Competing Risks - A Review

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The competing risks situation was first considered in the eighteenth century when small pox vaccination was being discovered and popularised.

D. Bernoulli (1761) posed the question : How much would the mortality be reduced or expected life be increased if the risk of death due to small pox is totally eliminated, the other risks persisting as before?

The demographers , over the years, have taken up this question enthusastically.

A strong motivation has come from the insurance business in which different premium rates have to be established for groups of persons exposed to different sorts of risks.

Of course now it is relevant in several disciplines - medicine, reliability engineering , economics , manufacturing and even the game of cricket (Crowder (2001)).

Let $X_1, X_2, \ldots, X_k$ denote the latent failure times of individuals subject to k risks, where X_i represents the age at death if cause i were the only cause of failure.

What is actually observed is the time to failure T, where $T = \min(X_1, X_2, \ldots, X_k)$, and the cause of failure δ where $\delta = j$ if $X_j = \min(X_1, X_2, \ldots, X_k)$.

The joint probability distribution of (T, δ) is specified by the set of k sub-distribution functions

$$F(i,t) = P[T \leq t, \delta = i],$$

or equivalently by the subsurvival functions

$$S(i,t) = P[T > t, \delta = i],\ i = 1, 2, \ldots, k.$$

Let $H(t)$ and $S(t)$ denote , respectively the distribution function and the survivor function of T.

Let $f(i,t)$ denote the sub-density function corresponding to ith risk.

The density function of T is $h(t) = \sum_{i=1}^{k} f(i,t)$

$p_i = F(i,\infty)$ is the probability of failure due to the ith risk.

The cause specific hazard rate be given by $\lambda(i,t) = \frac{f(i,t)}{S(t)}$.

Note that if $X_1, X_2, \ldots, X_k$ are independent and identically distributed random variables with a common distribution function $F(x)$, then it is uniquely determined by the distribution function of the minimum, that is,

$$H(x) = P[T \leq x] = 1 - [1 - F(x)]^k.$$

When $X's$ are not identically distributed , let $F_i(x)$ denote the distribution of X_i.

If $X's$ are independent, but not identically distributed , Berman (1963) showed that the joint distribution of (T, δ) uniquely determines F_i, the distribution function of X_i.

$$F_i(x) = 1 - \exp[1 - \int_{-\infty}^{x} [1 - \sum_{j=1}^{k} F(j,t)]^{-1} dF(i,t)],$$

Keyfitz et al (1972) have strongly argued that risks to human life can be grouped into four non-overlapping sets, which are independent of each other.

In this, and other such situations, one can use methods based on the apriori independence assumption while dealing with pooled risks.

If the latent failure times are not independent, then the underlying distributions are not identifiable on the basis of (T, δ) (see Tsiatis (1975), Crowder (1991, 1993)).

Theorem (Tsiatis (1975)) Suppose that the set of sub-survival functions $S(j,t)$ is given for some model with dependent risks. Then there exists a unique proxy model with independent risks yielding identical $S(j,t)$. It is defined by

$\bar{G}^*(t) = \prod_{j=1}^{k} \bar{G}_j(t_j)$, where

$\bar{G}_j(t) = exp(-\int_0^t \lambda(j,s)ds)$.

The independence or otherwise of the latent lifetimes $(X_1, X_2, \ldots, X_k)$ cannot be statistically tested from any data collected on (T, δ).

Also, the marginal distribution functions $F_i(x)$ may not represent the probability distribution of lifetimes in any practical situation.

Elimination of jth risk may change the environment in such a way that $F_i(x)$ does not represent the lifetime of X_i in the changed scenario.

In view of the above considerations, unless one can assume independence, it is necessary to suggest appropriate models, develop methodology and carry out the data analysis in terms of the observable random variables (T, δ) alone.

For recent work on these lines see Deshpande (1990), Aras and Deshpande (1992),Aly, Kochar and McKeague (1994), Sun and Tiwari (1995), Deshpande and Dewan (2000) , Crowder (2001), Kalbfleisch and Prentice (2002), Dewan and Kulathinal (2004), and Dewan, Deshpande and Kulathinal (2004).

Locally Most Powerful Rank Tests

Suppose $k = 2$, that is , a unit is exposed to two risks of failure denoted by 1 and 0.

