isid/ms/2010/06
September 10, 2010
http://www.isid.ac.in/~statmath/eprints

# On the construction of nested orthogonal arrays

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# On the Construction of Nested Orthogonal Arrays

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**Abstract. Nested orthogonal arrays are useful in obtaining space-filling designs for an experimental set up consisting of two experiments, the expensive one of higher accuracy to be nested in a larger inexpensive one of lower accuracy. Systematic construction methods of some families of symmetric and asymmetric nested orthogonal arrays were provided recently in [1]. In this paper, we provide some more methods of construction of nested orthogonal arrays.**

**Keywords**: *Orthogonal arrays; Hadamard matrices; Resolvable orthogonal arrays; Juxtaposition*

## 1. INTRODUCTION

An (ordinary) orthogonal array, $OA(N, k, s_1 \times s_2 \times \cdots \times s_k, g)$, having $N$ rows, $k$ columns, $s_1, \ldots, s_k$ symbols and strength $g$ $(2 \le g < k)$ is an $N \times k$ matrix with elements in the $i$th column from a set of $s_i \ge 2$ distinct symbols $(1 \le i \le k)$, in which all possible combinations of symbols appear equally often as rows in every $N \times g$ subarray.

In an $OA(N, k, s_1 \times \cdots \times s_k, g)$, if among $s_1, \ldots, s_k$, there are $w_i$ that equal $\mu_i$ $(1 \le i \le u)$, where $w_1, \ldots, w_u, \mu_1, \ldots, \mu_u$ are positive integers $(\mu_i \ge 2, 1 \le i \le u, w_1 + \cdots + w_u = k)$, then we will use the notation $OA(N, k, \mu_1^{w_1} \times \cdots \times \mu_u^{w_u}, g)$ for $OA(N, k, s_1 \times \cdots \times s_k, g)$. In particular, if $s_1 = s_2 = \cdots = s_k = s$, then the array reduces to a *symmetric* orthogonal array, denoted simply by $OA(N, k, s, g)$. Otherwise, the array is an *asymmetric* orthogonal array. Orthogonal arrays have been studied extensively and for a comprehensive account of the theory and applications of such arrays, a reference may be made to [3].

In recent years, considerable attention has been paid to experimental situations consisting of two experiments, the expensive one of higher accuracy being nested in a larger

and relatively less expensive one of lower accuracy. The higher accuracy experiment can, for instance, correspond to a smaller physical experiment while the lower accuracy one can be a larger computer experiment. The modeling and analysis of data from such nested experiments has been addressed by several authors (see e.g., [4], [6], [7], [8]). Nested orthogonal arrays are useful in designing such nested experiments.
We now recall the definition of a nested orthogonal array.

**Definition**. *A nested orthogonal array, $NOA((N, M), k, (s_1 \times s_2 \times \cdots \times s_k, r_1 \times r_2 \times \cdots \times r_k), g)$, where $r_i \leq s_i$, with strict inequality for at least one $i$, $1 \leq i \leq k$, and $M < N$, is an orthogonal array $OA(N, k, s_1 \times \cdots \times s_k, g)$ which contains an $OA(M, k, r_1 \times \cdots \times r_k, g)$ as a subarray.*

If $s_1 = s_2 = \cdots = s_k = s$ and $r_1 = r_2 = \cdots = r_k = r$, then one obtains a *symmetric* nested orthogonal array, denoted by $NOA((N, M), k, (s, r), g)$, where $M < N$ and $r < s$. Otherwise, the array is an *asymmetric* nested orthogonal array.

As noted in [1], in the context of asymmetric nested orthogonal arrays, the above definition does not preclude the possibility of the existence of an asymmetric nested orthogonal array wherein the smaller orthogonal array is a symmetric orthogonal array, nested within a larger asymmetric orthogonal array.

The question of existence of symmetric nested orthogonal arrays has been examined in detail in [5], where some examples of such arrays can also be found. Methods of construction of several families of symmetric and asymmetric nested orthogonal arrays have been provided recently in [1]. In this communication, some more methods of construction of nested orthogonal arrays are provided.

