A Review Of Some Statistical Tests Under Independent Competing Risks Model

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Abstract

Suppose an individual is exposed to k risks of failure. Assume that the latent failure times associated with these risks are independently distributed. We review various tests based on heuristics to test for the hypotheses of stochastic dominance, proportionality of competing risks hazards and for ordering of hazard rates. We also review tests based on martingale theory for independent latent risks. We illustrate our tests on two data sets .

1 Introduction

The competing risks situation was first considered in the eighteenth century when small pox vaccination was being discovered and popularised. D. Bernoulli (1761) posed the question : How much would the mortality be reduced or expected life be increased if the risk of death due to small pox is totally eliminated, the other risks persisting as before? The demographers , over the years, have taken up this question enthusastically. A strong motivation has come from the insurance business in which different premium rates have to be established for groups of persons exposed to different sorts of risks. Of course now it is relevant in several disciplines - medicine, reliability engineering , economics , manufacturing and even the game of cricket (Crowder (2001)).

Let $X_1, X_2, \ldots, X_k$ denote the latent failure times of individuals subject to k risks, where X_i represents the age at death if cause i were the only cause of failure. What is actually observed is the time to failure T, where $T = \min(X_1, X_2, \ldots, X_k)$, and the cause of failure δ where $\delta = j$ if $X_j = \min(X_1, X_2, \ldots, X_k)$. Note that if $X_1, X_2, \ldots, X_k$ are independent and identically distributed random variables with a common distribution function $F(x)$, then it is uniquely determined by the distribution function of the minimum, that is,

$$P[T \leq x] = 1 - [1 - F(x)]^k.$$

If $X's$ are independent, but not identically distributed , Berman (1963) showed that the joint distribution of (T, δ) uniquely determines F_i, the distribution function of X_i.

$$F_i(x) = 1 - \exp[1 - \int_{-\infty}^{x} [1 - \sum_{j=1}^{k} H_j(t)]^{-1} dH_i(t)],\ i = 1, 2, \ldots, k,$$

where $H_j(x) = P[\delta = j, T \leq x]$.

If the latent failure times are not independent, then the underlying distributions are not identifiable on the basis of (T, δ) (see Tsiatis (1975), Crowder (1991, 1993)). The underlying risks may not always be independent. But Keyfitz et al (1972) have strongly argued that risks to human life can be grouped into four non-overlapping sets, which are independent of each other. In this, and other such situations, one can use methods based on the apriori independence assumption while dealing with pooled risks.

Throughout this paper we will assume that the latent failure times are independently distributed. In section 2 we look at the locally most powerful rank tests for equality of distribution functions of independent latent failure times. In sections 3, 4 and 5, respectively, we look at tests based on heuristics to test for the hypotheses of stochastic dominance, proportionality of competing risks hazards and for ordering of hazard rates . In section 6 we review tests based on martingale theory for independent latent risks. Finally we illustrate our tests on two data sets.

2 Locally Most Powerful Rank Tests

We will restrict ourselves to the case when $k = 2$. Let X and Y denote the hypothetical lifetimes due to two risks with corresponding distribution functions F and G, survival functions $\bar{F}$ and $\bar{G}$, density functions f and g and failure rate functions $r_F = f/\bar{F}$ and $r_G = g/\bar{G}$, respectively. We observe $T = \min(X, Y)$ and $\delta = I(X > Y)$. On the basis of data $(T_1, \delta_1), \ldots, (T_n, \delta_n)$ on n independent individuals we wish to test the null hypothesis

$$H_{01} : F(x) = G(x), \text{ for all } x \geq 0 \tag{1}$$

against various alternatives discussed below.

