether invariants, we would have got the position observable as m^{-1} times the time-independent term in the second entry in Eq.(34).

3. Quantum Systems

We now take up a systematic study of the 'quantum systems' defined as supmech Hamiltonian systems with non-supercommutative system algebras. Theorem (2) of I dictates these systems to have a standard symplectic structure characterized by a universal real parameter of the dimension of action; we shall identify it with the Planck constant $\hbar$. We first treat quantum systems in the general algebraic setting. We then employ the CC condition to show that they inevitably have Hilbert space based realizations, generally admitting commutative superselection rules. The autonomous development of the Hilbert space QM of 'standard quantum systems' (those with finitely generated system algebras) is then presented. This is followed by straightforward treatments of Hilbert space quantum mechanics of material particles and of quantum - classical correspondence (the latter highlighting the transparency resulting from the fact that both QM and classical mechanics are subdisciplines of supmech).

3.1. The general algebraic formalism for quantum systems

Formally, a *quantum system* is a supmech Hamiltonian system $(\mathcal{A}, \mathcal{S}_1, \omega, H)$ in which the system algebra $\mathcal{A}$ is non-supercommutative and ω is the *quantum symplectic form* ω_Q given by [see Eq.(44) of I]

$$\omega_Q = -i\hbar\omega_c \tag{35}$$

where ω_c is the canonical 2-form of $\mathcal{A}$ defined by Eq.(39) of I. [We have, in the terminology of section 3.3 of I, the quantum symplectic structure with parameter $b = -i\hbar$. If the superalgebra $\mathcal{A}$ is not 'special' (i.e. not restricted to have only inner superderivations), we have a generalized symplectic structure as mentioned at the end of section 4 in I.] This is the only place where we put the Planck constant 'by hand' (the most natural place to do it — such a parameter is *needed* here to give the symplectic form ω_Q the dimension of action); its appearance at all conventional places (canonical commutation relations, Heisenberg and Schrödinger equations, etc) will be automatic.

The *quantum Poisson bracket* implied by the quantum symplectic form (34) is [see Eq.(43) of I]

$$\{A, B\} = (-i\hbar)^{-1}[A, B]. \tag{36}$$

Recalling that the bracket [,] represents a supercommutator, the bracket on the right in Eq.(36) is an anticommutator when both A and B are odd/fermionic and a commutator in all other situations with homogeneous A,B.

A *quantum canonical transformation* is an automorphism Φ of the system algebra $\mathcal{A}$ such that $\Phi^*\omega_Q = \omega_Q$. Now, by Eq.(12) of I,

$$(\Phi^*\omega_Q)(X_1, X_2) = \Phi^{-1}[\omega_Q(\Phi_* X_1, \Phi_* X_2)] \tag{37}$$

where X_1, X_2 are inner superderivations, say, D_A and D_B. We have [recalling Eq.(3) of I]

$$(\Phi_* D_A)(B) = \Phi[D_A(\Phi^{-1}(B)] = \Phi([A, \Phi^{-1}(B)]) = [\Phi(A), B]$$

which gives

$$\Phi_* D_A = D_{\Phi(A)}. \tag{38}$$

Eq.(37) above and Eq.(39) of I (i.e. $\omega_c(D_A, D_B) = [A, B]$) now give

$$\Phi(i[A, B]) = i[\Phi(A), \Phi(B)] \tag{39}$$

which shows, quite plausibly, that quantum canonical transformations are (in the present algebraic setting — we have not yet come to the Hilbert space) the automorphisms of the system algebra preserving the quantum PBs.

The evolution of a quantum system in time is governed, in the Heisenberg picture, by the noncommutative Hamilton's equation (49) of I which now becomes the familiar *Heisenberg equation* of motion

$$\frac{dA(t)}{dt} = (-i\hbar)^{-1}[H, A(t)]. \tag{40}$$

In the Schrödinger picture, the time dependence is carried by the states and the evolution equation (51) of I takes the form

$$\frac{d\phi(t)}{dt}(A) = (-i\hbar)^{-1}\phi(t)([H, A]) \tag{41}$$

which may be called the *generalized von Neumann equation.*

We shall call two quantum systems $\Sigma = (\mathcal{A}, \mathcal{S}_1, \omega, H)$ and $\Sigma' = (\mathcal{A}', \mathcal{S}_1', \omega', H')$ *equivalent* if they are equivalent as noncommutative Hamiltonian systems. (See section 3.4 of I.)

Note. In the abstract algebraic framework, the CC condition is to be kept track of. We shall see in the following subsection that this condition permits us to obtain Hilbert space based realizations of quantum systems (which have the CC condition built in them as shown in section 2.2 above).

3.2. Inevitability of the Hilbert space

Given a quantum system $\Sigma = (\mathcal{A}, \mathcal{S}_1, \omega, H)$, any other quantum system $\Sigma' = (\mathcal{A}', \mathcal{S}_1', \omega', H')$, equivalent to Σ as a noncommutative Hamiltonian system, is physically equivalent to Σ and may be called a realization of Σ. By a Hilbert space realization of Σ we mean an equivalent quantum system $\hat{\Sigma} = (\hat{\mathcal{A}}, \hat{\mathcal{S}}_1, \hat{\omega}, \hat{H})$ of the type treated in section 3.4 of I (with the condition of the irreducibility of the $\mathcal{A}$-action on $\mathcal{H}$ relaxed). This amounts to (a) constructing a quantum triple $(\hat{\mathcal{H}}, \hat{\mathcal{D}}, \hat{\mathcal{A}})$ in which the algebra $\hat{\mathcal{A}}$ is isomorphic, as a topological *-algebra, to the system algebra $\mathcal{A}$ and (b) obtaining the other three ingredients of $\hat{\Sigma}$ so as to have the desired equivalent noncommutative hamiltonian system. From the above definition it is clear that, such a realization, if it exists, is unique up to equivalence. The precise statement about the existence of these realizations appears in theorem (1) below.

Part (a) is the problem of obtaining a faithful *-representation of the *-algebra $\mathcal{A}$. Some good references for the treatment of relevant mathematical concepts are (Powers [38], Dubin and Hennings [19], Horuzhy [27]). By a *-representation of a *-algebra $\mathcal{A}$ we mean a triple $(\mathcal{H}, \mathcal{D}, \pi)$ where $\mathcal{H}$ is a (separable) Hilbert space, $\mathcal{D}$ a dense linear subset of $\mathcal{H}$ and π a *-homomorphism of $\mathcal{A}$ into the operator algebra $L^+(\mathcal{D})$ (the largest *-algebra of operators on $\mathcal{H}$ having $\mathcal{D}$ as an invariant domain) satisfying the relation

$$(\chi, \pi(A)\psi) = (\pi(A^*)\chi, \psi) \quad \text{for all} \quad A \in \mathcal{A} \text{ and } \quad \chi, \psi \in \mathcal{D}.$$

The operators $\pi(A)$ induce a topology on $\mathcal{D}$ defined by the seminorms $\|.\|_S$ (where S is any finite subset of $\mathcal{A}$) given by

$$\|\psi\|_S = \sum_{A \in S} \|\pi(A)\psi\| \tag{42}$$

where $\|.\|$ is the Hilbert space norm. The mappings $\pi(A) : \mathcal{D} \to \mathcal{D}$ are continuous in this topology for all $A \in \mathcal{A}$. The representation π is said to be closed if $\mathcal{D}$ is complete in the induced topology. Given a *-representation π of $\mathcal{A}$, there exists a unique minimal closed extension $\bar{\pi}$ of π (called the closure of π).

The representation π is said to be irreducible if its weak commutant $\pi_w'(\mathcal{A})$, defined as the set of bounded operators C on $\mathcal{H}$ satisfying the condition

$$(C^*\psi, A\chi) = (A^*\psi, C\chi) \quad \text{for all } A \in \mathcal{A} \quad \text{and} \quad \psi, \chi \in \mathcal{D}$$

consists of complex multiples of the unit operator.

Once we have the triple $(\hat{\mathcal{H}}, \hat{\mathcal{D}}, \hat{\pi})$ where $\hat{\pi}$ is a faithful *-representation of $\mathcal{A}$, we have the quantum triple $(\hat{\mathcal{H}}, \hat{\mathcal{D}}, \hat{\mathcal{A}})$ where $\hat{\mathcal{A}} = \hat{\pi}(\mathcal{A})$. The construction of $\hat{\omega}$ and $\hat{H}$ is then immediate :

$$\hat{\omega} = -i\hbar\hat{\omega}_c, \quad \hat{H} = \hat{\pi}(H) \tag{43}$$

where $\hat{\omega}_c$ is the canonical form on $\hat{\mathcal{A}}$. The construction of the Hilbert space-based realization of the quantum system Σ is then completed by obtaining $\hat{\mathcal{S}}_1 = \mathcal{S}_1(\hat{\mathcal{A}})$ such that the pair $(\mathcal{O}(\hat{\mathcal{A}}), \hat{\mathcal{S}}_1)$ satisfies the CC condition.

We shall build up our arguments such that no new assumptions will be involved in going from the abstract algebraic setting to the Hilbert space setting; emergence of the Hilbert space formalism will be automatic.

To this end, we shall exploit the fact that the CC condition guarantees the existence of plenty of (pure) states of the algebra $\mathcal{A}$. Given a state ϕ on $\mathcal{A}$, a standard way to obtain a representation of $\mathcal{A}$ is to employ the so-called GNS construction. Some essential points related to this construction are recalled below :

(i) Considering the given algebra $\mathcal{A}$ as a complex vector space, one tries to define a scalar product on it using the state ϕ : $(A, B) = \phi(A^*B)$. This, however, is not positive definite if the set

$$L_\phi = \{A \in \mathcal{A};\ \phi(A^*A) = 0\} \tag{44}$$

(which can be shown to be a left ideal of $\mathcal{A}$) has nonzero elements in it. On the quotient space $\mathcal{D}_\phi^{(0)} = \mathcal{A}/L_\phi$, the object

$$([A], [B]) = \phi(A^*B) \tag{45}$$

is a well defined scalar product. Here $[A] = A + L_\phi$ denotes the equivalence class of A in $\mathcal{D}_\phi^{(0)}$. One then completes the inner product space $(\mathcal{D}_\phi^{(0)}, (,))$ to obtain the Hilbert space $\mathcal{H}_\phi$; it is separable if the topological algebra $\mathcal{A}$ is separable.
(ii) One obtains a representation $\pi_\phi^{(0)}$ of $\mathcal{A}$ on the pair $(\mathcal{H}_\phi, \mathcal{D}_\phi^{(0)})$ by putting

$$\pi_\phi^{(0)}(A)[B] = [AB]; \tag{46}$$

it can be easily checked to be a well defined *-representation. We denote by π_ϕ the closure of the representation $\pi_\phi^{(0)}$); the completion $\mathcal{D}_\phi$ of $\mathcal{D}_\phi^{(0)}$ in the $\pi_\phi^{(0)}$-induced topology acts as the common invariant domain for the operators $\pi_\phi(A)$.
(iii) The original state ϕ is represented as a vector state in the representations $\pi_\phi^{(0)}$ and π_ϕ by the vector $\chi_\phi = [I]$ (the equivalence class of the unit element of $\mathcal{A}$); indeed, we have, from Eq.(45),

$$\begin{aligned} \phi(A) &= ([I], [A]) = ([I], \pi_\phi^{(0)}(A)[I]) \\ &= (\chi_\phi, \pi_\phi^{(0)}(A)\chi_\phi) = (\chi_\phi, \pi_\phi(A)\chi_\phi). \end{aligned} \tag{47}$$

The triple $(\mathcal{H}_\phi, \mathcal{D}_\phi, \pi_\phi)$ satisfying Eq.(47) is referred to as the GNS representation of $\mathcal{A}$ induced by the state ϕ; it is determined uniquely, up to unitary equivalence, by the state ϕ. It is irreducible if and only if the state ϕ is pure.