## 2. PRELIMANIRIES AND NOTATION

We first introduce some notation. For a positive integer $m$, $\mathbf{1}_m$, $I_m$ and $\mathbf{0}_m$ respectively, denote an $m \times 1$ vector with all elements equal to 1, an identity matrix of order $m$ and $m \times 1$ null vector. $A'$ will denote the transpose of a matrix $A$. For a pair of matrices $E = (e_{ij})$ and $F$, of orders $m \times n$ and $u \times v$, respectively, $E \otimes F$ will denote their Kronecker (tensor) product, i.e, $E \otimes F$ is an $mu \times nv$ matrix given by $(e_{ij}F)$.

A square matrix $H_n$ of order $n$ with entries $\pm 1$ is called a Hadamard matrix if $H_n H_n' = nI_n$. A positive integer $n$ is called a Hadamard number if $H_n$ exists. $H_n$ trivially exists for $n = 1, 2$ and a necessary condition for the existence of a Hadamard matrix of order $n > 2$ is that $n \equiv 0 \pmod 4$. Note that if $H_n$ is a Hadamard matrix, then we also have $H_n' H_n = nI_n$. From the definition of a Hadamard matrix, it is seen easily that a Hadamard matrix remains so if any of its rows or columns is multiplied by $-1$. Therefore, without loss of generality, one can write a Hadamard matrix with its first column consisting of only $+1$'s. For more details on Hadamard matrices, see e.g., [4].

Finally, an ordinary orthogonal array $OA(N, k, s_1 \times \cdots \times s_k, g)$ is called *tight* if the number of rows of the array attains the Rao's lower bound on the number of rows; for details on Rao's bounds, see e.g., [3]. In particular, Rao's bounds for arrays of strength two and three are given respectively, by

$$N \geq 1 + \sum_{i=1}^{k} (s_i - 1), \text{ if } g = 2 \tag{1}$$

$$N \geq 1 + \sum_{i=1}^{k} (s_i - 1) + (s^* - 1) \left\{ \sum_{i=1}^{k} (s_i - 1) - (s^* - 1) \right\}, \text{ if } g = 3, \tag{2}$$

where $s^* = \max_{1 \leq i \leq k} s_i$.

## 3. SYMMETRIC NESTED ORTHOGONAL ARRAYS

Barring one family, all the symmetric nested orthogonal arrays constructed in [1] have both $s$ and $r$ as powers of 2. In practice however, situations arise when both $s$ and $r$ are not necessarily powers of 2; for example, a popular choice is $s = 3, r = 2$. Thus, it is important to find arrays where both $s$ and $r$ are not powers of 2. To that end, we have the following result.

**Theorem 1**. *Let $s \geq 3$ be an integer. Then there exists a symmetric nested orthogonal array $NOA((s^4, 8), 4, (s, 2), 3)$. Furthermore, $k = 4$ is the maximum number of columns that these arrays can accommodate.*

*Proof.* The desired (symmetric) nested orthogonal array can be constructed by considering the $s^4 \times 4$ matrix $A$, whose rows are all possible 4-tuples with elements $0, 1, \ldots, s-1$ (say) and observing that the $8 \times 4$ matrix $B$ shown below, is a submatrix of $A$:

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}'.$$

It is easy to check that $B$ is a (symmetric) $OA(8,4,2,3)$ and therefore $A$ is an $NOA((s^4,8), 4,(s,2),3)$. Clearly, in order that this nested orthogonal array exists, it is necessary that both the larger and smaller orthogonal arrays exist individually. By (2), in an ordinary orthogonal array $OA(8,k,2,3)$, $k \leq 4$ and thus, the assertion about the maximum number of columns in the nested orthogonal array follows. $\square$