We begin with testing procedures based on likelihood theory. The likelihood function is given by (see Miller (1981))

$$L(t_1,\ldots,t_n;\delta_1,\ldots,\delta_n) = \prod_{i=1}^{n}[g(t_i)\bar{F}(t_i)]^{\delta_i}[f(t_i)\bar{G(t_i)}]^{1-\delta_i}. \tag{2}$$

Let us parametrize the problem by writing $G(x) = F_\theta(x)$, such that the null hypothesis that the two distributions are identical is given by $\theta = 0$. Then, it is easy to see that the locally most powerful test for testing H_{01} against the alternative that $H_1 : \theta > 0$ is based on the statistic

$$\sum_{i=1}^{n}\delta_i[\frac{\bar{f}(t_i)}{f(t_i)} + \frac{f^*(t_i)}{\bar{F}(t_i)}] - \sum_{i=1}^{n}\frac{f^*(t_i)}{\bar{F}(t_i)} \tag{3}$$

where

$$\begin{aligned} f^*(x) &= \frac{\partial}{\partial\theta}F_\theta(x)|_{\theta=0}, \\ \bar{f}(x) &= \frac{\partial}{\partial\theta}f_\theta(x)|_{\theta=0}. \end{aligned} \tag{4}$$

Large values of the statistic are significant. Using standard results (see, Rao (1973) pp 455) it follows that the statistic has limiting normal distribution.

Parametric estimation and testing based on independent latent failures can be studied by using any reasonable parametric form for the distribution of the latent failure times (see, e.g., David and Moeschberger (1978)).

Next we consider the locally most powerful rank tests for the above problem. These would use the ranks of $T_1, T_2, \ldots, T_n$ among themselves and the corresponding identifiers of the cause of failure. Let $T_{(1)} \le T_{(2)} \le \ldots \le T_{(n)}$ denote the ordered failure times. Let

$$W_i = \begin{cases} 1 & \text{if } T_{(i)} \text{ corresponds to a } Y \text{observation,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Thus W_i identifies the cause of failure corresponding to the ith ordered failure time $T_{(i)}$. Let R_j be the rank of T_j among $T_1, \ldots, T_n$. Let

$$\underline{R}' = (R_1, R_2, \ldots, R_n), \;\; \underline{W}' = (W_1, W_2, \ldots, W_n)$$

denote the vector of ranks and indicator functions corresponding to ordered minima. The likelihood of ranks is given by

$$P(\underline{R}, \underline{W}, \theta) = \int \dots \int_{0<t_1<\dots<t_n<\infty} \prod_{i=1}^{n} [f_\theta(t_i)\bar{F}(t_i)]^{w_i} [f(t_i)\bar{F}_\theta(t_i)]^{1-w_i} dt_1 dt_2 \dots dt_n \tag{6}$$

Under the null hypothesis H_{01}, that is, $\theta = 0$

$$P(\underline{R}, \underline{W}, 0) = \frac{1}{n!2^n} = P(R_1 = r_1, \dots, R_n = r_n, W_1 = w_1, \dots, W_n = w_n) \tag{7}$$

where $(r_1, \dots, r_n)$ is a permutation of $(1, \dots, n)$ and $(w_1, \dots, w_n)$ is a vector of $0's$ and $1's$. Under H_{01}, $\underline{R}$ and $\underline{W}$ are independent.

Theorem 1 *Under regularity conditions (see Puri and Sen (1971)) the LMP rank test for testing* $H_{01} : \theta = 0$ *against* $H_1 : \theta > 0$ *is based on the statistic*

$$V = \sum_{j=1}^{n} [a_j w_j - b_j(1 - w_j)] \tag{8}$$

where

$$\begin{aligned} a_j &= n!2^n \int \dots \int_{0<t_1,\dots<t_n<\infty} \frac{\bar{f}(t_j)}{f(t_j)} \prod_{i=1}^{n} [f(t_i)\bar{F}(t_i)dt_i] \\ b_j &= n!2^n \int \dots \int_{0<t_1,\dots<t_n<\infty} \frac{f^*(t_j)}{\bar{F}(t_j)} \prod_{i=1}^{n} [f(t_i)\bar{F}(t_i)dt_i] \end{aligned} \tag{9}$$

In particular,

(i) if $F_\theta(x) = \frac{e^{x+\theta}}{1+e^{x+\theta}}$, the logistic distribution, then

$$V = \sum_{i=1}^{n} (1 - c_j) W_j$$

where

$$c_j = \frac{1}{2n+1} + \sum_{k=2}^{j} \frac{2n(2n-2)\dots(2n-2k+4)}{(2n+1)(2n-1)\dots(2n-2k+3)} \tag{10}$$

which is the analog in the competing risks set up for Wilcoxon-Mann-Whitney statistic for the two sample location problem. The statistic is remarkably simple in the two sample complete data problem compared to what is obtained here for the competing risks data. We would therefore like to study certain tests based on some simpler statistics. Also, the logistic distribution , being on $(-\infty, \infty)$ is not an appropriate model for lifetimes.