When n units are put to trial , the data consists of $(T_i, \delta_i^*), i = 1, \ldots, n$ where $\delta^* = 2 - \delta$.

Suppose we wish to test the hypothesis

$H_0 : F(1, t) = F(2, t),$ for all t.

Under the null hypothesis the two risks are equally effective at all ages in the prevailing environment. However, the alternative hypothesis is that the two risks are not equally effective atleast at some ages.

The likelihood function is given by (see , Aras and Deshpande (1992))

$$L(\underline{t}, \underline{\delta^*}) = \prod_{i=1}^{n} [f(1, t_i)]^{\delta_i^*} [f(2, t_i)]^{1-\delta_i^*} \qquad (1)$$

where $\underline{t} = (T_1, \ldots, T_n), \underline{\delta^*} = (\delta_1^*, \ldots, \delta_n^*)$.

If $F(i, t)$ depends upon the parameter θ , then inference about it can be based on the above likelihood function. When T and δ^* are independent then Deshpande (1990) proposed the model $F(1, t) = \theta H(t), \;\; F(2, t) = (1 - \theta)H(t)$. Here $\theta = P[\delta^* = 1]$.

Then the likelihood reduces to

$$L(\theta, \underline{T}, \underline{\delta^*}) = \theta^{\sum_{i=1}^{n} \delta_i^*} (1-\theta)^{\sum_{i=1}^{n} (1-\delta_i^*)} \prod_{i=1}^{n} h(t_i).$$

Testing of the hypothesis $F(1, t) = F(2, t)$ reduces to testing that $\theta = 1/2$.

The obvious statistic is the sign statistic

$U_1 = \frac{1}{n}\sum_{i=1}^{n} \delta_i^*$.

nU_1 has $B(n, \theta)$ distribution and there exist optimal estimation and testing procedures based on it.

However, if $F(1, t)$ and $F(2, t)$ depend on a parameter θ in a more complicated manner then one needs to look at locally most powerful rank tests.

Let $f(1, t) = f(t, \theta), f(2, t) = h(t) - f(t, \theta)$ where $h(t)$ and $f(t, \theta)$ are known density functions and incidence density such that $f(t, \theta_0) = \frac{1}{2}h(t)$.

Let $T_{(1)} \leq T_{(2)} \leq \ldots \leq T_{(n)}$ denote the ordered failure times. Let

$$W_i = \begin{cases} 1 & \text{if } T_{(i)} \text{ corresponds to first risk,} \\ 0 & \text{otherwise.} \end{cases}$$

Let R_j be the rank of T_j among $T_1, \ldots, T_n$.

Let $\underline{R}' = (R_1, R_2, \ldots, R_n)$, $\underline{W}' = (W_1, W_2, \ldots, W_n)$ denote the vector of ranks and indicator functions corresponding to ordered minima.

The likelihood of $(\underline{R}, \underline{W})$ is given by

$$\begin{aligned} & P(\theta, \underline{R}, \underline{W}) \\ = \; & \int_{0<t_1<\ldots<t_n<\infty} \prod_{i=1}^{n} [f(t_i,\theta)]^{w_i}[h(t_i) - f(t_i,\theta)]^{1-w_i} dt_i \end{aligned}$$

Theorem 1 *If $f'(t,\theta)$ is the derivative of $f(t,\theta)$ with respect to θ, then the locally most powerful rank test for $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$ is given by: reject H_0 for large values of*

$L_c = \sum_{i=1}^{n} w_i a_i$, *where*

$a_i = \int \ldots \int_{0<t_1<\ldots<t_n<\infty} \frac{f'(t_i,\theta_0)}{f(t_i,\theta_0)} \prod_{i=1}^{n} [f(t_i, \theta_0) dt_i].$

(i) If Deshpande's model holds with $\theta_0 = 1/2$, then sign test is the LMPR test.

(ii) If $f(1,t) = \frac{1}{2}g(t,0)$ and $f(2,t) = \frac{1}{2}g(t,\theta)$, $\theta > 0$ and $g(t,\theta)$ is the logistic density function $g(t,\theta) = \frac{e^{(x-\theta)}}{[1+e^{(x-\theta)}]^2}$, then the LMPR test is based on the statistic

$$W^+ = \sum_{i=1}^{n} W_i R_i,$$

which is the analogue of the Wilcoxon signed rank statistic for competing risks data.