**Remark**. Taking $s=3$ in Theorem 1, one gets an $NOA((81,8),4,(3,2),3)$. It may be noted that this array cannot be obtained via an application of Lemma 3 in [5]. An $NOA((s^5,(s-1)^5,5,(s,s-1),4)$ exists for every integer $s \geq 3$ ([5]). For $s=3$, one thus obtains an $NOA((243,32),5,(3,2),4)$. Applying Lemma 3 in [5] to this nested orthogonal array yields an $NOA((81,16), 4,(3,2),3)$. However, in this nested array, the smaller array has more rows than that in the corresponding array in Theorem 1 above. Moreover, an application of either Theorem 1 or Theorem 2 in [5] shows that in an $NOA((81,16), k,(3,2),3)$, $k \leq 5$ and thus, one has $k \leq 5$. This upper bound on $k$ is not attained by the array $NOA((81,16),4,(3,2),3)$.

## 4. ASYMMETRIC NESTED ORTHOGONAL ARRAYS

4.1. *Use of Hadamard matrices.*

We make use Hadamard matrices to obtain some families of asymmetric nested orthogonal arrays of strength two and three. Let $u \geq 4$ be a Hadamard number and $H_u$ be a Hadamard matrix of order $u$. Write $H_u$ as $H_u = [\mathbf{1}_u\ A^*]$. Let $A$ be a $u \times (u-1)$ matrix obtained by replacing the $-1$'s in $A^*$ by 0. Then $A$ is a symmetric orthogonal array $OA(u,u-1,2,2)$ of strength two with symbols 0 and 1. Let $\bar{A}$ be a $u \times (u-1)$ matrix obtained by interchanging the two symbols in $A$. Let $t,m$ be integers where $t \geq 3$ and $2 \leq m < t$. Consider the $tu \times u$ matrix $B$ given by

$$\begin{bmatrix} A' & \bar{A}' & \cdots & A' & \bar{A}' & \cdots & A' & \bar{A}' \\ \mathbf{0}'_u & \mathbf{1}'_u & \cdots & (m-2)\mathbf{1}'_u & (m-1)\mathbf{1}'_u & \cdots & (t-2)\mathbf{1}'_u & (t-1)\mathbf{1}'_u \end{bmatrix}'.$$

Then, it is easy to verify that $B$ is an asymmetric $NOA((tu,mu),u,(t\times 2^{u-1},m\times 2^{u-1}),2)$ of strength two. The first $mu$ rows of $B$ form an $OA(mu,u,\ m\times 2^{u-1},2)$ while $B$ is an $OA(tu,u,t\times 2^{u-1},2)$.

If $t$ and $m$ are both even integers, then $B$ is an $NOA((tu,mu),u,(t\times 2^{u-1},m\times 2^{u-1}),3)$ of strength three and in such a case, $u-1$ is the maximum number of 2-symbol columns that such an array can accommodate. The assertion about the strength of the array being 3 follows from the well known fact that $[A' \quad \bar{A}']'$ is an $OA(2u, u-1, 2, 3)$ and that about the maximum number of columns follows from the fact that by (2), in an $OA(mu, k, m \times 2^k), 3)$, $k \leq u-1$. We thus have

**Theorem 2**. *The existence of a Hadamard matrix of order $u$ implies the existence of an asymmetric $NOA((tu, mu), u, (t \times 2^{u-1}, m \times 2^{u-1}), 2)$. Furthermore, if $t$ and $m$ are both even integers, then $B$ is an $NOA((tu, mu), u, (t \times 2^{u-1}, m \times 2^{u-1}), 3)$ and $u-1$ is the maximum number of 2-symbol columns that such an array can accommodate.*

**Example 1**. To illustrate Theorem 2, first let $t = 3, m = 2$. Then,

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}'$$

and

$$B = \begin{bmatrix} \mathbf{0}' & \mathbf{1}' & 2\mathbf{1}' \\ A' & \bar{A}' & A' \end{bmatrix}'.$$

Clearly, $B$ is an asymmetric nested orthogonal array $NOA((12, 8), 4, (3 \times 2^3, 2^4), 2)$. where the first 8 rows of $B$ form a symmetric $OA(8, 4, 2, 2)$ while all the 12 rows form an asymmetric $OA(12, 4, 3 \times 2^3, 2)$.