(ii) If $F_\theta(x) = 1 - e^{-(1+\theta)x}$, $\theta > 0$, the exponential distribution, then the LMP rank statistic is

$$nU_1 = \sum_{j=1}^{n} W_j = \sum_{j=1}^{n} \delta_j. \tag{11}$$

Here, the statistic U_1 is the usual sign statistic, the proportion of deaths due to second cause.

Note that under exponential alternatives we have proportional hazards and hence T and δ are independent (see Kochar and Proschan (1991)). Since $P(\underline{T} = \underline{t}) = \frac{1}{n!}$, both under the null and the alternative hypotheses, $\underline{T}$ does not provide any additional information for discriminating between the null and the alternative hypotheses. Thus for exponential, or any other proportional hazards alternatives, where $\underline{T}$ and $\underline{W}$ are independent, one cannot improve upon the sign test in the class of all rank tests. But for other alternatives the additional information is useful. Most of these alternatives have been extensively studied for the two sample problem - here we review non-parametric procedures relevant to competing risks data.

We see that in the rank set up the relevant simple statistics will be $nU_1 = \sum_{i=1}^{n} \delta_i$ and $W^+ = \sum_{i=1}^{n} \delta_i R_i$. The first is the sign statistic and the second may be regarded as an adaptation of the Wilcoxon signed rank statistic to the competing risks data, being the sum of the ranks of those lifetimes which were terminated due to the second risk. We detail below certain tests proposed on heuristic grounds which are based on linear combinations of these two.

Before we look at various tests on heuristic grounds let us look at some general results.

Lemma 1 Whenever T and δ are independent, we have

(i) $P[\sum_{i=1}^{n} R_i\delta_i = k] = P[\sum_{i=1}^{n} iW_i = k]$,

(ii) $W_1, W_2, \ldots, W_n$ are independent and identically distributed with $P[W_i = 1] = P[W_i = 0] = 1/2, i = 1, \ldots, n$,

(iii) $W^+ = \sum_{i=1}^{n} R_i\delta_i$ is symmetric about $n(n+1)/4$,

(iv) The moment generating function of $S = \sum_{i=1}^{n} a(R_i)\delta_i = \sum_{i=1}^{n} a_iW_i$ is given by

$$M_S(t) = \frac{1}{2^n}\prod_{i=1}^{n}(1 + e^{ta_i}), \tag{12}$$

(v)

$$\begin{aligned} E[aW^+ + bU_1] &= an(n+1)/2 + b/2, \\ V[aW^+ + bU_1] &= a^2n(n+1)(2n+1)/24 + b * 2n/4. \end{aligned} \tag{13}$$

It should be seen that an improvement over the sign test is possible only when T and δ are dependent under the alternative hypotheses.

3 Tests for stochastic dominance

Here we look at tests for testing

$$H_{01} : F(x) = G(x) \text{ for all } x \tag{1}$$

against the alternative

$$H_{A1} : F(x) \leq G(x) \text{ for all } x, \;\; F(x) < G(x) \text{ for some } x. \tag{2}$$

Under the alternative $X's$ tend to be stochastically larger than $Y's$. This means that under the alternative we expect failures due to risk 2 to occur more often than failures due to risk 1. Note that if $G(x) = F(x + \theta)$ or $G(x) = F((\theta + 1)x)$, then the alternative H_1 implies H_{A1}.

Consider two pairs (T_i, δ_i) and (T_j, δ_j) simultaneoulsly. In the following table we exhibit the correspondence between the ordering of $X's$ and the $Y's$ with the values of δ_i, δ_j and the ordering between T_i and T_j.