This construction (on a single state), however, does not completely solve our problem because a GNS representation is generally not faithful; for all $A \in L_\phi$, we have obviously $\pi_\phi(A) = 0$. It is faithful if the state ϕ is faithful (i.e. if $L_\phi = \{0\}$). Such a state, however, is not guaranteed to exist by our postulates.

A faithful but generally reducible representation of $\mathcal{A}$ can be obtained by taking the direct sum of the representations of the above sort corresponding to *all* the pure states ϕ. [For the construction of the direct sum of a possibly uncountable set of Hilbert spaces, see (Rudin [40]).] Let $\mathcal{K}$ be the Cartesian product of the Hilbert spaces $\{\mathcal{H}_\phi : \phi \in \mathcal{S}_1(\mathcal{A})\}$. A general element ψ of $\mathcal{K}$ is a collection $\{\psi_\phi \in \mathcal{H}_\phi; \phi \in \mathcal{S}_1(\mathcal{A})\}$; here ψ_ϕ will be called the component of ψ in $\mathcal{H}_\phi$. The desired Hilbert space $\mathcal{H}$ consists of those elements ψ in $\mathcal{K}$ which have an at most countable set of nonzero components ψ_ϕ which, moreover, satisfy the condition

$$\sum_\phi \|\psi_\phi\|^2_{\mathcal{H}_\phi} < \infty.$$

The scalar product in $\mathcal{H}$ is given by

$$(\psi, \psi') = \sum_\phi (\psi_\phi, \psi'_\phi)_{\mathcal{H}_\phi}.$$

The direct sum of the representations $\{(\mathcal{H}_\phi, \mathcal{D}_\phi, \pi_\phi); \phi \in \mathcal{S}_1(\mathcal{A})\}$ is the representation $(\mathcal{H}, \mathcal{D}, \pi)$ where $\mathcal{H}$ is as above, $\mathcal{D}$ is the subset of $\mathcal{H}$ consisting of vectors ψ with $\psi_\phi \in \mathcal{D}_\phi$ for all $\phi \in \mathcal{S}_1(\mathcal{A})$ and, for any $A \in \mathcal{A}$,

$$\pi(A)\psi = \{\pi_\phi(A)\psi_\phi; \phi \in \mathcal{S}_1(\mathcal{A})\}.$$

Now, given any two different elements A_1, A_2 in $\mathcal{O}(\mathcal{A})$, let ϕ_0 be a pure state (guaranteed to exist by the CC condition) such that $\phi_0(A_1) \neq \phi_0(A_2)$. Let $\psi_0 \in \mathcal{H}$ be the vector with the single nonzero component $(\psi_0)_{\phi_0} = \chi_{\phi_0}$. For any $A \in \mathcal{A}$, we have

$$(\psi_0, \pi(A)\psi_0) = (\chi_{\phi_0}, \pi_{\phi_0}(A)\chi_{\phi_0}) = \phi_0(A).$$

This implies

$$(\psi_0, \pi(A_1)\psi_0) \neq (\psi_0, \pi(A_2)\psi_0), \quad \text{hence } \pi(A_1) \neq \pi(A_2)$$

showing that the representation $(\mathcal{H}, \mathcal{D}, \pi)$ is faithful.

The Hilbert space $\mathcal{H}$ obtained above may be non-separable (even if the spaces $\mathcal{H}_\phi$ are separable); this is because the set $\mathcal{S}_1(\mathcal{A})$ is generally uncountable. To obtain a faithful representation of $\mathcal{A}$ on a separable Hilbert space, we shall use the separability of $\mathcal{A}$ as a topological algebra. Let $\mathcal{A}_0 = \{A_1, A_2, A_3, ...\}$ be a countable dense subset of $\mathcal{A}$ consisting of nonzero elements. The CC condition guarantees the existence of pure states ϕ_j (j=1,2,...) such that

$$\phi_j(A_j^* A_j) \neq 0, \quad j = 1, 2, ... \tag{48}$$

Now consider the GNS representations $(\mathcal{H}_{\phi_j}, \mathcal{D}_{\phi_j}, \pi_{\phi_j})$ (j=1,2,...). Eq.(48) guarantees that

$$\pi_{\phi_j}(A_j) \neq 0, \quad j = 1, 2, ... \tag{49}$$

Indeed

$$\begin{aligned} 0 \neq \phi_j(A_j^* A_j) &= (\chi_{\phi_j}, \pi_{\phi_j}(A_j^* A_j)\chi_{\phi_j}) \\ &= (\pi_{\phi_j}(A_j)\chi_{\phi_j}, \pi_{\phi_j}(A_j)\chi_{\phi_j}). \end{aligned}$$

Now consider the direct sum $(\mathcal{H}', \mathcal{D}', \pi')$ of these representations. To show that π' is faithful, we must show that, for any nonzero element A of $\mathcal{A}$, $\pi'(A) \neq 0$. This is guaranteed by Eq.(49) because, $\mathcal{A}_0$ being dense in $\mathcal{A}$, A can be arranged to be as close as we like to some A_j in $\mathcal{A}_0$.

The representation π', is, in general, reducible. To obtain a faithful irreducible representation, we should try to obtain the relations $\pi(A_j) \neq 0$ (j= 1,2,..) in a single GNS representation π_ϕ for some $\phi \in \mathcal{S}_1(\mathcal{A})$. To this end, let $B^{(k)} = A_1 A_2 ... A_k$ and choose $\phi^{(k)} \in \mathcal{S}_1(\mathcal{A})$ such that

$$\phi^{(k)}(B^{(k)*}B^{(k)}) \neq 0.$$

In the GNS representation $(\mathcal{H}_{\phi^{(k)}}, \mathcal{D}_{\phi^{(k)}}, \pi_{\phi^{(k)}})$, we have

$$0 \neq \pi_{\phi^{(k)}}(B^{(k)}) = \pi_{\phi^{(k)}}(A_1)...\pi_{\phi^{(k)}}(A_k)$$

which implies

$$\pi_{\phi^{(k)}}(A_j) \neq 0, \quad j = 1, ..., k. \tag{50}$$

This argument works for arbitrarily large but finite k. If the $k \to \infty$ limit of the above construction leading to a limiting GNS representation $(\underline{\mathcal{H}}, \underline{\mathcal{D}}, \underline{\pi})$ exists, giving

$$\underline{\pi}(A_j) \neq 0, \quad j = 1, 2, ..., \tag{51}$$

then, by an argument similar to that for π' above, one must have $\underline{\pi}(A) \neq 0$ for all non-zero A in $\mathcal{A}$ showing faithfulness of $\underline{\pi}$.

Note. For system algebras generated by a finite number of elements (this covers all applications of QM in atomic physics), a limiting construction is not needed; the validity of Eq.(50) for sufficiently large k is adequate. [Hint : Take the generators of the algebra $\mathcal{A}$ as some of the elements of $\mathcal{A}_0$.]

Coming back to the general case, we have, finally, the faithful (but generally not irreducible) representation $(\hat{\mathcal{H}}, \hat{\mathcal{D}}, \hat{\pi})$ of $\mathcal{A}$; $\hat{\pi}(\mathcal{A}) \equiv \hat{\mathcal{A}}$ is the object desired in part (a) above. Since $\hat{\pi}$ is faithful, $\hat{\mathcal{A}}$ is an isomorphic copy of $\mathcal{A}$. There is a bijective correspondence $\phi \leftrightarrow \hat{\phi}$ between $\mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\hat{\mathcal{A}})$ [restricting to a bijection between $\mathcal{S}_1(\mathcal{A})$ and $\mathcal{S}_1(\hat{\mathcal{A}})$] such that

$$< \hat{\phi}, \hat{A} > = < \phi, A > \quad \text{for all } A \in \mathcal{A} \tag{52}$$

where $\hat{A} = \hat{\pi}(A)$. This equation implies that, since the pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1)$ satisfies the CC condition, so will the pair $(\mathcal{O}(\hat{\mathcal{A}}), \hat{\mathcal{S}}_1)$. We have, finally, a Hilbert space realization $\hat{\Sigma} = (\hat{\mathcal{A}}, \hat{\mathcal{S}}_1, \hat{\omega}, \hat{H})$ of the quantum system $\Sigma = (\mathcal{A}, \mathcal{S}_1, \omega, H)$.

Note, from Eq.(52), that

$$\hat{\phi} = (\hat{\pi}^{-1})^T(\phi). \tag{53}$$

When $\hat{\pi}$ is irreducible (equal to π_{ϕ_0}, say, where $\phi_0 \in \mathcal{S}_1(\mathcal{A})$), pure states of $\hat{\mathcal{A}}$ are vector states $\hat{\phi}_\psi$ corresponding to normalized vectors $\psi \in \hat{\mathcal{D}}$:

$$\hat{\phi}_\psi(\hat{A}) = (\psi, \hat{A}\psi) = (\psi, \hat{\pi}(A)\psi). \tag{54}$$

These normalized vectors are of the form

$$\psi_B = N_B^{1/2}[B], \quad B \in \mathcal{A}, \quad B \notin L_{\phi_0} \tag{55}$$

[see equations (45) and (46)] where $N_B = [\phi_0(B^*B)]^{-1}$. Putting $\hat{\phi} = \hat{\phi}_{\psi_B}$ in Eq.(52), we have

$$\begin{aligned} < \phi, A > &= < \hat{\phi}_{\psi_B}, \hat{A} > = (\psi_B, \hat{A}\psi_B) = N_B \ ([B], \hat{\pi}(A)[B]) \\ &= N_B \ \phi_0(B^*AB) \equiv \phi_B(A) \end{aligned} \tag{56}$$

where we have defined the linear functional ϕ_B on $\mathcal{A}$ by

$$\phi_B(A) = N_B \ \phi_0(B^*AB) \quad \text{for all } A \in \mathcal{A}. \tag{57}$$

Equations (53) and (56) now give

$$\hat{\phi}_{\psi_B} = (\hat{\pi}^{-1})^T(\phi_B) \ \text{ for all } B \in \mathcal{A}, B \notin L_{\phi_0}. \tag{58}$$

It is instructive to verify directly that the objects $\phi_B(A)$ of Eq.(57) depend only on the equivalence class [B] and are genuine elements of $\mathcal{S}_1(\mathcal{A})$ when $\phi_0 \in \mathcal{S}_1(\mathcal{A})$.