(iii) In case of Lehmann type alternative defined by $F(1,t) = [\frac{H(t)}{2}]^\theta$, $F(2,t) = H(t) - [\frac{H(t)}{2}]^\theta$, the LMPR test is based on scores $a_i = E(E_{(j)})$ where $E_{(j)}$ is the jth order statistic from a random sample of size n from the standard exponential distribution.

But for more complicated families of distributions, e.g., Gumbel (1960), the scores are complicated and need to be solved using numerical integration (see Aras and Deshpande (1992)).

Tests for bivariate symmetry

Suppose that the latent failure times X and Y are dependent with their joint distribution given by $F(x,y)$.

On the basis of n independent pairs (T_i, δ_i^*) we want to test whether the forces of two risks are equivalent against the alternative that the force of one risk is greater than that of the other.

That is we test the null hypothesis of bivariate symmetry

$H_0 : F(x,y) = F(y,x)$ for every (x,y).

Theorem 3.1: Under the null hypothesis of bivariate symmetry we have

(i)$F(1,t) = F(2,t)$ for all t,

(ii)$S(1,t) = S(2,t)$ for all t,

(iii) $\lambda(1,t) = \lambda(2,t)$ for all t,

(iv) $P[\delta^* = 1] = P[\delta^* = 0]$

(v) T and δ^* are independent.

Consider the following alternatives

$H_1 : \lambda(1, t) < \lambda(2, t)$; $H_2 : F(1, t) < F(2, t;)$

$H_3 : S(1, t) > S(2, t)$.

All these alternatives say that risk II is more potent than risk I at all ages t in some stochastic sense.

Sen (1979) considered fixed sample and sequential tests for the null hypothesis of bivariate symmetry of the joint distribution of (X, Y).

The alternatives are expressed in terms of $\pi_1(t) = Pr[\delta^* = 1|T = t]$, the conditional probability that the failure is due to first risk , given that failure occurs at time t.

He derived optimal score statistics for such parametric situations. But the statistics cannot be used without the knowledge of the joint distribution $F(x, y)$.

We look at various distribution-free test procedures for testing H_0 against above alternatives.

For testing H_0 against H_1 consider

$$\psi(t) = F(1, t) - F(2, t) = \int_0^t S(t)[\lambda(1, u) - \lambda(2, u)]du.$$

H_1 holds iff the above function is non-increasing in t.

Consider the following measure of deviation between H_0 and H_1,

$$\Delta = \int_{0<x<y<\infty} [\psi(x) - \psi(y)]\, dF(x)\, dF(y),$$

and its empirical estimator Δ_n as the test statistic where Δ_n is given by

$$\Delta_n = \int_{0<x<y<\infty} [\psi_n(x) - \psi_n(y)]\, dF_n(x)\, dF_n(y),$$

where $F_{1n}(t) = \frac{1}{n}\sum_{j=1}^{n} I\{\delta_j = 1, T_j \le t\}$,

$F_n(t) = \frac{1}{n}\sum_{j=1}^{n} I\{T_j \le t\}$,

and $\psi_n(t) = 2F_{1n}(t) - F_n(t)$

are the empirical estimators of F_1, F and ψ, respectively. Then

$\Delta_n = \frac{1}{n^3}[\frac{n(n^2-1)}{6} - 2\sum_{i=1}^{n}(i-1)(n-i+1)W_i]$.

Under H_0, $\Delta = 0$ but under the alternative $\Delta > 0$.

Large values of the statistic are significant.

Rejecting H_0 for large values of Δ_n is equivalent to rejecting it for small values of the statistic

$U_2 = \sum_{i=1}^{n-1} i(n-i)W_{i+1}$.

The exact null distribution of U_2 can be obtained following the approach of Hettmansperger (1984 pp 35).

For $n = 5(1)20$, the 1% and 5% critical values of U_2 are given in second and fourth columns of Table 2.1 and corresponding exact significance levels in third and fifth columns respectively.