Next, let $t = 6, u = 2$ and $A$ as the $OA(4, 3, 2, 2)$ exhibited above. The array $B$ shown below in transposed form is an $NOA((24, 8), 4, (6 \times 2^3, 2^4), 3)$:

$$\begin{bmatrix} 0011 & 1100 & 0011 & 1100 & 0011 & 1100 \\ 0101 & 1010 & 0101 & 1010 & 0101 & 1010 \\ 0110 & 1001 & 0110 & 1001 & 0110 & 1001 \\ 0000 & 1111 & 2222 & 3333 & 4444 & 5555 \end{bmatrix}'.$$

The first 8 rows of the above array is a tight $OA(8, 4, 2, 3)$ while all the 24 rows form an $OA(24, 4, 6 \times 2^3, 3)$.

We now construct another family of asymmetric nested orthogonal arrays using Hadamard matrices. As before, let $H_u$ be a Hadamard matrix of order $u \geq 4$ and let $A$ and $\bar{A}$ be the 2-symbol orthogonal arrays $OA(u, u-1, 2, 2)$ derived from $H_u$ and described in the beginning of this section.

Let $\boldsymbol{c} = (0, 1, \ldots, u-1)'$ and define a $2u \times (u+1)$ matrix $B$ as

$$B = \begin{bmatrix} \boldsymbol{c} & \mathbf{0}_u & A \\ \boldsymbol{c} & \mathbf{1}_u & \bar{A} \end{bmatrix}.$$

For $0 \le i \le u-1$, let $\boldsymbol{a}_i'$ be the $i$th row of $A$ and $\boldsymbol{b}_i'$ be the $i$th row of $\bar{A}$. Define the $(u-2) \times 1$ vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as $\boldsymbol{\alpha} = (2, 3, \ldots, u-1)'$ and $\boldsymbol{\beta} = (u, u+1, \ldots, 2u-3)'$ and let

$$C = \begin{bmatrix} \mathbf{0}_{u-2} & \boldsymbol{\alpha} & \mathbf{1}_{u-2} \otimes \boldsymbol{a}_0' \\ \mathbf{0}_{u-2} & \boldsymbol{\beta} & \mathbf{1}_{u-2} \otimes \boldsymbol{b}_0' \\ \mathbf{1}_{u-2} & \boldsymbol{\alpha} & \mathbf{1}_{u-2} \otimes \boldsymbol{a}_1' \\ \mathbf{1}_{u-2} & \boldsymbol{\beta} & \mathbf{1}_{u-2} \otimes \boldsymbol{b}_1' \\ 2\mathbf{1}_{u-2} & \boldsymbol{\alpha} & \mathbf{1}_{u-2} \otimes \boldsymbol{a}_2' \\ 2\mathbf{1}_{u-2} & \boldsymbol{\beta} & \mathbf{1}_{u-2} \otimes \boldsymbol{b}_2' \\ \vdots & & \\ (u-1)\mathbf{1}_{u-2} & \boldsymbol{\alpha} & \mathbf{1}_{u-2} \otimes \boldsymbol{a}_{u-1}' \\ (u-1)\mathbf{1}_{u-2} & \boldsymbol{\beta} & \mathbf{1}_{u-2} \otimes \boldsymbol{b}_{u-1}' \end{bmatrix}.$$

We then have the following result.