Proposition 3.1 *Given the pair ($\mathcal{A}$, $\mathcal{S}_1$) of the system algebra $\mathcal{A}$ and its set of pure states $\mathcal{S}_1$, a state $\phi \in \mathcal{S}_1$ and an element $B \in \mathcal{A}$ such that $B \notin L_\phi$, the linear functional $\phi_B : \mathcal{A} \to \mathbb{C}$ defined by Eq.(57) (with ϕ_0 replaced by ϕ) (a) depends only on the equivalence class $[B] \equiv B + L_\phi$ of B, and (b) is a pure state of $\mathcal{A}$.*

Proof. (a) We must show that, for all $K \in L_\phi$ and all $A \in \mathcal{A}$,

$$\phi_B(A) = \phi_{B+K}(A) = N_{B+K}\phi[(B+K)^*A(B+K)].$$

This is easily seen by using the Schwarz inequaliy

$$|\phi(C^*D)|^2 \leq \phi(C^*C)\ \phi(D^*D) \ \text{ for all } C, D \in \mathcal{A}$$

and the relation $\phi(K^*K) = 0$.
(b) Positivity and normalization of the functional ϕ_B are easily proved showing that it is a state. [Note that the positivity of ϕ_B holds only with the convention $(AB)^* = B^*A^*$ and not with $(AB)^* = (-1)^{\epsilon_A\epsilon_B} B^*A^*$; see the note in the beginning of section 2.] To show that it is a pure state, we shall prove that the GNS representation $(\mathcal{H}_B, \mathcal{D}_B, \pi_B)$ induced by the state ϕ_B is unitarily equivalent to the GNS representation $(\mathcal{H}, \mathcal{D}, \pi)$ induced by the pure state ϕ (and is, therefore, irreducible).

Writing, for $A, B \in \mathcal{A}$,

$$[A] \equiv A + L_\phi,\ [A]_B \equiv A + L_{\phi_B},\ \chi = [I],\ \chi_B = [I]_B,$$

we have

$$\begin{aligned} (\chi_B, \pi_B(A)\chi_B)_{\mathcal{H}_B} &= \phi_B(A) = N_B\phi(B^*AB) \\ &= N_B(\chi, \pi(B^*AB)\chi)_{\mathcal{H}}. \end{aligned} \tag{59}$$

The object ψ_B of Eq.(55) is a normalized vector in $\mathcal{D}$. Since π is irreducible, the set $\{\pi(A)\psi_B; A \in \mathcal{A}\}$ (with B fixed) is dense in $\mathcal{D}$. Moreover, the set $\{\pi_B(A)\chi_B; A \in \mathcal{A}\}$ is dense in $\mathcal{D}_B$. We define a mapping $U : \mathcal{D} \to \mathcal{D}_B$ by

$$U\pi(A)\psi_B = \pi_B(A)\chi_B \ \text{ for all } A \in \mathcal{A}. \tag{60}$$

Now, with $B \in \mathcal{A}$ fixed and any $A, C \in \mathcal{A}$, we have

$$\begin{aligned} (\pi_B(A)\chi_B, \pi_B(C)\chi_B)_{\mathcal{H}_B} &= (\chi_B, \pi_B(A^*C)\chi_B)_{\mathcal{H}_B} \\ &= N_B(\chi, \pi(B^*A^*CB)\chi)_{\mathcal{H}} \\ &= (\psi_B, \pi(A^*C)\psi_B)_{\mathcal{H}} \\ &= (\pi(A)\psi_B, \pi(C)\psi_B)_{\mathcal{H}} \end{aligned} \tag{61}$$

showing that U is an isometry; by standard arguments, it extends to a unitary mapping from $\mathcal{H}$ to $\mathcal{H}_B$ mapping $\mathcal{D}$ onto $\mathcal{D}_{\mathcal{B}}$. This proves the desired unitary equivalence of π and π_B implying that ϕ_B is a pure state. □

The proof of part (b) above has yielded a useful corollary :

Corollary (3.2). *The GNS representations induced by the states ϕ and ϕ_B of proposition (3.1) are related through a unitary mapping as in Eq.(60).*

Having obtained the quantum triple $(\hat{\mathcal{H}}, \hat{\mathcal{D}}, \hat{\mathcal{A}})$ with the locally convex topology on $\hat{\mathcal{D}}$ as described above, a mathematically rigorous vrsion of Dirac's bra-ket formalism (Roberts [39], Antoine [2], A. Böhm [10], de la Madrid [17]) based on the Gelfand triple

$$\hat{\mathcal{D}} \subset \hat{\mathcal{H}} \subset \hat{\mathcal{D}}' \tag{62}$$

where $\hat{\mathcal{D}}'$ is the dual space of $\hat{\mathcal{D}}$ with the strong topology (Kristensen, Mejlbo and Thue Poulsen [33]) defined by the seminorms p_W given by

$$p_W(F) = sup_{\psi \in W} \; |F(\psi)| \;\; \text{for all } F \in \mathcal{D}'$$

for all bounded sets W of $\hat{\mathcal{D}}$; the triple (62) constitutes the *canonical rigged Hilbert space* based on $(\hat{\mathcal{H}}, \hat{\mathcal{D}})$ (Lassner [30]). The space $\hat{\mathcal{D}}'$ (the space of continuous linear functionals on $\hat{\mathcal{D}}$) is the space of bra vectors of Dirac. The space of kets is the space $\hat{\mathcal{D}}^{\times}$ of continuous antilinear functionals on $\hat{\mathcal{D}}$. [An element $\chi \in \mathcal{H}$ defines a continuous linear functional F_χ and an antilinear functional K_χ on $\hat{\mathcal{H}}$ (hence on $\hat{\mathcal{D}}$) given by $F_\chi(\psi) = (\chi, \psi)$ and $K_\chi(\psi) = (\psi, \chi)$; both the bra and ket spaces, therefore, have $\mathcal{H}$ as a subset.]

When $\hat{\pi}$ is irreducible, the (unnormalized) vectors in $\hat{\mathcal{D}}$ representing pure states of $\hat{\mathcal{A}}$ have unrestricted superpositions allowed between them; they constitute a coherent set in the sense of (Bogolubov [9]) (which means that they, as a set, cannot be represented as a union of two nonempty mutually orthogonal sets). We can now follow the reasoning employed in the proof of lemma (4.2) in (Bogolubov [9]) to conclude that, in the general case (when $\hat{\pi}$ may be reducible), the Hilbert space $\hat{\mathcal{H}}$ can be expressed as a direct sum of mutually orthogonal coherent subspaces :

$$\hat{\mathcal{H}} = \bigoplus_\alpha \hat{\mathcal{H}}_\alpha \tag{63}$$

such that each of the $\hat{\mathcal{D}}_\alpha \equiv \hat{\mathcal{D}} \cap \mathcal{H}_\alpha$ is a coherent set on which $\hat{\mathcal{A}}$ acts irreducibly (but not necessarily faithfully) and $\hat{\mathcal{D}} = \cup_\alpha \hat{\mathcal{D}}_\alpha$. [Introduce an equivalence relation $\sim$ in $\hat{\mathcal{D}}$: $\psi \sim \chi$ if there is a coherent subset $\mathcal{C}$ in $\hat{\mathcal{D}}$ to which both ψ, χ belong. This gives the equivalence classes $\hat{\mathcal{D}}_\alpha$ in $\hat{\mathcal{D}}$. Define $\hat{\mathcal{H}}_\alpha$ as the closure of $\hat{\mathcal{D}}_\alpha$ in $\hat{\mathcal{H}}$, etc.] The breakup (63) implies the breakup $\hat{\pi} = \oplus_\alpha \hat{\pi}_\alpha$ where each triple $(\hat{\mathcal{H}}_\alpha, \hat{\mathcal{D}}_\alpha, \hat{\pi}_\alpha)$ is an irreducible (but not necessarily faithful) representation of $\mathcal{A}$. For every $A \in \mathcal{A}$ and $\psi = \{\psi_\alpha \in \hat{\mathcal{D}}_\alpha\} \in \hat{\mathcal{D}}$, we have

$$\hat{\pi}(A)\psi = \{\hat{\pi}_\alpha(A)\psi_\alpha\}. \tag{64}$$

This situation corresponds to the existence of superselection rules; the subspaces $\hat{\mathcal{H}}_\alpha$ are referred to as coherent subspaces or superselection sectors. The projection operators P_α for the subspaces $\hat{\mathcal{H}}_\alpha$ belong to the center of $\hat{\mathcal{A}}$. [To show this, it is adequate to show that, for any

$\hat{A} \equiv \hat{\pi}(A) \in \hat{\mathcal{A}}$ and $\psi = \{\psi_\alpha\} \in \hat{\mathcal{D}}$, $\hat{A}P_\alpha\psi = P_\alpha\hat{A}\psi$. Using Eq.(64), each side is easily seen to be equal to $\hat{\pi}_\alpha(A)\psi_\alpha$.]

Operators of the form

$$Q = \sum_\alpha a_\alpha P_\alpha, \quad a_\alpha \in \mathbb{R} \tag{65}$$

serve as superselection operators. Any two such operators obviously commute. We have, therefore, a formalism in which there is a natural place for superselection rules which are restricted to be commutative.

We have proved the following theorem.