TABLE 2.1

n	$\alpha \approx 0.01$	$\alpha \approx 0.01$	$\alpha \approx 0.05$	$\alpha \approx 0.05$
5	-	-	0	0.062500
6	-	-	0	0.031250
7	0	0.015625	6	0.046875
8	7	0.015625	14	0.046875
9	8	0.011719	22	0.054688
10	18	0.011719	34	0.054688
11	28	0.013672	50	0.050781
12	38	0.010254	67	0.051758
13	54	0.010498	90	0.051514
14	74	0.010010	118	0.052734
15	98	0.010132	148	0.050293
16	126	0.010010	186	0.051300
17	158	0.010010	228	0.050308
18	198	0.010269	278	0.050613
19	242	0.010311	332	0.049911
20	290	0.010050	396	0.050467

Under $H_0,\ E[U_2] = \frac{n(n^2-1)}{12},\ \ Var[U_2] = \frac{n(n^4-1)}{120}.$

By the Central Limit Theorem it can be shown that, under H_0, $n^{\frac{1}{2}}\{\frac{U_2}{n^3} - \frac{1}{12}\} \xrightarrow{\mathcal{L}} N(0, \frac{1}{120})$.

After change of order of integration Δ can also be expressed as

$$\Delta = \int_0^\infty S^2(x)F(x)[\lambda_2(x) - \lambda_1(x)]\, dx.$$

It is seen from this representation that the test based on U_2 is equivalent to the test V of Yip and Lam (1992) proposed for the case of independent risks without censoring. They have not discussed its small sample exact null distribution.

Deshpande (1990) proposed two tests for testing H_0 versus H_2 on heuristic grounds. The first test is the Wilcoxon signed rank type statistic

$$W^+ = \sum_{i=1}^n (1 - \delta_i^*)R_i.$$

It is argued that W^+ will be large when the alternative H_2 is true, there being a greater incidence of the second risk upto any fixed time t.

This consideration leads to another test based on the U-statistic

$$
\begin{aligned}
U_3 &= \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \phi_3(T_i, \delta_i^*, T_j, \delta_j^*) \\
&= \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} (n - R_i + 1)\delta_i^*,
\end{aligned}
$$

where ϕ_3 is given by

$$
\phi_3(T_i, \delta_i^*, T_j, \delta_j^*) = \begin{cases} 1 & \text{if } \delta_j^* = 0, T_i > T_j, \\ & \text{or } \ \delta_i^* = 0, T_i < T_j, \\ 0 & \text{otherwise.} \end{cases}
$$

$E(U_3) = 1/2$ under H_0 and strictly larger than $1/2$ under H_2.

U_3 is the same as the statistic proposed earlier to test for H_0 against the alternative H_{A1} of stochastic dominance of distribution functions of independent latent failure times (see, Deshpande and Dewan (2005)). It is also consistent for testing bivariate symmetry against dominance of incidence functions.

For testing H_0 against H_2, one can consider the measure of deviation

$$F(2,t) - F(1,t),$$

which is non-negative under H_2. Then

$$\int_0^\infty [F(2,t) - F(1,t)] dH(t)$$

$$= P[\delta_1^* = 0, T_1 \leq T_2] - \frac{1}{2},$$

and its U-statistic estimator is the statistic U_3 discussed above.

Similarly for testing H_0 against H_3 consider the measure of deviation $S(1,t) - S(2,t)$, which is non-negative under H_3. Then

$$\int_0^\infty [S(1,t) - S(2,t)] dH(t) = P[\delta_1^* = 1, T_1 > T_2] - \frac{1}{2}.$$

Consider the kernel

$$\phi_4(T_i,\delta_i,T_j,\delta_j) = \begin{cases} 1 & \text{if } \delta_i^* = 1, T_i > T_j, \\ & \text{or } \ \delta_j^* = 1, T_i < T_j, \\ 0, & \text{otherwise,} \end{cases}$$

and the corresponding U-statistic is given by

$$\begin{aligned} U_4 &= \frac{1}{\binom{n}{2}} \sum_{1\le i<j\le n} \phi_4(T_i,\delta_i,T_j,\delta_j) \\ &= \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} (R_i - 1)\delta_i^*. \end{aligned}$$

This statistic was earlier proposed by Bagai et al (1989) to test for equality of failure rates of independent latent competing risks.