TABLE 1

	$\delta_i = 1, \delta_j = 1$	$\delta_i = 1, \delta_j = 0$	$\delta_i = 0, \delta_j = 1$	$\delta_i = 0, \delta_j = 0$
$T_i > T_j$	$X_i > Y_i, X_j > Y_j$ $Y_i > Y_j$	$X_i > Y_i, X_j < Y_j$ $Y_i > X_j$	$X_i < Y_i, X_j > Y_j$ $X_i > Y_j$	$X_i < Y_i, X_j < Y_j$ $X_i > X_j$
$T_i < T_j$	$X_i > Y_i, X_j > Y_j$ $Y_i < Y_j$	$X_i > Y_i, X_j < Y_j$ $Y_i < X_j$	$X_i < Y_i, X_j > Y_j$ $X_i < Y_j$	$X_i < Y_i, X_j < Y_j$ $X_j > X_i$

It is seen that in each case one can order X and Y in only three of the four possible types of pairs.

Define a kernel function

$$\phi_2(T_i, \delta_i, T_j, \delta_j) = \begin{cases} 1 & \text{if } \delta_i = 1, T_i < T_j \\ & \text{or } \delta_i = 1, T_i > T_j, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

and construct a U-statistic based on it

$$U_2 = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \phi_2(T_i, \delta_i, T_j, \delta_j) = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} (n - R_i)\delta_i. \tag{4}$$

Or consider a modified version

$$U_2^* = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} (n - R_i + 1)\delta_i = \frac{1}{\binom{n}{2}} S_2, (say). \tag{5}$$

This would count the number of pairs (X_i, Y_i) for which $(X_i > Y_i)$ also. It is easy to see that the moment generating function of S_2 under H_{01} is given by

$$M_{S_2}(t) = 2^{-n} \prod_{j=1}^{n} (1 + e^{jt}) \tag{6}$$

which is same as that of the Wilcoxon-signed rank statistic under H_{01} (see Hettmansperger (1984)). Hence tables of critical values for the Wilcoxon-signed rank statistic can be used for S_2 also.

From the results on U-statistics it follows that under H_{01} the asymptotic distribution of $\sqrt{n}(U_2 - E(U_2))$ is Normal with mean zero and variance 1/3. Further, using Slutsky's theorem, it is easy to see that U_2 and U_2^* are asymptotically equivalent. Infact $U_2 = \sum_{i=1}^{n}(n - R_i)\delta_i$ is a linear combination of U_2^* and the sign statistic U_1. This test has been proposed by Bagai, Deshpande and Kochar (1989a).

4 Tests for proportionality of competing risks hazard rates

Analysis of data in reliability and mortality studies is often carried out under the proportional hazards model due to Cox (1972). However, it has been observed (see, e.g., Begg et al (1984)) that the ratio of hazard rates is not constant over all ages as assumed under Cox's model. In male mice cancer data due to Hoel (1972), the adverse effect of cancer appears only in the long term. Sengupta and Deshpande (1995) proposed a test for testing the null hypothesis

$$H_{02} : \frac{r_F(x)}{r_G(x)} = \text{constant for all } x \tag{1}$$

against the alternative

$$H_{A2} : \frac{r_F(x)}{r_G(x)} \text{ is increasing in } x \tag{2}$$

Under the null hypothesis failures due to first and second risk occur at the same relative rate throughout whereas under the alternative the failures due to risk 1 occur at a relatively faster rate as age increases.

$\frac{r_F(x)}{r_G(x)}$ is increasing in x implies

$$f(x_2)\bar{G}(x_2)\bar{F}(x_1)g(x_1) - f(x_1)\bar{G}(x_1)\bar{F}(x_2)g(x_2) \geq 0 \text{ for } x_1 < x_2.$$

Integrating the difference above over the range $x_1 < x_2$, we get

$$\Delta_2(F,G) = Pr(Y_1 < X_1, Y_1 < X_2 < Y_2) - Pr(X_1 < Y_1, X_1 < Y_2 < X_2) \tag{3}$$

where X_1, X_2 and Y_1, Y_2 are independent random variables from F and G, respectively. $\Delta_2(F, G) = 0$ under H_{02} and positive under H_{A2}.