Theorem(1). *Given a quantum system $\Sigma = (\mathcal{A}, \mathcal{S}_1, \omega, H)$ (where the system algebra $\mathcal{A}$ is supposedly separable as a topological algebra), the following holds true.*
(a) *The system algebra $\mathcal{A}$ admits a faithful *-representation $(\hat{\mathcal{H}}, \hat{\mathcal{D}}, \hat{\pi})$ in a separable Hilbert space $\hat{\mathcal{H}}$ giving the quantum triple $(\hat{\mathcal{H}}, \hat{\mathcal{D}}, \hat{\mathcal{A}})$ with $\hat{\mathcal{A}} = \hat{\pi}(\mathcal{A})$.*
(b) *With pure states defined through Eq.(52) and the quantum symplectic form $\hat{\omega}$ and the Hamiltonian operator $\hat{H}$ given by Eq.(43), this provides the Hilbert space based realization $\hat{\Sigma} = (\hat{\mathcal{A}}, \hat{\mathcal{S}}_1, \hat{\omega}, \hat{H})$ of the quantum system Σ. This realization supports a rigorous version of the Dirac bra-ket formalism based on the canonical rigged Hilbert space (62).*
(c) *When $\mathcal{A}$ is generated by a finite number of elements, it is possible to have the faithful *-repesentation $\hat{\pi}$ of part (a) irreducible. In this case pure states of $\hat{\mathcal{A}}$ are the vector states corresponding to the normalized elements of $\hat{\mathcal{D}}$.*
(d) *In the general case, the Hilbert space $\hat{\mathcal{H}}$ of (a) above can be expressed as a direct sum (63) of mutually orthogonal subspaces (superselection sectors) such that each $\mathcal{H}_\alpha$ is an irreducible invariant subspace for the opertor algebra $\hat{\mathcal{A}}$, each set $\mathcal{D}_\alpha$ is coherent and $\hat{\mathcal{D}} = \cup_\alpha \mathcal{D}_\alpha$. The superselection operators (65) constitute a real subalgebra of the center of $\hat{\mathcal{A}}$.*

We shall call a quantum system with a finitely generated system algebra a *standard quantum system*. According to theorem (1), such a system admits a Hilbert space based realization with the system algebra represented faithfully and irreducibly and there are no superselection rules. All quantum systems consisting of a finite number of particles (in particular all quantum systems in atomic physics) obviously belong to this class.

3.3. Hilbert space quantum mechanics of standard quantum systems

We shall now consider Hilbert space based realizations of standard quantum systems and relate the supmech treatment of their kinematics and dynamics in section 3.1 to the traditional Hilbert space based formalism.

We first consider the implementation of symplectic mappings in such realizations. The main result is contained in the following theorem.

Theorem (2). *Let* $\Sigma = (\mathcal{A}, \mathcal{S}_1, \omega, H)$ *and* $\Sigma' = (\mathcal{A}', \mathcal{S}_1', \omega', H')$ *be two equivalent standard quantum systems; the equivalence is described by the symplectic mappings* $\Phi = (\Phi_1, \Phi_2)$ *[which means that* $\Phi_1 : \mathcal{A} \to \mathcal{A}'$ *is an isomorphism of unital *-algebras such that* $\Phi^*\omega' = \omega$ *and* $\Phi_2 : \mathcal{S}_1 \to \mathcal{S}_1'$ *is a bijection such that* $< \Phi_2(\phi), \Phi_1(A) > = < \phi, A >$ *for all* $\phi \in \mathcal{S}_1$ *and* $A \in \mathcal{A}$*]. Given their Hilbert space realizations* $\hat{\Sigma} = (\hat{\mathcal{A}}, \hat{\mathcal{S}}_1, \hat{\omega}, \hat{H})$ *and* $\hat{\Sigma}' = (\hat{\mathcal{A}}', \hat{\mathcal{S}_1}', \hat{\omega}', \hat{H}')$ *[the respective representations of system algebras being* $(\hat{\mathcal{H}}, \hat{\mathcal{D}}, \hat{\pi})$ *and* $(\hat{\mathcal{H}}', \hat{\mathcal{D}}', \hat{\pi}')$*], there exists a unitary mapping* $U : \hat{\mathcal{H}} \to \hat{\mathcal{H}}'$ *mapping* $\hat{\mathcal{D}}$ *onto* $\hat{\mathcal{D}}'$ *implementing the given equivalence with*

$$\hat{\pi}'(\Phi_1(A)) = U\hat{\pi}(A)U^{-1} \text{ for all } A \in \mathcal{A};\ \psi' = U\psi \tag{66}$$

where $\psi \in \hat{\mathcal{D}}$ *and* $\psi' \in \hat{\mathcal{D}}'$ *are representative vectors for the states* $\phi \in \mathcal{S}_1$ *and* $\Phi_2(\phi) \in \mathcal{S}_1'$ *respectively.*

Proof. Since the quantum systems are standard, their pure states are represented by normalized vectors in $\hat{\mathcal{D}}$ and $\hat{\mathcal{D}}'$. Let $\phi \in \mathcal{S}_1$, $\phi' = \Phi_2(\phi)$ and $\psi \in \hat{\mathcal{D}}$ and $\psi' \in \hat{\mathcal{D}}'$ are normalized vectors such that $\phi_\psi = (\hat{\pi}^{-1})^T(\phi)$ and $\phi_{\psi'} = ([\hat{\pi}']^{-1})^T(\phi')$ are the corresponding vector states in $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}_1}'$ respectively. Writing $\hat{A} = \hat{\pi}(A)$ for $A \in \mathcal{A}$ and $\hat{A}' = \hat{\pi}'(A')$ for $A' = \Phi_1(A) \in \mathcal{A}'$, we have

$$\begin{aligned}(\psi', \hat{A}'\psi')_{\hat{\mathcal{H}}'} &= < \phi_{\psi'}, \hat{\pi}'(A') > = < \phi, A > \\ &= < \phi_\psi, \hat{\pi}(A) > = (\psi, \hat{A}\psi)_{\hat{\mathcal{H}}}\end{aligned} \tag{67}$$

for all $A \in \mathcal{A}$ and all $\phi \in \mathcal{S}_1$.

Let $\{\chi_r\}$ (r = 1,2,...) be an orthonormal basis in $\hat{\mathcal{H}}$ (with all $\chi_r \in \hat{\mathcal{D}}$), $\phi_r \in \mathcal{S}_1$ the state reprented by the vector χ_r, $\phi_r' = \Phi_2(\phi_r)$ and $\chi_r' \in \hat{\mathcal{D}}'$ a normalized vector representing the state ϕ_r'. Define a mapping $U : \hat{\mathcal{H}} \to \hat{\mathcal{H}}'$ such that $U\chi_r = \chi_r'$ (r= 1,2,...). Putting $\psi = \chi_s$ and $\psi' = \chi_s'$ in Eq.(67), we have (dropping the subscripts on the scalar products)

$$(U\chi_s, \hat{A}'U\chi_s) = (\chi_s, \hat{A}\chi_s).$$

Writing similar equations with χ_s replaced by $(\chi_r + \chi_s)/\sqrt{2}$ and $(\chi_r + i\chi_s)/\sqrt{2}$ we obtain the relation

$$(\chi_r, U^\dagger\hat{A}'U\chi_s) = (\chi_r, \hat{A}\chi_s)$$

(for arbitrary r and s) which implies

$$U^\dagger\hat{A}'U = \hat{A} \text{ for all } A \in \mathcal{A}.$$

Now, for A = I, we must have $A' = I$ (the mapping Φ_1 being an isomorphism of the unital algebra $\mathcal{A}$ onto $\mathcal{A}'$); this gives $U^\dagger U = I$ or, remembering the invertibility of the mapping Φ_2, $U^\dagger = U^{-1}$. We have, therefore, $\hat{A}' = U\hat{A}U^{-1}$. The condition (39) implies

$$U(i[\hat{A}, \hat{B}])U^{-1} = i[U\hat{A}U^{-1}, U\hat{B}U^{-1}]$$

which permits U to be taken as a linear and, therefore, unitary operator.

Now let $\psi = \sum a_r \chi_r$. We have

$$U\psi = \sum a_r U\chi_r = \sum a_r \chi_r' \equiv \psi''.$$

This gives, employing the Dirac notation for projectors,

$$|\psi'' >< \psi''| = U|\psi >< \psi|U^{-1} = |\psi' >< \psi'|$$

where the last step follows from Eq.(67) (with $\hat{A}' = U\hat{A}U^{-1}$.) and the CC condition. It follows that ψ'' is an acceptable representative of the state represented by ψ' implying that we can consistently take $\psi' = U\psi$. $\square$

We shall say, in the context of the above theorem, that the mappings (Φ_1, Φ_2) are *unitarily implemented*. Taking $\Sigma' = \Sigma$ in the theorem, we have

Corollary (3.3). *Given two Hilbert space realizations of a standard quantum system, the mappings describing their equivalence as supmech Hamiltonian systems can be implemented unitarily.*

Taking $\hat{\Sigma}' = \hat{\Sigma}$ in corollary (3.3), we have

Corollary (3.4). *In a Hilbert space realization of a standard quantum system, a quantum canonical transformation can be implemented unitarily.*

We shall henceforth drop the tildes and take $\Sigma = (\mathcal{A}, \mathcal{S}_1, \omega, H)$ directly as a Hilbert space realization of a standard quantum system; here $\mathcal{A}$ is now an Op*-algebra based on the pair $(\mathcal{H}, \mathcal{D})$ constituting a quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$. In concrete applications, there is some freedom in the choice of $\mathcal{D}$. When $\mathcal{A}$ is generated by a finite set of fundamental observables $F_1, .., F_n$, a good choice is, in the notation of Dubin and Hennings [19], $\mathcal{D} = C^\infty(F_1, .., F_n)$ (i.e. intersection of the domains of all polynomials in $F_1, .., F_n$).

We have now $\mathcal{A}$ as our system algebra; its states are given by the subclass of density operators ρ on $\mathcal{H}$ for which $|Tr(\rho\bar{A})| < \infty$ (where the overbar indicates closure of the operator) for all observables A in $\mathcal{A}$[21]; the quantity $Tr(\rho\bar{A}) \equiv \phi_\rho(A)$ (where ϕ_ρ is the state represented by the density operator ρ) is the expectation value of the observable A in the state ϕ_ρ. Pure states are the subclass of these states consisting of one-dimensional projection operators $|\psi >< \psi|$ where ψ is any normalized element of $\mathcal{D}$.

The density operators representing states, being Hermitian operators, are also observables. A density operator ρ is the observable corresponding to the property of the system being in the state ϕ_ρ. Given two states represented by density operators ρ_1 and ρ_2, we have the quantity $w_{12} = Tr(\rho_1\rho_2)$ defined (representing the expectation value of the observable ρ_1 in the state

ρ_2 and vice versa) which has the natural interpretation of transition probability from one of the states to the other (the two are equal because $w_{12} = w_{21}$). When $\rho_i = |\psi_i><\psi_i|$ (i = 1,2) are pure states, we have $Tr(\rho_1\rho_2) = |(\psi_1, \psi_2)|^2$ — the familiar text book expression for the transition probability between two pure quantum states.

Note. Recalling the stipulation in section 2.1 about probabilities in the formalism, it is desirable to represent the quantities w_{12} as bonafide probabilities in the standard form (1) employing an appropriate PObVM [which, in the present Hilbert space setting, should be a traditional POVM (positive operatorvalued measure)]. It is clearly adequate to have such a representation for the case of pure states with $\rho_j = |\psi_j><\psi_j|$ (j = 1,2), say. To achieve this, let $\phi = \phi_{\rho_1}$ and $\{\chi_r; r = 1, 2, ...\}$ an orthonormal basis in $\mathcal{H}$ having $\chi_1 = \psi_2$. The desired POVM is obtained by taking, in the notation of section 2.1,

$$\Omega = \{\chi_r; r = 1, 2, ...\}, \quad \mathcal{F} = \{\text{All subsets of } \Omega\} \tag{68}$$

and, for $E = \{\chi_r; r \in J\} \in \mathcal{F}$ where J is a subset of the positive integers,

$$\nu(E) = \sum_{r \in J} |\chi_r><\chi_r|. \tag{69}$$

We now have $w_{12} = |(\psi_1, \psi_2)|^2 = p_\phi(E)$ of Eq.(1) with $\phi = \phi_{\rho_1}$ and $E = |\chi_1><\chi_1| = |\psi_2><\psi_2|$.