Aly et al (1994) proposed Kolmogrov-Smirnov type tests for testing the equality of two competing risks against the alternatives H_1 and H_2. Here we discuss their approach. Consider

$$\begin{aligned} \psi^*(t) &= F(2,t) - F(1,t) \\ &= \int_0^t S(u)(\lambda(2,u) - \lambda(1,u))du. \end{aligned}$$

Under H_0, $\psi^*(t) = 0$. H_1 holds iff $\psi^*(t)$ is nondecreasing in t .

Let $\psi_n^*(t)$ be its empirical estimator. Consider

$D_{1n} = \sup_{0 \le s < t < \infty} \{\psi_n^*(t) - \psi_n^*(s)\}$.

Large values of the statistic are significant.

If one is interested in a general two sided alternative $F(1, t) \neq F(2, t)$ for some t or equivalently $\lambda_1(t) \neq \lambda_2(t)$ for some t , then one can use the Kolmogrov-Smirnov type statistic

$D_n = \sup_{t \ge 0} |\psi_n^*(t)|$.

Under $H_0, \sqrt{n} D_n$ converges in distribution to $\sup_{0 \le x \le 1} |W(x)|$. This test is consistent against arbitrary departures from H_0.

For testing H_0 against H_2, Aly , Kochar and McKeague (1994) proposed the statistic

$D_{2n} = \sup_{0 \le t \infty} \psi_n(t)$.

Large values of D_{2n} are significant for testing H_0 against H_2.

For exact null and asymptotic distributions of these statistics see Aly et al (1994).

Censored data

Most of the above tests can be generalized to the case when the data is right censored.

Let C be the censoring random variable independent of the latent failure times X and Y.

Denote the survival function of C by S_c and assume that $S_C(t) > 0$ for all t.

Now the available information consists of $(\tilde{T}_i, \tilde{\delta}_i)$, $i = 1, 2, \cdots, n$ where

$\tilde{T} = \min(T, C)$ and $\tilde{\delta} = \delta^* I(T \leq C)$.

Aly et al (1994) generalised the function ψ^* so as to capture departures of H_0 from H_2 in case of censored data. Let

$$\phi(t) = \int_0^t S(u-)(S_C(u-))^{\frac{1}{2}}(\lambda(2, u) - \lambda(1, u))du,$$

which is the ψ^* function when there is no censoring.

The integrand $S_C(u-)^{1/2}$ is the function required to compensate for censoring in order that the D statistics remain asymptotically distribution-free.

Under $H_0, \phi(t) = 0$.

H_1 holds iff $\phi(t)$ is increasing in t.

An obvious choice of ϕ_n is

$$\phi_n(t) = \int_0^t \widehat{S}(u-)(\widehat{S}_C(u-))^{1/2}\, d(\widehat{\Lambda}_1 - \widehat{\Lambda}_2)(u),$$

where $\widehat{S}$ and $\widehat{S}_C$ are the product limit estimators of $\bar{S}$ and S_C, and $\widehat{\Lambda}_j$ is the Aalen estimator of the cumulative CSHR function

$$\Lambda_j(t) = \int_0^t \lambda_j(u)\, du.$$

A suitable modification of U_2 to censored data is given by the statistic

$$K_n = \int_{0<x<y<\infty} (\phi_n(x) - \phi_n(y))\, d\widehat{S}(x)\, d\widehat{S}(y),$$

where ϕ_n and $\widehat{S}$ are as defined above. K_n can be expressed as

$K_n = \sum_{i|\tilde{\delta}_i \neq 0} \phi_n(i)(2\widehat{S}(\tilde{T}_{(i-1)})-1)(\widehat{S}(\tilde{T}_{(i-1)})\frac{1}{n-i+1}$.

Large values of K_n are significant for testing H_0 against H_1.

Aly et al (1994) showed that, under H_0,

$n^{1/2}\phi_n \xrightarrow{\mathcal{L}} W(S(.))$ where W(.) is the standard Brownian Motion.

Using the continuous mapping theorem, we have $n^{1/2}K_n \xrightarrow{\mathcal{L}} N(0, \frac{1}{30})$. Sun and Tiwari (1995) modified the statistic U_3 so that it can be used for censored data. Consider

$$\begin{aligned} V &= \int_0^\infty (F(2,t) - F(1,t))dH(t) \\ &= \int_0^\infty [S(t)]^2 d(\Lambda_2(t) - \Lambda_1(t)). \end{aligned}$$

A natural estimator of V is given by V_n where

$$V_n = \int_0^\infty [\widehat{S}(t-)]^2 d(\widehat{\Lambda}_2(t) - \widehat{\Lambda}_1(t)),$$

where $\widehat{S}(t)$ and $\widehat{\Lambda}_j$ are as defined above.