**Theorem 3**. *The matrix* $D = \begin{bmatrix} B \\ C \end{bmatrix}$ *is an asymmetric nested orthogonal array* $NOA((2u^2-2u, 2u), u+1, (u \times (2u-2) \times 2^{u-1}, u \times 2^u), 2)$. *Furthermore,* $u+1$ *is the maximum number of columns that such an array can accommodate.*

*Proof*. First observe that $B$ as above is an asymmetric orthogonal array, $OA(2u, u+1, u \times 2^u, 2)$ of strength two. Furthermore, this array is *tight* as the lower bound in (1) is attained. In $B$, the first column has $u$ symbols, $0, 1, \ldots, (u-1)$ and the remaining $u$ columns have two symbols each, 0 and 1. Also, it is easy to see that $C$ is an asymmetric orthogonal array $OA(2u^2-4u, u+1, u \times (2u-4) \times 2^{u-1}, 2)$, where the first column has $u$ symbols, $0, 1, \ldots, (u-1)$, the second column has $(2u-4)$ symbols, $2, 3, \ldots, (2u-3)$ and the remaining columns have two symbols each, 0 and 1. It then follows that $D$ is an asymmetric nested orthogonal array with the stated parameters, where $B$ is the smaller array, nested within $D$. The claim of the maximum number of columns being $u+1$ follows from the fact that $B$ is a tight array.

□

**Example 2**. Letting $u = 4$ in Theorem 3, one obtains an asymmetric nested orthogonal array $NOA((24,8), 5, (4 \times 6 \times 2^3, 4 \times 2^4), 2)$ displayed below in transposed form:

$$\begin{bmatrix} 0123 & 0123 & 0000 & 1111 & 2222 & 3333 \\ 0000 & 1111 & 2345 & 2345 & 2345 & 2345 \\ 0011 & 1100 & 0011 & 0011 & 1100 & 1100 \\ 0101 & 1010 & 0011 & 1100 & 0011 & 1100 \\ 0110 & 1001 & 0011 & 1100 & 1100 & 0011 \end{bmatrix}'.$$

The first 8 rows of the above array constitute an $OA(8, 5, 4 \times 2^4, 2)$, while all the 24 rows form an $OA(24, 5, 4 \times 6 \times 2^3, 2)$.

Similarly, taking $u = 8$, one obtains an $NOA((112, 16), 9, (8 \times 14 \times 2^7, 8 \times 2^8), 2)$.

4.2. *Use of resolvable arrays.*

We now present some asymmetric nested orthogonal arrays of strength two obtained by exploiting the resolvability of orthogonal arrays. Let $A$ be an $OA(N, k, s_1 \times \cdots \times s_k, 2)$, such that its rows can be partitioned into $s_1$ sets of $N/s_1$ rows each, say $A_1, A_2, \ldots, A_{s_1}$, and where each $A_i$ $(1 \le i \le s_1)$ is an orthogonal array of strength *unity*. Such an orthogonal array is called resolvable. This means that for $1 \le i \le s_1$, $A_i$ is an $OA(N/s_1, k, s_1 \times \cdots \times s_k, 1)$ of strength one.

Let $t, m,\ s_1 \le m < t$ be integers such that $s_1$ divides both $t$ and $m$. Consider the $tN/s_1 \times (k+1)$ matrix $B$ given by

$$B = \begin{bmatrix} \mathbf{0} & A_1 \\ \mathbf{1} & A_2 \\ \vdots & \\ (s_1-1)\mathbf{1} & A_{s_1} \\ \vdots & \\ (m-s_1)\mathbf{1} & A_1 \\ (m-s_1+1)\mathbf{1} & A_2 \\ \vdots & \\ (m-1)\mathbf{1} & A_{s_1} \\ \vdots & \\ (t-s_1)\mathbf{1} & A_1 \\ (t-s_1+1)\mathbf{1} & A_2 \\ \vdots & \\ (t-1)\mathbf{1} & A_{s_1} \end{bmatrix},$$

where $\mathbf{0}$ and $\mathbf{1}$ are $N/s_1 \times 1$ vectors of all zeros and all ones, respectively. Then, we have the following result.

**Theorem 4**. *The array $B$ above is an $NOA((tN/s_1, mN/s_1), k+1, (t\times s_1\times\cdots\times s_k, m\times s_1\times\cdots\times s_k), 2)$.*

*Proof.* From the resolvability of the array $A$, it is easy to see that $B$ is an $OA(tN/s_1, k+1, t\times s_1\times\cdots\times s_k, 2)$. Also, the first $mN/s_1$ rows of $B$ form an $OA(mN/s_1, k+1, m\times s_1\times\cdots\times s_k, 2)$. □

The following example illustrates Theorem 4.

