ON TESTING DEPENDENCE BETWEEN TIME TO FAILURE AND CAUSE OF FAILURE VIA CONDITIONAL PROBABILITY

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Introduction

Competing risks data: $T =$ failure time
$\delta =$ an indicator of the failure type, $\delta \in \{0, 1\}$

Censored data:

Acturial studies:

Planning studies: T could be the time to first employment and δ indicating the employment obtained

Joint distribution of T and δ

Subsurvival functions:

$$S_i(t) = P[T \geq t, \delta = i],\ i = 0, 1$$

Subdistribution functions:

$$F_i(t) = P[T \geq t, \delta = i],\ i = 0, 1$$

The survival function of T is given by

$$S(t) = P[T \geq t] = S_0(t) + S_1(t)$$

The distribution function of T is given by

$$F(t) = P[T < t] = F_0(t) + F_1(t)$$

Consider the following two conditional probabilities

$$\begin{aligned} \Phi_1(t) &= pr(\delta = 1 \mid T \geq t) = S_1(t)/S(t) \text{ and} \\ \Phi_0^*(t) &= pr(\delta = 0 \mid T < t) = F_0(t)/F(t), \end{aligned}$$

$$\Phi_1(t) = \phi, \forall\, t > 0$$

is equivalent to independence of T and δ.

Similarly $\Phi_0^*(t)$ constant is equivalent to independence of T and δ.

T and δ can be studied separately.

If T and δ are independent then

$$S_i(t) = S(t)P[\delta = i].$$

Thus the hypothesis of equality of incidence functions, or that of equality of cause-spectific hazard rates reduces to testing whether $P[\delta = 1] = P[\delta = 0] = 1/2$, a Bernoulli trial with success probability half. The problem from two-dimensions reduces to one-dimensions.

Few definitions:

Definition 1 *X_2 is Right Tail Increasing in X_1, $RTI(X_2 \mid X_1)$, if*

$$P[X_2 > t_2 \mid X_1 > t_1]$$

is increasing in t_1 for all t_2.

Definition 2 *X_2 is Left Tail Decreasing in X_1, $LTD(X_2 \mid X_1)$, if $pr(X_2 \leq t_2 \mid X_1 \leq t_1)$ is decreasing in t_1 for all t_2.*

Definition 3 *X_1 and X_2 are Positively Quadrant Dependent, $PQD(X_1, X_2)$, if*

$$P[X_1 > t_1, X_2 > t_2] \geq P[X_1 > t_1]P[X_2 > t_2], \forall t_1, t_2.$$

Definition 4 *Let $K(x_1, x_2)$ be the joint probability density function of X_1 and X_2. Then X_1 and X_2 are Totally Positive of Order 2, TP_2, if*

$$f(s_1, t_1)f(s_2, t_2) \geq f(s_2, t_1)f(s_1, t_2)$$

for all $s_1 < s_2, t_1 < t_2$.

$$TP_2 \Longrightarrow RTI(X_2 \mid X_1) \Longrightarrow PQD(X_1, X_2)$$

Similarly

$$TP_2 \Longrightarrow LTD(X_2 \mid X_1) \Longrightarrow PQD(X_1, X_2)$$

But no hierarchy between $RTI(X_2 \mid X_1)$ and $LTD(X_2 \mid X_1)$

Dependence of T and δ and Monotonicity of $\Phi_1(t)$

(1) Independence of T and δ is equivalent to

(a) $\Phi_1(t) = \phi = pr(\delta = 1)$, for all $t > 0$, a constant and

(b) $\Phi_0^*(t) = 1 - \phi = \phi_0 = pr(\delta = 0)$, for all $t > 0$, a constant.

(2) $PQD(\delta, T)$ is equivalent to

(a) $\Phi_1(t) \geq \Phi_1(0) = \phi,\ for\ all\ t > 0$, and

(b) $\Phi_0^*(t) \geq \Phi_0^*(\infty) = 1 - \phi,\ for\ all\ t > 0$.