The unitarily implemented Φ_2 actions (quantum canonical transformations) on states leave the transition probabilities invariant [in fact, they leave transition amplitudes invariant : $(\psi', \chi') = (\psi, \chi)$]. Note that, in contrast with the traditional formalism of QM, invariance of transition probabilities under the fundamental symmetry operations of the theory is not postulated but proved in the present setting. The fundamental symmetry operations themselves came as a matter of course from the basic premises of the theory : noncommutative symplectics — exactly as the classical canonical transformations arise naturally in the traditional commutative symplectics.

A symmetry implemented (in the unimodal sense, as defined in section 3.4 of I) by a unitary operator U acts on a state vector $\psi \in \mathcal{D}$ according to $\psi \to \psi' = U\psi$ and (when its action is transferred to operators) on an operator $A \in \mathcal{A}$ according to $A \to A'$ such that, for all $\psi \in \mathcal{D}$,

$$(\psi', A\psi') = (\psi, A'\psi) \quad \Rightarrow A' = U^{-1}AU. \tag{70}$$

For an infinitesimal unitary transformation, $U \simeq I + i\epsilon G$ where G is an even, Hermitian element of $\mathcal{A}$ [this follows from the condition $(U\phi, U\psi) = (\phi, \psi)$ for all $\phi, \psi \in \mathcal{D}$]. Considering the transformation $A \to A'$ in Eq.(70) as a quantum canonical transformation, generated (through PBs) by an element $T \in \mathcal{A}$, we have

$$\delta A = -i\epsilon[G, A] = \epsilon\{T, A\} \tag{71}$$

giving $T = -i(-i\hbar)G = -\hbar G$ and

$$U \simeq I - i\frac{\epsilon}{\hbar}T. \tag{72}$$

It is the appearance of $\hbar$ in Eq.(72) which is responsible for its appearance at almost all conventional places in QM.

The quantum canonical transformation representing evolution of the system in time is implemented on the state vectors by a one-parameter family of unitary operators [in the form $\psi(t) = U(t-s)\psi(s)$] generated by the Hamiltonian operator H : $U(\epsilon) \simeq I - i\frac{\epsilon}{\hbar}H$. This gives, in the Schrödinger picture, the Schrödinger equation for the evolution of pure states :

$$i\hbar\frac{d\psi(t)}{dt} = H\psi(t). \tag{73}$$

In the Heisenberg picture, we have, of course, the Heisenberg equation of motion (40), which is now an operator equation.

We had seen in the previous subsection that quantum triples provide a natural setting for a mathematically rigorous development of the Dirac bra-ket formalism. For later use, we recall a few points relating to this formalism which hold good when the space $\hat{\mathcal{D}}$ is nuclear (Gelfand and Vilenkin [21]).

A self-adjoint operator A in $\mathcal{A}$ in a rigged Hilbert space (with nuclear rigging as mentioned above) has complete sets of generalized eigenvectors [eigenkets $\{|\lambda>; \lambda \in \sigma(A)$, the spectrum of A $\}$ and eigenbras $\{<\lambda|; \lambda \in \sigma(A)\}$] :

$$A|\lambda> = \lambda|\lambda>; \quad <\lambda|A = \lambda <\lambda|;$$

$$\int_{\sigma(A)} d\mu(\lambda)|\lambda><\lambda| = I \tag{74}$$

where I is the unit operator in $\mathcal{H}$ and μ is a unique measure on $\sigma(A)$. These equations are to be understood in the sense that, for all $\chi, \psi \in \mathcal{D}$,

$$<\chi|A|\lambda> = \lambda <\chi|\lambda>; \quad <\lambda|A|\psi> = \lambda <\lambda|\psi>;$$

$$\int_{\sigma(A)} d\mu(\lambda) <\chi|\lambda><\lambda|\psi> = <\chi|\psi>.$$

The last equation implies the expansion (in eigenkets of A)

$$|\psi> = \int_{\sigma(A)} d\mu(\lambda)\ |\lambda><\lambda|\psi>. \tag{75}$$

More generally, one has complete sets of generalized eigenvectors associated with finite sets of commuting self-adjoint operators.

3.4. Quantum mechanics of localizable elementary systems (massive particles)

A *quantum elementary system* is a standard quantum system which is also an elementary system. The concept of a quantum elementary system, therefore, combines the concept of quantum symplectic structure with that of a relativity scheme. The basic entities relating to an elementary system are its fundamental observables which generate the system algebra $\mathcal{A}$. For quantum elementary systems, this algebra $\mathcal{A}$ has the quantum symplectic structure as described in section 3.1. All the developments in section 2.5 can now proceed with the Poisson brackets (PBs) understood as quantum PBs of Eq.(36). Since the system algebra is finitely generated, theorem (1) guarantees the existence of a Hilbert space-based realization of such a system involving a quantum triple $(\hat{\mathcal{H}}, \hat{\mathcal{D}}, \hat{\mathcal{A}})$ where $\hat{\mathcal{A}}$ is a faithful irreducible representation of $\mathcal{A}$ based on the pair $(\hat{\mathcal{H}}, \hat{\mathcal{D}})$. We shall drop the hats and call the quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$.

The relativity group G_0 (or its projective group $\hat{G}_0$) has a Poisson action on $\mathcal{A}$ and a transitive action on the set $\mathcal{S}_1(\mathcal{A})$ of pure states of $\mathcal{A}$. We have seen above that, in the present setting, a symmetry operation can be represented as a unitary operator on $\mathcal{H}$ mapping $\mathcal{D}$ onto itself. A symmetry group is then realized as a unitary representation on $\mathcal{H}$ having $\mathcal{D}$ as an invariant domain. For an elementary system the condition of transitive action on $\mathcal{S}_1$ implies that this representation must be irreducible. (There is no contradiction between this requirement and that of invariance of $\mathcal{D}$ because $\mathcal{D}$ is not a closed subspace of $\mathcal{H}$ when $\mathcal{H}$ is infinite dimensional.)

Note. We now have a formal justification for the direct route to the Hilbert space taken in the traditional treatment of QM of elementary systems, namely, employment of projective unitary irreducible representations of the relativity group G_0. This is the simplest way to satisfy the condition of transitive action of G_0 on the space of pure states and simultaneously satisfy the CC condition.

By a (quantum) *particle* we shall mean a localizable (quantum) elementary system. We shall consider only nonrelativistic particles. The configuration space of a nonrelativistic particle is the 3-dimensional Euclidean space $\mathbb{R}^3$. The fundamental observables for such a system were identified, in section 2.5, as the mass (m) and Cartesian components of position (X_j), momentum (P_j) and spin(S_j) (j = 1,2,3) satisfying the PB relations in equations (26,27,13). The mass m will be treated, as before, as a positive parameter. The system algebra $\mathcal{A}$ of the particle is the *-algebra generated by the fundamental observables (taken as hermitian) and the unit element. Since it is an ordinary *-algebra (i.e. one not having any fermionic objects), the supercommutators reduce to ordinary commutators. Recalling Eq.(36), the PBs mentioned above now take the form of the commutation relations

$$[X_j, X_k] = 0 = [P_j, P_k], \quad [X_j, P_k] = i\hbar\delta_{jk}I \qquad (A)$$

$$[S_j, S_k] = i\hbar\epsilon_{jkl}S_l, \quad [S_j, X_k] = 0 = [S_j, P_k]. \qquad (B) \qquad (76)$$

We now consider explicit construction of the quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ for these objects. We shall first consider the spinless particles ($\mathbf{S} = 0$); for these, we need to consider only the Heisenberg commutation relations (76A) [often referred to as the *canonical commutation relations* (CCR)]. Since the final construction is guaranteed to be unique upto unitary equivalence, we can allow ourselves to be guided by considerations of simplicity and plausibility.

Eq.(12), written (with n = 3) for a pure state (represented by a normalized vector $\psi \in \mathcal{D}$) now takes the form (writing μ_ψ for μ_{ϕ_ψ})

$$(\psi, X_j\psi) = \int_{\mathbb{R}^3} x_j d\mu_\psi(x)$$

which shows that the scalar product in $\mathcal{H}$ involves integration over $\mathbb{R}^3$ with respect to a measure. The group of space translations is to be represented unitarily in $\mathcal{H}$ (being a subgroup of the Galilean group). The simplest choice (which eventually works well as we shall see) is to take $\mathcal{H} = L^2(\mathbb{R}^3, dx)$ and the unitary operators U(a) representing space translations as given by

$$[U(a)\psi](x) = \psi(x - a) \qquad (77)$$

[which is a special case of of the relation $[U(g)\psi](x) = \psi(T_g^{-1}x)$; these operators are unitary when the transformation T_g of $\mathbb{R}^3$ preserves the Lebesgue measure]. Recalling Eq.(72), we have, for an infinitesimal translation,

$$\delta\psi = -\frac{i}{\hbar}\mathbf{a}.\mathbf{P}\psi = -\mathbf{a}.\bigtriangledown\psi$$

giving the operators P_j representing momentum components as

$$(P_j\psi)(x) = -i\hbar\frac{\partial\psi}{\partial x_j}. \qquad (78)$$

Taking the position operators X_j to be the multiplication operators given by

$$(X_j\psi)(x) = x_j\psi(x), \qquad (79)$$

the CCR of Eq.(76A) are satisfied.

We now have [21]

$$\mathcal{D} = C^\infty(X_j, P_j, ; j = 1, 2, 3) = \mathcal{S}(\mathbb{R}^3)$$

The operators U(a) clearly map the domain $\mathcal{D} = \mathcal{S}(\mathbb{R}^3)$ onto itself. With this choice of $\mathcal{D}$, the operators X_j and P_j given by equations (79) and (78) are essentially self adjoint; we denote their self adjoint extensions by the same symbols.