In the absence of censoring V_n reduces to the statistic U_3.

Sun and Tiwari (1995) discussed the asymptotic distribution of V_n.

Aly et al (1994) extended D_{1n} and D_{2n} to included censored data.

SIMULATION RESULTS

Given below are the results of a simulation study done for power comparisons of various tests for the uncensored case listed above.

Random samples were generated from absolutely continuous bivariate exponential (ACBVE) due to Block and Basu (1974) with density

$$f(x,y) = \begin{cases} \frac{\lambda\lambda_1(\lambda_2+\lambda_0)}{\lambda_1+\lambda_2} e^{-\lambda_1 x-(\lambda_2+\lambda_0)y} & \text{if } x<y, \\ \frac{\lambda\lambda_2(\lambda_1+\lambda_0)}{\lambda_1+\lambda_2} e^{-\lambda_2 y-(\lambda_1+\lambda_0)x} & \text{if } x>y, \end{cases}$$

where $(\lambda_0, \lambda_1, \lambda_2)$ are the parameters and $\lambda = \lambda_0 + \lambda_1 + \lambda_2$.

The CSHR's are $\lambda_j(t) = \frac{\lambda_j\lambda}{\lambda_1+\lambda_2}, \quad j = 1, 2.$

Under H_1, $\lambda_1 < \lambda_2$.

X and Y are independent if and only if $\lambda_0 = 0$.

We set $\lambda_1 = 1$ and consider $\lambda_2 = 1.0, 1.4, 1.8, 2.2$ indicating larger and larger departures from H_0.

The case $\lambda_2 = 1.0$ corresponds to the null hypothesis. $n = 100$ and there are 10000 replications.

TABLE 3.1 - UNCENSORED CASE

	λ_2			
Test	1.0	1.4	1.8	2.2
D_1	3.76	41.98	82.53	96.83
D_2	4.85	47.71	86.98	98.14
U_2	5.09	44.60	83.96	96.92
U_3	4.79	43.06	80.54	95.42
U_4	4.99	42.67	80.81	95.77
$Sign$	4.39	49.50	88.29	98.66

Next we look at the censored case. The censoring distribution was exponential with parameters 1 and 3, respectively. We use asymptotic critical levels of 5 percent. Results are based on $5,000$ replications.

TABLE 3.2

Observed levels and powers of K_n at an asymptotic level of 5 percent. The underlying distribution of (X, Y) is ACBVE with $\lambda_1 = 1$.

(a) CENSORED (EXP(1))

	n=50		n=100	
λ_2	$\lambda_0 = 0$	$\lambda_0 = 1$	$\lambda_0 = 0$	$\lambda_0 = 1$
1.0	.0218	.0312	.0360	.0376
1.5	.1864	.2192	.3732	.4302
2.0	.4482	.5080	.7862	.8342
2.5	.6928	.7414	.9546	.9704

(b) CENSORED (EXP(3))

	n=50		n=100	
λ_2	$\lambda_0 = 0$	$\lambda_0 = 1$	$\lambda_0 = 0$	$\lambda_0 = 1$
1.0	.0048	.0124	.0084	.0172
1.5	.0342	.0762	.1012	.1834
2.0	.1202	.1986	.3216	.4860
2.5	.2496	.3774	.5862	.7344

From the table it is clear that the asymptotic critical levels give conservative tests for the censored case, with the effect increasing as the censoring becomes more severe.

There is slight effect on the levels or the power due to lack of independence of X and Y in the presence of censoring.

The results are comparable with the results for test proposed by Aly et al (1994) for the lightly censored case.

Remarks

(i) The various tests are consistent against their intended alternatives.

(ii We can also use these tests for the hypothesis $\lambda_1(t) = \lambda_2(t)$ against the alternative that cause-specific hazards are ordered.

(iii) The tests are distribution-free under H_0. The null distribution of the tests U_3 and U_4 is same as in the case of independent latent failures (see, Deshpande and Dewan (2003)).

(iv) It is important to note that T and δ continue to be independent under the null hypothesis of bivariate symmetry .