(3) $RTI(\delta \mid T)$ is equivalent to $\Phi_1(t) \uparrow t$.

(4) Subsurvival functions $S_i(t)$ being TP_2 is equivalent to $\Phi_1(t) \uparrow t$.

(5) $LTD(\delta \mid T)$ is equivalent to $\Phi_0^*(t) \downarrow t$.

(6) Subdistribution function $F_i(t)$ being TP_2 is equivalent to $\Phi_0^*(t) \downarrow t$.

Examples:

In the clinical trials carried out to study the performance of intrauterine device where termination of the device could be due to several reasons it is often of interest to know the chances of termination due to a specific reason given that the device was intact for some specified period. In such situation, conditional probabilities are of interest.

$\Phi_1(t)$ constant

1. Bivariate exponential distributions: Block and Basu (1974), Farlie-Gumbel-Morgenstern bivariate exponential distribution, Gumbel Type A distribution

2. Constant warning-constant inspection model: a warning is emitted at time $X-d$ before it fails, where $d < 1$ is a constant

3. Hoel data (1972): the data were obtained from a laboratory experiment on male mice which had received a radiation dose of 300r at an age of 5 to 6 weeks. The death occurred due to cancer ($\delta = 1$), or other causes ($\delta = 0$).

$\Phi_1(t)$ increasing

1. Random sign censroing (Cooke *et al.*, 1993): lifetime of a unit, X, is censored by $Z = X - W\eta$, where $0 < W < X$ is the time at which a warning is emitted by the unit before its failure, and η is a random variable taking values $\{-1, 1\}$ and is independent of X. When $W = aX$ for some $0 < a < 1$ and X is assumed to be exponential, it is easy to see that

2. Nair data (1993): the data are times to failure (in millions of operations) of 37 switches which could fail in two modes (A or B)

$\Phi_1(t)$ decreasing

1. Proportional warning-constant inspection model: similar to the constant warning-constant inspection model except that the warning is emitted at time X/η if the component fails at X and where $\eta > 1$ is a constant

Hazard rate ordering and aging

Let $r_i(t)$ and $h_i(t)$ denote crude and cause-specific hazard rates, respectively,$i = 0, 1$. Then

$$\begin{aligned} r_i(t) &= \frac{f_i(t)}{S_i(t)} \\ h_i(t) &= \frac{f_i(t)}{S(t)}. \end{aligned}$$

The overall hazard rate of T is

$$r(t) = f(t)/S(t) = h_1(t) + h_2(t)$$

where $f_i(.), f(.)$ are densities corresponding to $S_i(.)$ and $S(.)$, respectively. Note that, $r_1(t)$ is the hazard rate of the conditional survival function of T given $\delta = 1$, $S_1(t)/\phi$.

Theorem 1 $\Phi_1(t) \uparrow t$ *is equivalent to* $r_1(t) \leq r(t) \leq r_0(t)$.

This means that given that the failure is due to risk 1, the failure rate of a unit is less than the overall failure rate while given that the failure is due to risk 0, the failure rate is higher than the overall failure.

Another interesting result stated below connects the monotonicity of $\Phi_1(t)$ with the ordering between two survival functions.

Theorem 2 $\Phi_1(t) \uparrow t$ *implies that the survival function of* T *given* $\delta = 1$ *is larger than the survival function of* T *given* $\delta = 0$, *i.e.,* $S_1(t)/\phi \geq S_0(t)/(1-\phi)$.

It is important to note that the hazard rates $r_1(t)$ and $r_0(t)$ corresponds to the above two distributions.

Theorem 3 $\Phi_1(t) \uparrow t$ *is equivalent to* $h_1(t) \leq r(t)$ *and* $h_0(t) \geq (1-\Phi_1(t))r(t)$.

The above theorem implies that

$h_1(t) \leq \frac{\Phi_1(t)}{1-\Phi_1(t)} h_0(t)$.

It is interesting and also useful to express the cause-specific hazard rate in terms of $\Phi_1(t)$. This enables one to study the aging through the properties of $\Phi_1(t)$.