The space $\mathcal{S}(\mathbb{R}^3)$ is nuclear [9] and the rigged Hilbert space

$$\mathcal{S}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \subset \mathcal{S}'(\mathbb{R}^3)$$

satisfies the conditions for the validity of the results stated at the end of section 3.3. We shall make use of the complete sets of generalized eigenvectors of the operators X_j. Let $x = (x_1, x_2, x_3)$, $dx = dx_1 dx_2 dx_3$ and $|x>, <x|$ the simultaneous eigenkets and eigenbras of the operators X_j (j= 1,2,3):

$$X_j|x> = x_j|x>, \quad <x|X_j = <x|x_j, \quad x_j \in \mathbb{R}, \quad j = 1, 2, 3; \tag{80}$$

they form a complete set providing a resolution of identity in the form

$$I = \int_{R^3} |x> dx <x|. \tag{81}$$

Given any vector $|\psi> \in \mathcal{D}$, the corresponding wave function appearing in Eq.(79) is $\psi(x) \equiv <x|\psi>$; we have, indeed,

$$(X_j\psi)(x) = <x|X_j|\psi> = x_j\psi(x).$$

Recalling the discussion of localization in section 2.4, the localization observable P(D) corresponding to a Borel set D in $\mathbb{R}^3$ is represented as the operator

$$P(D) = \int_D |x> dx <x|. \tag{82}$$

[The required properties of P(D) are easily verified.] Given the particle in the state corresponding to $|\psi> \in \mathcal{D}$, the probability that it will be found in the domain D is given by

$$<\psi|P(D)|\psi> = \int_D <\psi|x> dx <x|\psi> = \int_D |\psi(x)|^2 dx \tag{83}$$

giving the traditional Born interpretation of the wave function ψ.

The pair $(\mathcal{H}, \mathcal{D}) = (L^2(R^3), \mathcal{S}(R^3))$ with operators X_j and P_j as constructed above is known as the *Schrödinger representation* of the CCR (76A).

The self adjoint operators P_j, X_j generate the unitary groups of operators $U(a) = exp(-ia.P)$ and $V(b) = exp(-ib.X)$ (where $a.P = \sum_j a_j P_j$ etc. and we have put $\hbar = 1$.) which satisfy the Weyl commutation relations

$$\begin{aligned} U(a)U(b) &= U(b)U(a) = U(a+b), \;\; V(a)V(b) = V(b)V(a) = V(a+b) \\ U(a)V(b) &= e^{ia.b}V(b)U(a). \end{aligned} \tag{84}$$

For all $\psi \in \mathcal{D}$, we have

$$(U(a)\psi)(x) = \psi(x-a), \quad (V(b)\psi)(x) = e^{-ib.x}\psi(x); \tag{85}$$

this is referred to as the Schrödinger representation of the Weyl commutation relations. According to the uniqueness theorem of von Neumann [50], the irreducible representation of the Weyl commutation relations is, up to unitary equivalence, uniquely given by the Schrödinger representation (86).

Note. (i) Not every representation of the CCR (76A) with essentially self adjoint X_j and P_j gives a representation of the Weyl commutation relation. [For a counterexample, see Inoue [28], example (4.3.3).] A necessary and sufficient condition for the latter to materialize is that the harmonic oscillator Hamiltonian operator $H = P^2/(2m) + kX^2/2$ be essentially self adjoint. In the Schrödinger representation of the CCR obtained above, this condition is satisfied [22,19]

(ii) The von Neumann uniqueness theorem serves to confirm/verify, in the present case, the uniqueness (up to equivalence) of the Hilbert space realization of a standard quantum system mentioned in sections 3.2 and 3.3. Taking the opposite view, given the uniqueness (up to unitary equivalence) of the Hilbert space realizations of the algebraic quantum system corresponding to a nonrelativistic massive spinless particle and the remark (i) above, we have an alternative proof of the von Neumann uniqueness theorem.

Quantum dynamics of a free nonrelativistic spinless particle is governed, in the Schrödinger picture, by the Schrödinger equation (73) with $\psi \in \mathcal{D} = \mathcal{S}(\mathbb{R}^3)$ and with the Hamiltonian (29) [where $\mathbf{P}$ is now the operator in Eq.(78)]:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \bigtriangledown^2 \psi. \tag{86}$$

Explicit construction of the projective unitary representation of the Galilean group G_0 in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, dx)$ and Galilean covariance of the free particle Schrödinger equation (86) have been treated in the literature [4, 43, 15].

When external forces are acting, the Hamiltonian operator has the more general form (30). Restricting V in this equation to a function of $\mathbf{X}$ only (as is the case in common applications), and proceeding as above, we obtain the traditional Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = [-\frac{\hbar^2}{2m} \bigtriangledown^2 + V(\mathbf{X})]\psi \tag{87}$$

where $\mathbf{X}$ is now the position operator of Eq.(79).

It should be noted that, in the process of obtaining the Schrödinger equation (87) for a nonrelativistic spinless particle with the traditional Hamiltonian operator, we did not use the classical Hamiltonian or Lagrangian for the particle. No *quantization* algorithm has been employed; the development of the quantum mechanical formalism has been autonomous, as promised.

From this point on, the development of QM along the traditional lines can proceed.

For nonrelativistic particles with $m > 0$ and spin $s \geq 0$, we have $\mathcal{H} = L^2(\mathbb{R}^3, C^{2s+1})$ and $\mathcal{D} = \mathcal{S}(\mathbb{R}^3, \mathbb{C}^{2s+1})$. The treatment of spin being standard, we skip the details.

4. QUANTUM-CLASSICAL CORRESPONDENCE

It will now be shown that supmech permits a transparent treatment of quantum-classical correspondence. In contrast to the general practice in this domain, we shall be careful about the domains of operators and avoid some usual pitfalls in the treatment of the $\hbar \rightarrow 0$ limit.

Our strategy will be to start with a quantum Hamiltonian system, transform it to an isomorphic supmech Hamiltonian system involving phase space functions and $\star$-products [Weyl-Wigner-Moyal formalism (Weyl [44], Wigner [46], Moyal [35])] and show that, in this latter Hamiltonian system, the subclass of phase space functions in the system algebra which go over to smooth functions in the $\hbar \rightarrow 0$ limit yield the corresponding classical Hamiltonian system. For simplicity, we restrict ourselves to the case of a spinless nonrelativistic particle though the results obtained admit trivial generalization to systems with phase space R^{2n}.

In the existing literature, the works on quantum-classical correspondence closest to the present treatment are those of Liu [31,32], Gracia-Bondía and Várilly [23] and Hörmander [26]; some results from these works, especially Liu [31,32], are used below [mainly in obtaining equations (93) and (96)]. The reference (Bellissard and Vitot [6]) is a comprehensive work reporting on some detailed features of quantum-classical correspondence employing some techniques of noncommutative geometry; its theme, however, is very different from ours.

In the case at hand, we have the quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ where $\mathcal{H} = L^2(\mathbb{R}^3), \mathcal{D} = \mathcal{S}(\mathbb{R}^3)$ and $\mathcal{A}$ is the system algebra of a spinless Galilean particle treated in section 3.4 as a standard quantum system. As in Eq.(87), we shall take the potential function V to be a function of $\mathbf{X}$ only. For $A \in \mathcal{A}$ and ϕ, ψ normalized elements in $\mathcal{D}$, we have the well defined quantity

$$(\phi, A\psi) = \int \int \phi^*(y) K_A(y, y') \psi(y') dy dy'$$

where the kernel K_A is a (tempered) distribution. Recalling the definition of Wigner function [46,48] corresponding to the wave function ψ :

$$W_\psi(x, p) = \int_{\mathbb{R}^3} exp[-ip.y/\hbar] \psi(x + \frac{y}{2}) \psi^*(x - \frac{y}{2}) dy \tag{88}$$

and defining the quantity $A_W(x, p)$ by

$$A_W(x, p) = \int exp[-ip.y/\hbar] K_A(x + \frac{y}{2}, x - \frac{y}{2}) dy \tag{89}$$

(note that W_ψ is nothing but the quantity P_W where P is the projection operator $|\psi><\psi|$ corresponding to ψ) we have

$$(\psi, A\psi) = \int\int A_W(x,p)W_\psi(x,p)dxdp. \tag{90}$$

Whereas the kernels K_A are distributions, the objects A_W are well defined functions. For example,

$$\begin{aligned}
A &= I: \quad K_A(y,y') = \delta(y-y') \quad A_W(x,p) = 1 \\
A &= X_j: \quad K_A(y,y') = y_j\delta(y-y') \quad A_W(x,p) = x_j \\
A &= P_j: \quad K_A(y,y') = -i\hbar\frac{\partial}{\partial y_j}\delta(y-y') \quad A_W(x,p) = p_j.
\end{aligned}$$

The Wigner functions W_ψ are generally well-behaved functions. We shall use Eq.(90) to characterize the class of functions A_W and call them Wigner-Schwartz integrable (WSI) functions [i.e. functions integrable with respect to the Wigner functions corresponding to the Schwartz functions in the sense of Eq.(88)]. For the relation of this class to an appropriate class of symbols in the theory of pseudodifferential operators, we refer to Wong [57] and references therein.

The operator A can be reconstructed (as an element of $\mathcal{A}$) from the function A_W; for arbitrary $\phi, \psi \in \mathcal{D}$, we have

$$\begin{aligned}
&(\phi, A\psi) = \\
&(2\pi\hbar)^{-3}\int\int\int exp[ip.(x-y)/\hbar]\phi^*(x)A_W(\frac{x+y}{2}, p)\psi(y)dpdxdy.
\end{aligned} \tag{91}$$

Replacing, on the right hand side of Eq.(88), the quantity $\psi(x+\frac{y}{2})\psi^*(x-\frac{y}{2})$ by $K_\rho(x+\frac{y}{2}, x-\frac{y}{2})$ where $K_\rho(.,.)$ is the kernel of the density operator ρ, we obtain the Wigner function $\rho_W(x,p)$ corresponding to ρ. Eq.(90) then goes over to the more general equation

$$Tr(A\rho) = \int\int A_W(x,p)\rho_W(x,p)dxdp. \tag{92}$$

The Wigner function ρ_W is real but generally not non-negative.

Introducing, in $\mathbb{R}^6$, the notations ξ = (x,p), $d\xi = dxdp$ and $\sigma(\xi,\xi') = p.x' - x.p'$ (the symplectic form in $\mathbb{R}^6$), we have, for A,B $\in \mathcal{A}$

$$\begin{aligned}
(AB)_W(\xi) &= (2\pi)^{-6}\int\int exp[-i\sigma(\xi-\eta,\tau)]A_W(\eta+\frac{\hbar\tau}{4}). \\
&\quad .B_W(\eta-\frac{\hbar\tau}{4})d\eta d\tau \\
&\equiv (A_W \star B_W)(\xi).
\end{aligned} \tag{93}$$

The product $\star$ of Eq.(93) is the *twisted product* of Liu [31,32] and the $\star$- *product* of Bayen et al [5]. The associativity condition $A(BC) = (AB)C$ implies the corresponding condition $A_W \star (B_W \star C_W) = (A_W \star B_W) \star C_W$ in the space $\mathcal{A}_W$ of WSI functions which is a complex associative non-commutative, unital *-algebra (with the star-product as product and complex conjugation as involution). There is an isomorphism between the two star-algebras $\mathcal{A}$ and $\mathcal{A}_W$ as can be verified from equations (93) and (91).