(v) The statistic U_2 puts more weight on the middle observations and is less sensitive to the observations in the beginning and the end of the experiment.

On the other hand, U_3 puts more weight to later observations .

U_4 puts higher weight to observations in the beginning.

(vi) Deshpande and Dewan (2000) proposed tests for testing bivariate symmetry aginst dispersive asymmetry. Here the alternatives can be expressed in terms of ordering of sub-survival functions and ordering of sub-distributions of the maximum of observations and δ .

The statistic is a linear combination of two statistics, the first one is a U-statistic based on minimum and δ and the other one is a U-statistic based on maximum and δ.

The one based on minimum and δ is the statistic U_4

(vii) The statistics U_2, U_3, U_4 are all linear combinations of the sign statistic and the Wilcoxon-signed rank type statistic.

(viii) Tests proposed by Aly et al (1994) can be extended to the case of multiple risks in which any two of the cause-specific risks are to be compared. The statistic can be modified to test dominance of one risk over the other in a specified interval.

It is observed that many of the statistics which have been originally proposed in the context of independent risks continue to be useful for testing similar hypotheses regarding dependent risks. This raises the interesting possibility that in general such rank based procedures work whether the risks are independent or not.

Aly, E. A. A., Kochar, S. C. and McKeague, I. W. (1994). Some tests for comparing cumulative incidence functions and cause-specific hazard rates. *J. Amer. Statist. Assoc.*, **89**, 994-999.

Aras, G. and Deshpande, J.V. (1992). Statistical analysis of dependent competing risks. *Stats. and Decisions*, **10**, 323-336.

Block,H.W. and Basu, A.P. (1974). A continuous bivariate exponential distribution. *Journal Amer. Statist. Assoc.*, **69**, 1031-1037.

Cooke, R. M. (1996). The design of reliability databases, part II. *Reliability engineering and system safety* **51**, 209-223.

Crowder, M. J. (2001). *Classical competing risks*. Chapman and Hall/CRC, London.

Deshpande, J. V. (1990). A test for bivariate symmetry of dependent competing risks. *Biometrical Journal*, **32**, 736-746.

Deshpande, J.V. and Dewan, I. (2000). Testing bivariate symmetry against dispersive asymmetry. *Journal Indian Statist. Assoc.*, **38**, 227-249.

Deshpande, J.V. and Dewan, I. (2003). A review of some statistical tests under independent competing risks model. *Preprint*.

Dewan, I. and Kulathinal, S.B. (2003). Parametric models for sub-survival functions. Preprint.

Dewan I., Deshpande J.V. and Kulathinal S.B. (2004). On testing dependence between time to failure and cause of failure via conditional probabilities. *Scandanavian Journal of Statistics*, **31**, 2004, 79-92.

Gumbel , E.J. (1960). Bivariate exponential distribution. *J. Amer. Statist. Assoc.*, **55**, 698-707.

Hettmansperger , T.P. (1984). *Statistical Inference Based On Ranks* . John Wiley, New York.

Kalbfleisch, J.D. and Prentice, R.L. (1980). *The statistical analysis of failure time data*. John Wiley, New York.

Kalbfleisch, J.D. and Prentice, R.L. (2002). *The statistical analysis of failure time data*. Second Edition, John Wiley, New Jersey.

Kochar, S.C. (1995). A review of some distribution-free tests for the equality of cause specific hazard rates. *Analysis of Censored Data* , IMS Lecture Notes, **27**, 147 - 162.

Sen, P.K. (1979). Nonparametric tests for interchangeability under competing risks. In *Contributions to Statistics, Jaroslav Hajek Memorial Volume*, 211-228.Reidel, Dordrecht.

Sun, Y. and Tiwari, R.C. (1995). Comparing cause-specific hazard rates of a competing risks model with censored data. *Analysis of Censored Data* , IMS Lecture Notes, **27**, 255 - 270.

Tsiatis, A. (1975). A nonidentifiability aspect of the problem of competing risks. *Proc. Natl. Acad. Sci. U.S.A.*, **72**, 20-22.

Yip, P. and Lam, K.F. (1992). A class of non-parametric tests for the equality of failure rates in a competing risks model. *Comm. Statist. Theory Methods*, **21** , 2541-2556.