In the following table we take a relook at the possible arrangements of $X's$ and $Y's$ with not only pairwise orderings, but those in triples also in pairs as included in the above probabilities.

TABLE 2

	$\delta_i = 1, \delta_j = 1$	$\delta_i = 1, \delta_j = 0$	$\delta_i = 0, \delta_j = 1$	$\delta_i = 0, \delta_j = 0$
$T_i > T_j$	$Y_j < Y_i < X_i$ $Y_j < X_j$	$X_j < Y_i < X_i$ $X_j < Y_j$	$Y_j < X_i < Y_i$ $Y_j < X_j$	$X_j < X_i < Y_i$ $X_j < Y_j$
$T_i < T_j$	$Y_i < Y_j < X_j$ $Y_i < X_i$	$Y_i < X_j < Y_j$ $Y_i < X_i$	$X_i < Y_j < X_j$ $X_i < Y_i$	$X_i < X_j < Y_j$ $X_i < Y_i$

On heuristic basis we isolate those orderings which favour the alternative and define the kernel

$$\phi_3(T_i, \delta_i, T_j, \delta_j) = \begin{cases} 1 & \text{if } \delta_i = 1, \delta_j = 0, T_i < T_j \\ & \text{or } \ \delta_i = 0, \delta_j = 1, T_i > T_j, \\ -1 & \text{if } \delta_i = 1, \delta_j = 0, T_i > T_j \\ & \text{or } \ \delta_i = 0, \delta_j = 1, T_i < T_j, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The corresponding U-statistic is

$$U_3 = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \phi_3(T_i, \delta_i, T_j, \delta_j). \tag{5}$$

Large values of the statistic are significant.

It is easy to see that $E(U_3) = 2\Delta(F, G)$. Also

$$S_3 = \binom{n}{2} U_3 = \sum_{1 \le i < j \le n} (W_i - W_j) = \sum_{i=1}^{n} (n + 1 - 2i) W_i. \tag{6}$$

Therefore

$$S_3 = (n + 1)U_1 - 2W^+ \tag{7}$$

Hence S_3 is a linear combination of the sign statistic and the Wilcoxon signed rank type statistic W^+. Since $(T_1, \ldots, T_n)$ and $(\delta_1, \ldots, \delta_n)$ are independent under H_{02}, we have the moment generating function of S_3 under H_{02} is given by

$$M_{S_3(t)} = \prod_{i=1}^{n} [\frac{1}{a+1} + \frac{a}{a+1} \exp(n+1-2i)t]. \tag{8}$$

This gives $Var_{H_{02}} = \frac{4}{3} \frac{a}{(a+1)^2} \frac{n+1}{n^2-n}$. A consistent estimator of a is given by $\frac{U_1}{1-U_1}$ hence the asymptotic null distribution of $\{\frac{4}{3}U_1(1-U_1)\}^{-1/2}\sqrt{n}U_3$ is standard normal.

Deshpande and Sengupta (1995) have considered the above statistic and have also extended this to the case of more than two independent risks when all the risks other than the two important ones can be grouped together as a single risk and called risk 3.

5 Tests for equality of hazard rates

The next step would be to test if the constant a in H_{02} is unity, that is, we wish to test

$$H_{03} : r_F(x) = r_G(x), \text{ for all } x \tag{1}$$

against the alternative

$$H_{A3} : r_F(x) \leq r_G(x), \text{ for all } x \tag{2}$$

with a strict inequality over a set of non-zero probability.