Recalling that, in the quantum Hamiltonian system $(\mathcal{A}, \omega_Q, H)$ the form ω_Q is fixed by the algebraic structure of $\mathcal{A}$ and noting that, for the Hamiltonian H of Eq.(30) [with V = V($\mathbf{X}$)],

$$H_W(x, p) = \frac{p^2}{2m} + V(x), \tag{94}$$

we have an isomorphism between the supmech Hamiltonian systems $(\mathcal{A}, \omega_Q, H)$ and $(\mathcal{A}_W, \omega_W, H_W)$ where $\omega_W = -i\hbar\omega_c^{(W)}$; here $\omega_c^{(W)}$ is the canonical 2-form of the algebra $\mathcal{A}_W$. Under this isomorphism, the quantum mechanical PB (36) is mapped to the Moyal bracket

$$\{A_W, B_W\}_M \equiv (-i\hbar)^{-1}(A_W \star B_W - B_W \star A_W). \tag{95}$$

For functions f, g in $\mathcal{A}_W$ which are smooth and such that $f(\xi)$ and $g(\xi)$ have no $\hbar-$dependence, we have, from Eq.(93),

$$f \star g = fg - (i\hbar/2)\{f, g\}_{cl} + O(\hbar^2). \tag{96}$$

The functions $A_W(\xi)$ will have, in general, some $\hbar$ dependence and the $\hbar \to 0$ limit may be singular for some of them (Berry [8]). We denote by $(\mathcal{A}_W)_{reg}$ the subclass of functions in $\mathcal{A}_W$ whose $\hbar \to 0$ limits exist and are smooth (i.e. C^∞) functions; moreover, we demand that the Moyal bracket of every pair of functions in this subclass also have smooth limits. This class is easily seen to be a subalgebra of $\mathcal{A}_W$ closed under Moyal brackets. Now, given two functions A_W and B_W in this class, if $A_W \to A_{cl}$ and $B_W \to B_{cl}$ as $\hbar \to 0$ then $A_W \star B_W \to A_{cl}B_{cl}$; the subalgebra $(\mathcal{A}_W)_{reg}$, therefore, goes over, in the $\hbar \to 0$ limit , to a subalgebra $\mathcal{A}_{cl}$ of the commutative algebra $C^\infty(\mathbb{R}^6)$ (with pointwise product as multiplication). The Moyal bracket of Eq.(95) goes over to the classical PB $\{A_{cl}, B_{cl}\}_{cl}$; the subalgebra $\mathcal{A}_{cl}$, therefore, is closed under the classical Poisson brackets. The classical PB $\{,\}_{cl}$ determines the nondegenerate classical symplectic form ω_{cl}. [If $\{f, g\}_{cl} = \sigma^{\alpha\beta}\frac{\partial f}{\partial\xi^\alpha}\frac{\partial g}{\partial\xi^\beta}$, then $\omega_{cl} = \sigma_{\alpha\beta}d\xi^\alpha \wedge d\xi^\beta$ where the matrix $(\sigma_{\alpha\beta})$ is the inverse of the matrix $(\sigma^{\alpha\beta})$.] When $H_W \in (\mathcal{A}_W)_{reg}$ [which is the case for the H_W of Eq.(94)], the subsystem $(\mathcal{A}_W, \omega_W, H_W)_{reg}$ goes over to the supmech Hamiltonian system $(\mathcal{A}_{cl}, \omega_{cl}, H_{cl})$.

When the $\hbar \to 0$ limits of A_W and ρ_W on the right hand side of Eq.(92) exist (call them A_{cl} and ρ_{cl}), we have

$$Tr(A\rho) \to \int\int A_{cl}(x, p)\rho_{cl}(x, p)dxdp. \tag{97}$$

The quantity ρ_{cl} must be non-negative (and, therefore, a genuine density function). To see this, note that, for any operator $A \in \mathcal{A}$ such that $A_W \in (\mathcal{A}_W)_{reg}$, the object A^*A goes over to $\bar{A}_W * A_W$ in the Weyl-Wigner-Moyal formalism which, in turn, goes to $\bar{A}_{cl} A_{cl}$ in the $\hbar \to 0$ limit; this limit, therefore, maps non-negative operators to non-negative functions. Now if, in Eq.(97), A is a non-negative operator, the left hand side is non-negative for an arbitrarily small value of $\hbar$ and, therefore, the limiting value on the right hand side must also be non-negative. This will prove the non-negativity of ρ_{cl} if the objects A_{cl} in Eq.(97) realizable as classical limits constitute a dense set of non-negative functions in $C^\infty(M)$. This class is easily seen to include non-negative polynomials; good enough.

In situations where the $\hbar \to 0$ limit of the time derivative equals the time derivative of the classical limit [i.e. we have $A(t) \to A_{cl}(t)$ and $\frac{dA(t)}{dt} \to \frac{dA_{cl}(t)}{dt}$], the Heisenberg equation of motion for A(t) goes over to the classical Hamilton's equation for $A_{cl}(t)$. With a similar proviso, one obtains the classical Liouville equation for ρ_{cl} as the classical limit of the von Neumann equation.

Before closing this section, we briefly discuss an interesting point :

For commutative algebras, the inner derivations vanish and one can have only outer derivations. Classical mechanics employs a subclass of such algebras (those of smooth functions on manifolds). It is an interesting contrast to note that, while the quantum symplectics employ only inner derivations, classical symplectics employ only outer derivations. The deeper significance of this is related to the fact that the noncommutativity of quantum algebras is generally tied to the nonvanishing of the Planck constant $\hbar$. [This is seen most transparently in the star product of Eq.(93) above.] In the limit $\hbar \to 0$, the algebra becomes commutative (the star product of functions reduces to ordinary product) and the inner derivations become outer derivations (commutators go over to classical Poisson brackets implying that an inner derivation D_A goes over to the Hamiltonian vector field $X_{A_{cl}}$).

5. AXIOMS

We shall now write down a set of axioms covering the work presented in papers I and II. Before the statement of axioms, a few points are in order :

(i) These axioms are meant to be provisional; the 'final' axioms will, hopefully, be formulated (not necessarily by the present author) after a reasonably satisfactory treatment of quantum theory of fields and space-time geometry in an appropriately augmented supmech type framework has been given.

(ii) The terms 'system', 'observation', 'experiment' and a few other 'commonly used' terms will be assumed to be understood. The term 'relativity scheme' employed below will be understood to have its meaning as explained in section 2.5.

(iii) The 'universe' will be understood as the largest possible observable system containing every other observable system as a subsystem.
(iv) By an *experimentally accessible system* we shall mean one whose 'identical' (for all practical purposes) copies are reasonably freely available for repeated trials of an experiment. Note that the universe and its 'large' subsystems are not included in this class.
(v) The term 'system' will, henceforth will normally mean an experimentally accessible one. Whenever it is intended to cover the universe and/or its large subsystems (this will be the case in the first three axioms only), the term system* will be used.

The axioms will be labeled as **A1,..., A7**.

A1.(*Probabilistic framework; System algebra and states*)
(a) *System algebra; Observables.* A system* S has associated with it a (topological) superalgebra $\mathcal{A} = \mathcal{A}^{(S)}$ satisfying the conditions stated in section 3.4 of I. (Its elements will be denoted as A,B,...). Observables of S are elements of the subset $\mathcal{O}(\mathcal{A})$ of even Hermitian elements of $\mathcal{A}$.
(b) *States.* States of the system*, also referred to as the states of the system algebra $\mathcal{A}$ (denoted by the letters $\phi, \psi, ..$), are defined as (continuous) positive linear functionals on $\mathcal{A}$ which are normalized [i.e. $\phi(I) = 1$ where I is the unit element of $\mathcal{A}$]. The set of states of $\mathcal{A}$ will be denoted as $\mathcal{S}(\mathcal{A})$ and the subset of pure states by $\mathcal{S}_1(\mathcal{A})$. For any $A \in \mathcal{O}(\mathcal{A})$ and $\phi \in \mathcal{S}(\mathcal{A})$, the quantity $\phi(A)$ is to be interpreted as the expectation value of A when the system is in the state ϕ.
(c) Expectation values of odd elements of $\mathcal{A}$ vanish in every pure state (hence in every state).
(d) *Compatible completeness of observables and pure states.* The pair $(\mathcal{O}(\mathcal{A}),\ \mathcal{S}_1(\mathcal{A}))$ satisfies the CC condition described in section 2.2.
(e) *Experimental situations and probabilities.* An experimental situation (relating to observations on the system* S) has associated with it a positive observable-valued measure (PObVM) as defined in section 2.1; it associates, with measurable subset of a measurable space (the 'value space' of for the quantities being measured), objects called supmech events which have measure-like properties. Given the system prepared in a state ϕ, the probability of realization of a supmech event $\nu(E)$ is $\phi(\nu(E))$. It is stipulated that all probabilities in the formalism must be of this type.

A2. *Differential calculus; Symplectic structure.* The system algebra $\mathcal{A}$ of a system* S is such as to permit the development of superderivation-based differential calculus on it (as described in section 2 of I); moreover, it is equipped with a real symplectic form ω thus constituting a symplectic superalgebra $(\mathcal{A}, \omega)$ [more generally, a generalized symplectic superalgebra $(\mathcal{A}, \mathcal{X}, \omega)$ when the derivations are restricted to a distinguished Lie sub-superalgera $\mathcal{X}$ of the Lie superalgebra $SDer(\mathcal{A})$of the superderivations of $\mathcal{A}$].

A3. *Dynamics.* The dynamics of a system* S is described by an equicontinuous one-parameter family of canonical transformations generated by an even Hermitian element H (the Hamiltonian) of $\mathcal{A}$ which is bounded below in the sense that its expectation values in all pure states (hence in all states) are bounded below.

The mechanics described by the above-stated axioms will be referred to as Supmech. The triple $(\mathcal{A}, \omega, H)$ or, more precisely, the quadruple $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$ will be said to constitute a supmech Hamiltonian system.

A4. *Relativity scheme.* For systems admitting space-time description, the 'principle of relativity', as described in section 2.5, will be operative.

A5. *Elementary systems; Material particles.* (a) In every relativity scheme, material particles will be understood to be localizable elementary systems (as defined in sections 2.4 and 2.5).
(b) The system algebra for a material particle will be the one generated by its fundamental observables (as defined in section 2.5) and the identity element.

A6. *Coupled systems.* Given two systems S_1 and S_2 described as supmech Hamiltonian systems $(\mathcal{A}^{(i)}, \mathcal{S}_1^{(i)}, \omega^{(i)}, H^{(i)})$ (i=1,2), the coupled system $(S_1 + S_2)$ will be described as a supmech Hamiltonian system $(\mathcal{A}, \mathcal{S}_1, \omega, H)$ with

$$\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}, \quad \mathcal{S}_1 = \mathcal{S}_1(\mathcal{A}), \quad \omega = \omega^{(1)} \otimes I_2 + I_1 \otimes \omega^{(2)}$$

(where I_1 and I_2 are the unit elements of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ respectively) and H as in Eq.(100) of I.