**Example 3**. Consider a resolvable $OA(16, 8, 4^2\times 2^6, 2)$, displayed below in transposed form:

$$\left[\begin{array}{c|c|c|c}
0321 & 3012 & 0312 & 0132 \\
2103 & 0321 & 0312 & 1023 \\
0011 & 0011 & 1100 & 1010 \\
1010 & 1010 & 0110 & 1001 \\
0110 & 0110 & 0101 & 1100 \\
1100 & 0011 & 1100 & 0101 \\
1001 & 1001 & 0101 & 1100 \\
1010 & 0101 & 0110 & 0110
\end{array}\right]',$$

where each set of four rows forms a resolvable set. Thus, $s_1 = 4$. Following Theorem 4, we have an $NOA((4t, 4m), 9, (t\times 4^2\times 2^6, m\times 4^2\times 2^6), 2)$, where $t$ and $m$ are both multiples of 4 and $4\le m < t$. For example, taking $t = 8$ and $m = 4$, one gets an $NOA((32, 16), 9, (8\times 4^2\times 2^6, 4^3\times 2^6), 2)$.

A simple method of obtaining a resolvable orthogonal array is as follows: Let $A^* = OA(N, k, s_1\times s_2\times\cdots\times s_k, 2)$ denote an orthogonal array of strength two. Clearly, $N/s_1$ is an integer. Without loss of generality, let the first column of $A^*$ have symbols $0, 1, \ldots, s_1 - 1$. Permute the rows of $A^*$ such that the first $N/s_1$ rows each have 0 in the first column, the next $N/s_1$ rows have 1 in the first column, …, the last $N/s_1$ rows have the symbol $s_1 - 1$ in the first column. Deleting the first column of (the permuted) $A^*$ leaves a resolvable orthogonal array $OA(N, k-1, s_2\times\cdots\times s_k, 2) = A$, say, i.e., $A = [A_1'\ A_2'\ \cdots\ A_{s_1}']'$, where each $A_i$, as before, is an orthogonal array $OA(N/s_1, k-1, s_2\times\cdots\times s_k, 1)$ of strength unity. Using Theorem 4 and the resolvable orthogonal array just constructed, one thus gets the following corollary to Theorem 4.

**Corollary**. *The existence of an orthogonal array $OA(N, k, s_1\times s_2\times\cdots\times s_k, 2)$ implies* *the existence of a nested orthogonal array* $NOA((tN/s_1, mN/s_1), k, (t \times s_2 \times \cdots \times s_k, m \times s_2 \times \cdots \times s_k), 2)$*, where* $t, m$ *are integers and* $s_1$ *divides both* $t$ *and* $m$*.*

The following examples illustrate the above corollary.

**Example 4**. Consider the (ordinary) asymmetric orthogonal array $OA(12, 5, 3 \times 2^4, 2)$, say $A$, obtained by Wang and Wu [10]. Following the method described above and choosing $s_1 = 2$, we get a resolvable orthogonal array $OA(12, 4, 3 \times 2^3, 2)$, displayed below in transposed form:

$$\left[\begin{array}{cccccc|cccccc} 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]'.$$

Taking $m = 2$ in Theorem 4, we thus have an $NOA(((6t, 12), 5, (t \times 3 \times 2^3, 3 \times 2^4), 2)$, where $t \geq 4$ is an even integer. It was shown in [10] that in an $OA(12, k+1, 3 \times 2^k, 2)$, $k \leq 4$. In view of this result, one cannot add more 2-symbol columns in the arrays $NOA((6t, 12), 5, (t \times 3 \times 2^3, 3 \times 2^4), 2)$.