Note that H_{03} is equivalent to H_{01}, that is failure rates are equal iff distributions are equal but $r_F(x) \leq r_G(x)$ implies $F(x) \leq G(x)$ and not vice-versa. Hence tests which are consistent for testing H_{03} against H_{A3} are also consistent for testing H_{03} or H_{01} against a wider class H_{A1}. Under the alterantive H_{A3} failure rate due to the first risk is uniformly smaller than the failure rate due to second risk . Thus, under the alterantive H_{A3} we would expect the failures due to risk 2 to occur at an earlier stage as compared to the failures due to risk 1. Arrangements of the type $YYXX$ and $XYYX$ favour H_{A3} whereas arrangements of the type $XXYY$ and $YXXY$ would favour a smaller failure rate due to second risk. Kochar (1979) considered the following functional for testing H_{03} against H_{A3}

$$\Delta_3(F,G) = P[YYXX] + P[XYYX] - P[XXYY] - P[YXXY]. \qquad (3)$$

$P[YYXX]$ means $P[(Y_1 < Y_2 < X_1 < X_2)] + P[(Y_1 < Y_2 < X_2 < X_1)] + P[(Y_2 < Y_1 < X_1 < X_2)] + P[(Y_2 < Y_1 < X_2 < X_1)]$, where X_1, X_2, Y_1 and Y_2 are independent observations , the first two from F and the latter two from G. $\Delta_3(F,G) = 0$ under H_{03} and is positive under H_{A3}. Kochar proposed a test based on U-statistic estimator of $\Delta_3(F,G)$. From Table 2 we try to identify the combinations which would enable us to estimate $\Delta_(F,G)$ on the basis of competing risk data.

Define the kernel

$$\phi_4^*(T_i,\delta_i,T_j,\delta_j) = \begin{cases} 1 & \text{if } \delta_i = 1, T_i > T_j \\ & \text{or } \ \delta_j = 1, T_i < T_j, \\ -1 & \text{otherwise} \end{cases} \qquad (4)$$

Consider the U-statistic

$$U_4^* = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \phi_4^*(T_i,\delta_i,T_j,\delta_j). \qquad (5)$$

$E(U_4^*) = \Delta_3(F,G)$. However, we consider an equivalent statistic

$$U_4 = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \phi_4(T_i,\delta_i,T_j,\delta_j), \qquad (6)$$

where

$$\phi_4^*(T_i,\delta_i,T_j,\delta_j) = \begin{cases} 1 & \text{if } \delta_i = 1, T_i > T_j \\ & \text{or } \ \delta_j = 1, T_i < T_j, \\ 0 & \text{otherwise.} \end{cases} \qquad (7)$$

Large values of U_4 are significant and

$$\binom{n}{2} U_4 = \sum_{i=1}^{n} (R_i - 1)\delta_i = \sum_{i=1}^{n} (i-1) W_i. \tag{8}$$

which is again a linear combination of the sign statistic and the Wilcoxon signed rank type statistic. Being a U-statistic , the standardized version of U_4 will have asymptotically $N(0,1)$ distribution. This test has been discussed by Bagai, Deshpande and Kochar (1989b)

Remark We see that the heuristics lead us to linear combinations of type $aW^+ + bU_1$ in all the above cases. It is interesting that such simple statistics are able to discriminate between the various null hypotheses on one hand and the relevant alternatives involving competing risks on the other hand quite effectively. The exact null distributions of these statistics have been studied and all of them have limiting normal distributions.

In the next section we provide a summary of some other work where the proofs of aymptotic distributions of certain tests are based on the martingale central limit theorem.

6 Tests based on martingale theory

Yip and Lam (1992) proposed a class of nonparametric tests for testing equality of failure rates using martingale theory.

Let $\lambda_i(t), i = 1, 2$ denote the instantaneous transition rate, per individual, for a transition from state alive to dead from *ith* cause during the time interval $(0, t]$. Let $N_i(t)$ denote the number of transitions from alive to dead due to *ith* cause during the time interval $(0, t]$, $Y(t) = n - N(t-)$ is the number of survivors just before time t and $N(t) = N_1(t) + N_2(t)$ represents the total number of deaths from the two causes by time t. The Nelson-Aalen estimator for the cumulative hazard rate , $\Lambda_i(\tau) = \int_0^\tau \lambda_i(u)du$, is given by $\hat{\Lambda}_i(\tau) = \int_0^\tau \frac{dN_i(u)}{Y(u)}$, $i = 1, 2$.