Note. Theorem (2) in I implied restrictions on the possible situations when the interaction of two systems along the lines of the axiom **A6** can be consistently described. A consequence of this theorem is that *all* experimentally accessible systems in nature must have either supercommutative or non-supercommutative system algebras. The next axiom indicates the choice.

A7. *Quantum systems.* All (experimentally accessible) systems in nature have non-supercommutative system algebras (and hence are quantum systems); they have a quantum symplectic structure (as defined in section 3.3 of I) with the universal parameter $b = -i\hbar$.

Note. (i) The quantum systems were shown (in section 3.2) to have equivalent (as supmech Hamiltonian systems) Hilbert space based realizations (without introducing additional postulates); those having finitely generated system algebras were guaranteed to have their system algebras represented irreducibly in the Hilbert space.

(ii) Axioms A7 and A5(a) imply that all material particles are localizable elementary quantum systems. Since they have finitely generated system algebras, the corresponding supmech Hamiltonian systems are guaranteed to have Hilbert space based realizations with the system algebra represented faithfully and irreducibly. They can be treated as in section 3.4 without introducing any extra postulates; in particular, introduction of the Schrödinger wave functions with the traditional Born interpretation and the Schrödinger dynamics are automatic.

(iii) General quantum systems were shown in section 3.2 to admit commutative superselection rules.

10. CONCLUDING REMARKS

1. The central message of the first two papers in this series is this : Complex associative algebras are the appropriate objects for the development of a universal mechanics. The proposed universal mechanics— supmech — is constrained by the formalism (and empirical acceptability) to reduce to traditional quantum mechanics for all 'experimentally accessible' systems. It is worth re-emphasizing that, for an autonomous development of quantum mechanics, the fundamental objects are algebras and not Hilbert spaces.

2. A contribution of the present work expected to be of some significance for the algebraic schemes in theoretical physics and probability theory is the introduction of the condition of compatible completeness for observables and pure states [axiom A1(d)] which plays an important role in ensuring that the quantum systems defined algebraically in section 3.1, have faithful Hilbert space-based realizations. It is desirable to formulate necessary and/or sufficient conditions on the superalgebra $\mathcal{A}$ alone (i.e. without reference to states) so that the CC condition is automatically satisfied.

An interesting result, obtained in section 2.3, is that the superclassical systems with a finite number of fermionic generators generally do not satisfy the CC condition. This probably explains their non-occurrence in nature. It is worth investigating whether the CC condition is related to some stability property of dynamics.

3. Some features of the development of QM in the present work (apart from the fact that it is autonomous) should please theoreticians : there is a fairly broad-based algebraic formalism connected smoothly to the Hilbert space QM; there is a natural place for commutative superselection rules and for the Dirac's bra-ket formalism; the Planck constant is introduced 'by hand' at only one place (at just the right place : the quantum symplectic form) and it appears at all conventional places automatically. Moreover, once the concepts of localization, elementary system and standard quantum system are introduced at appropriate places, it is adequate to define a material particle as a localizable elementary quantum system ; 'everything else' — including the emergence of the Schrödinger wave functions with their traditional interpretation and the Schrödinger equation — is automatic.

4. The treatment of quantum-classical correspondence in section 4, illustrated with the example of a nonrelativistic spinless particle, makes clear as to how the subject should be treated in the general case : go from the traditional Hilbert space -based description of the quantum system to an equivalent (in the sense of a supmech hamiltonian system) phase space description in the Weyl-Wigner-Moyal formalism, pick up the appropriate subsets in the observables and states having smooth $\hbar \to 0$ limits and verify that the limit gives a commutative supmech Hamiltonian system (which is generally a traditional classical hamiltonian system).

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REFERENCES

[**1**] L.M. Alonso, Group Theoretical Foundations of Classical and Quantum Mechanics. II. Elementary Systems, J. Math. Phys. **20** (1979) 219-230.

[**2**] A.P. Antoine, J. Math. Phys. **10** (1969) 53-69,2276-2290.

[**3**] H. Bacry, Localizability and Space in Quantum Physics, Lecture Notes in Physics, vol 308, Springer-Verlag, Berlin, 1988.

[**4**] V. Bargmann, On unitary ray representations of continuous groups, Ann. Math. **59** (1954) 1-46.

[**5**] F. Bayen et al, Deformation theory and quantization (I. Deformations of symplectic structures; II. Physical applications), Annals of Phys. **110** (1978) 61,111.

[**6**] J. Bellissard, M. Vitot, Heisenberg's picture and non-commutative geometry of the semiclassical limit in quantum mechanics, Ann. Inst. Henri Poincaré **52** (1990) 175.

[**7**] F.A. Berezin, Superanalysis, edited by A.A. Kirillov, D. Reidel Pub. Co., Dordrecht, 1987.

[**8**] M. Berry, 'Some Quantum-Classical Asymptotics' in Chaos and Quantum Physics, Les Houches, session LII, 1989, J. Elsevier Science Publishers, 1991.

[**9**] N.N. Bogolubov, A.A. Logunov, I.T. Todorov, Introduction to Axiomatic Quantum Field Theory, Benjamin/Cummings, Reading, 1975.

[**10**] A. Böhm, The Rigged Hilbert Space and Quantum Mechanics, Lecture Notes in Physics, vol 78, Springer, Berlin, 1978.

[**11**] P. Busch, M. Grabowski, P.J. Lahti, Operational Quantum Physics, Springer-Verlag, Berlin, 1995.

[**12**] J.F. Cariñena, M. Santander, On the Projective Unitary Representations of Connected Lie Groups, J. Math. Phys. **16** (1975) 1416-1420.

[**13**] T. Dass, Symmetries, gauge fields, strings and fundamental interactions, vol. I: Mathematical techniques in gauge and string theories, Wiley Eastern Limited, New Delhi, 1993.

[**14**] T. Dass, A Stepwise Planned Approach to the Solution of Hilbert's Sixth Problem. I : Noncommutative Symplectic Geometry and Hamiltonian Mechanics. arXiv : 0909.4606 v2 [math-ph] (2009).

[**15**] T. Dass, S.K. Sharma, Mathematical Methods in Classical and Quantum Physics. Universities Press, Hyderabad, 1998.

[**16**] E.B. Davies, Quantum Theory of Open Systems, Academic Press, London, 1976.

[**17**] R. de la Madrid, The role of the rigged Hilbert space in quantum mechanics. *Eur. J. Phys.* **26** (2005) 287-312; ArXiv : quant-ph/0502053.

[**18**] C. DeWitt-Morette, K.D. Elworthy, A stepping stone to stochastic analysis, Phys. Rep. **77** (1981) 125-167.

[**19**] D.A. Dubin, M.A. Hennings Quantum Mechanics, Algebras and Distributions, Longman Scientific and Technical, Harlow, 1990.

[**20**] M. Dubois-Violette, 'Lectures on Graded Differential Algebras and Noncommutative Geometry' in Noncommutative Differential Geometry and its Aapplication to Physics (Shonan, Japan, 1999), pp 245-306. Kluwer Academic Publishers, 2001; arXiv: math.QA/9912017.

[**21**] I.M. Gelfand, N.J. Vilenkin, Generalized Functions, vol. IV, Academic Press, New York, 1964.

[**22**] J. Glimm, A. Jaffe, Quantum Physics: a Functional Integral Point of View, Springer Verlag, New York, 1981.

[**23**] J.M. Gracia-Bondía, J.C. Várilly, Phase space representation for Galilean quantum particles of arbitrary spin, J.Phys.A: Math.Gen. **21** (1988) L879-L883.

[**24**] V. Guillemin, S. Sternberg, Symplectic Techniques in Physics, Cambridge University Press, 1984.

[**25**] A.S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory, North Holland Publishing Corporation, Amsterdam, 1982.

[**26**] L. Hörmander, Weyl calculus of pseudodifferential operators, Comm. Pure Appl. Math. **32** (1979) 359-443.

[**27**] S.S. Horuzhy, Introduction to Algebraic Quantum Field Theory, Kluwer Academic Publishers, Dordrecht, 1990.

[**28**] A. Inoue, Tomita-Takesaki Theory in Algebras of Unbounded Operators, Springer, Berlin, 1998.

[**29**] P. Kristensen, L. Mejlbo, E. Thue Poulsen, Tempered distributions in infinitely many dimensions I. Canonical field operators, Comm. Math. Phys. **1** (1965) 175-214.

[**30**] G. Lassner, Algebras of unbounded operators and quantum dynamics, Physica **124A** (1984) 471-480.

[**31**] K.C. Liu, J. Math. Phys. **16** (1975) 2054.

[**32**] K.C. Liu, J. Math. Phys. **17** (1976) 859.

[**33**] G.W. Mackey, Imprimitivity for representations of locally compact groups, Proc. Nat. Acad. Sci. U.S. **35** (1949) 537-545.

[**34**] Y. Matsushima, Differentiable Manifolds, Marcel Dekker, New York, 1972.

[**35**] J.E. Moyal, Quantum mechanics as a statistical theory, Proc. Camb. Phil. Soc. **45** (1949) 99-124.

[**36**] T.D. Newton, E.P. Wigner, Localized states for elementary systems, Rev. Mod. Phys. **21** (1949) 400-406.

[**37**] K.R. Parthasarathy, An Introduction to Quantum Stochastic Calculus, Birkhaüser, Basel, 1992.

[**38**] R.T. Powers, Self-adjoint algebras of unbounded operators, *Comm. Math. Phys.* **21** (1971) 85-124.

[**39**] J.E. Roberts, J. Math. Phys. **7** (1966) 1097-1104.

[**40**] W. Rudin, Functional Analysis, Tata McGraw-Hill, New Delhi, 1974.

[**41**] J.-M. Souriau, Structure of Dynamical Systems, a Symplectic View of Physics, Birkhäuser, Boston, 1997.

[**42**] E.C.G. Sudarshan, N. Mukunda *Classical Dynamics : A Modern Perspective.* Wiley, New York, 1974.

[**43**] V.S. Varadarajan, Geometry of Quantum Theory, 2nd ed., Springer-Verlag, New York, 1985.

[**44**] H. Weyl, Theory of Groups and Quantum Mechanics, Dover, New York, 1949.

[**45**] A.S. Wightman, On the localizability of quantum mechanical systems, Rev. Mod. Phys. **34** (1962) 845-872.

[**46**] E.P. Wigner, On the quantum correction for thermodynamic equilibrium, Phys. Rev. **40** (1932) 749-759.

[**47**] E.P. Wigner, Unitary representations of the inhomogeneous Lorentz group, Ann. Math.(N.Y.) **40** (1939) 149-204.

[**48**] M.W. Wong, Weyl Transforms, Springer, New York, 1998.