For $t = 4, 6$ for example, one obtains an $NOA((24, 12), 5, (4 \times 3 \times 2^3, 3 \times 2^4), 2)$ and an $NOA((36, 12), 5, (6 \times 3 \times 2^3, 3 \times 2^4), 2)$, respectively.

**Example 5**. Next, consider an $OA(20, 9, 5 \times 2^8, 2)$ given in [10]. Following the construction described above and again choosing $s_1 = 2$, one obtains a resolvable orthogonal array $OA(20, 8, 5 \times 2^7, 2)$, displayed below:

$$\left[\begin{array}{ccccc|ccccc} 00 & 11 & 22 & 33 & 44 & 00 & 11 & 22 & 33 & 44 \\ 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 \\ 01 & 10 & 11 & 01 & 00 & 10 & 01 & 00 & 10 & 11 \\ 00 & 01 & 10 & 01 & 11 & 11 & 01 & 10 & 10 & 00 \\ 01 & 00 & 01 & 10 & 11 & 10 & 11 & 01 & 10 & 00 \\ 01 & 01 & 01 & 01 & 10 & 10 & 10 & 10 & 01 & 10 \\ 01 & 10 & 00 & 11 & 10 & 01 & 01 & 11 & 00 & 10 \\ 00 & 10 & 01 & 11 & 01 & 11 & 10 & 10 & 00 & 01 \end{array}\right]'.$$

From Theorem 4 therefore, we get an $NOA((10t, 20), 9, (t \times 5 \times 2^7, 5 \times 2^8), 2)$, where $t \geq 4$ is an even integer. It is known ([10]) that in an $OA(20, k+1, 5 \times 2^k, 2)$, $k \leq 8$ and hence, no further 2-symbol columns can be added to such nested orthogonal arrays.

With $t = 4$ for example, one gets an $NOA((40,20), 9, (5 \times 4 \times 2^7, 5 \times 2^8), 2)$ with a maximum number of columns.

4.3. *Arrays by juxtaposition.*
A simple but effective method of construction of (ordinary) asymmetric orthogonal arrays, leading to several new asymmetric orthogonal arrays was proposed by Suen [9]. His method can be described as follows: Let $L_1 = OA(N_1, k+1, u \times s_1 \times \cdots \times s_k, 2)$ and $L_2 = OA(N_2, k+1, v \times s_1 \times \cdots \times s_k, 2)$ be two orthogonal arrays of strength two each such that $N_1/u = N_2/v$, where the $u$ symbols in the first column of $L_1$ are $0, 1, \ldots, u-1$, the $v$ symbols in the first column of $L_2$ are $u, u+1, \ldots, u+v-1$, and for $1 \leq i \leq k$, the $s_i$ symbols in the $(i+1)$st column of both $L_1$ and $L_2$ are $0, 1, \ldots, s_i - 1$. Then, the array $L = [L_1' \;\; L_2']'$ is an $OA(N_1 + N_2, (u+v) \times s_1 \times \cdots \times s_k, 2)$.

From the very method of construction, it easily seen that $L$ in fact is a nested orthogonal array, $NOA((N_1 + N_2, N_1), k+1, ((u+v) \times s_1 \times \cdots \times s_k, u \times s_1 \times \cdots \times s_k), 2)$. The orthogonal array $L_1$ is nested within the larger orthogonal array $L$. All the orthogonal arrays in Table 1 of [9] are thus nested asymmetric orthogonal arrays. For example, taking $L_1 = OA(24, 15, 2 \times 6 \times 2^{13}, 2)$ and $L_2 = OA(36, 15, 3 \times 6 \times 2^{13}, 2)$, obtained by deleting 5 columns from an $OA(36, 20, 6 \times 3 \times 2^{18}, 2)$ and permuting the first two columns, one gets an $NOA((60, 24), 15, (6 \times 5 \times 2^{13}, 6 \times 2^{14}), 2)$.

**ACKNOWLEDGEMENT**

This work was supported by the Indian National Science Academy under the Senior Scientist scheme of the academy. The support is gratefully acknowledged.

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