For testing the null hypothesis $\lambda_1 = \lambda_2 \; = \lambda$ (say) consider the stochastic process $R(t)$ defined by

$$R(t) = \int_0^t W(u)[d\hat{\Lambda}_1(u) - d\hat{\Lambda}_2(u)], \tag{1}$$

where $W(u)$ is a locally bounded predictable process.

Under H_0, $R(t)$ is a zero mean martingale with variance

$$2E[\int_0^t \{W(u)\}^2(\frac{1}{Y(u)})\lambda(u)du] \tag{2}$$

which can be estimated by

$$S(t)^2 = \int_0^t (W(u))^2 \frac{dN(u)}{(Y(u))^2}, \tag{3}$$

where $dN(u) = dN_1(u) + dN_2(u)$. Under H_0 we can estimate the common value of $\Lambda(u)$ by

$$\hat{\Lambda}(t) = \int_0^t \frac{dN_1(u) + dN_2(u)}{2Y(u)}. \tag{4}$$

From the martingale central limit theorem it follows that as $n \to \infty$ the statistic $R(\tau)/S(\tau)$ is asymptotically a standard normal variable.

The choice of the weight function $W(u)$ determines the weight attached to each time point in the comparison of λ_1 and λ_2. For $W(u) = Y(u)$, $R(\tau) = N_1(\tau) - N_2(\tau)$. This is equivalent to the sign test discussed before. For $W(u) = (Y(u))^2$, it gives more weight to the earlier part of the experiment. $W(u) = Y(u)N(u-)$ gives more weight to the later part of the experiment and is equivalent to U_4. $W(u) = Y^2(u)N(u-)$ puts more weight to the centre and less weight to both ends of the experiment.

Yip and Lam (1992) have added that the statistic $R(\tau)/S(\tau)$ can also be used to test the hypothesis $\lambda_1(t) = \lambda_2(t), t > 0$ against the alternative that intensities are proportional, that is, $\lambda_1(t) = \theta\lambda_1(t), t > 0$.

Yip and Lam (1993) extended their results for testing for equality of k failure rates.

7 Two examples

We illustrate our testing procedures on two data sets due to King (1971) and Boag (1949). The data sets are given in Crowder (2001).

King's Data . The data consists of breaking strength (mg) of 23 wire connections. There are two types of failure : breakage at the bonded end and breakage along the wire itself. The response variable T is breaking strength or loading, which takes the place of failure time. We assume that the two risks of failure act independently. $\delta = 1$ denotes breakage along the wire and $\delta = 0$ denotes breakage at the bonded end.

We carry out various tests of hypotheses discussed above. The null distribution of $\binom{n}{2}U_2$ and $\binom{n}{2}U_4$ is identical. From the tables for Wlicoxon signed rank statistic for $n = 22$, we get the critical value is 178, whereas the two statistics take values 99 and 121, respectively. However, Nelson (1982) pointed out that there are two breakages due to bond with breaking strengths 0. These two zeros must correspond to faulty bonds. He also expressed some doubt about the value 3150. We recalculated the statistics after deleting these three values. $\binom{n}{2}U_2$ and $\binom{n}{2}U_4$ take values 89 and 101, respectively. The exact cutoff point for $n = 19$ is 137. In both the cases the null hypothesis of equality of distributions is not rejected . One concludes that breakage of wire occurs at the same time as breakage of bonds. + However U_3 is -0.09 in the first case and -0.06 in the second case. Hence, the hypothesis that the ratio of failure rates is constant is accepted.

Boag's data The data consists of survival times (in months) for 121 breast cancer patients. It comes from the clinical records of one hospital from the years 1929 to 1938. The causes of death are cancer (1) and others(0). We want to test whether cancer occurs earlier compared to other risks, whether it has a faster failure rate. Boag compared fits of the log-normal distribution with others, e.g., the exponential. We want to know if proportional hazards is an appropriate model for this data. $U_1 = 0.64, U_2 = 0.81, U_3 = 0.33$ and $U_4 = 0.8$. Since the sample size is large we use limiting distributions of the statistic. Standardised U_2 is significant rejecting the null hypothesis of equality of distributions. Failures due to cancer occur more often. Standardised U_3 is significant. So the ratio of failure rates is not proportional - and hence T and δ are not independent.

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