Theorem 4 *(a)* $h_1(t) = -\Phi_1'(t) + \Phi_1(t)r(t)$, *where* $\Phi_1'(t)$ *is the first derivative of* $\Phi_1(t)$ *with respect to* t.

(b) If $\Phi_1(t)$ *is monotone increasing and concave then* $h_1(t)$ *is an increasing function of* t, *provided* $r(t)$ *is IFR.*

In case of independent latent lifetimes, the survival function/hazard rate of X is expressed in terms of $h_1(t)$. If $h_1(t)$ is IFR then X will also have IFR distribution.

Analogous results for reverse hazard rate and $\phi_0^*(t)$.

Test statistic and its distribution

General dependence between T and δ

To test H_0 : T and δ are independent against the alternative of dependence

$H_0 : \Phi_1(t)$is a constant
$H_1 : \Phi_1(t)$is not a constant

A test based on Kendall's τ

Concordance:
$Ti > Tj,\ \delta_i = 1, \delta_j = 0$ or
$Ti < Tj,\ \delta_i = 0, \delta_j = 1$

Discordance:
$Ti > Tj,\ \delta_i = 0, \delta_j = 1$ or
$Ti < Tj,\ \delta_i = 1, \delta_j = 0$
Define the kernel

$$\psi(T_i, \delta_i, T_j, \delta_j) = \begin{cases} 1 & \text{if } T_i > T_j,\ \delta_i = 1, \delta_j = 0 \\ & \text{or } T_i < T_j,\ \delta_i = 0, \delta_j = 1 \\ -1 & \text{if } T_i > T_j,\ \delta_i = 0, \delta_j = 1 \\ & \text{or } T_i < T_j,\ \delta_i = 1, \delta_j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

U-statistic is given by

$$U_k = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \psi(T_i, \delta_i, T_j, \delta_j).$$

$$E(U_k) \;=\; 2P(\delta = 1) + 4\int_0^\infty S(t)dS1(t)$$

Under H_0, $E(U_k) = 0$

Let R_j be the rank of T_j. Let $T_{(1)} < T_{(2)} < T_{(n)}$ be the ordered $T_i's$. Let

$$W_j \;=\; \begin{cases} 1 & \text{if } T_{(j)} \text{ corresponds to } \delta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $V_k = \binom{n}{2} U_k$ can be written as

$$\begin{aligned} V_k &= \sum_{j=1}^{n} (2R_j - n - 1)\delta_j \\ &= \sum_{j=1}^{n} (2j - n - 1)W_j \\ &= \sum_{j=1}^{n} a_j W_j \qquad (1) \end{aligned}$$

where $a_j = 2j - n - 1$.

Under H_0, $P[W_i = 1] = \phi$, $P[W_i = 0] = 1 - \phi$. Moment generating function of V_k, under H_0:

$$M(t) = \prod_{j=1}^{n} [\phi \, e^{t(2j-n-1)} + (1 - \phi)].$$

Theorem 5 *Under H_0, $\sqrt{n}U_k$ converges in distribution to $N(0, \sigma^2)$, where*

$$\widehat{\sigma^2} = \frac{4}{3}\widehat{\phi}(1 - \widehat{\phi})$$

as $n \to \infty$,

A test procedure for testing H_0 against H_1 is then: reject H_0 at 5% level of significance if $\mid Z = \sqrt{n}U_k/\widehat{\sigma} \mid$ is larger than 1.96, a cut-off point of standard normal distribution.

Testing independence against $PQD(T, \delta)$

$$H_0 : T \text{ and } \delta \text{ are independent}$$

against the alternative

$H_2 : T$ and δ are positive quadrant dependent.

Test based on $\phi_1(t)$

$$H_2 : \Phi_1(t) \geq \phi \text{ for all } t$$

with strict inequality for some t.

Consider

$$\begin{aligned}\Delta_3(S_1, S) &= \int_0^\infty [S_1(t) - \phi S(t)] dF(t) \\ &= pr(T_2 > T_1, \delta_2 = 1) - \phi/2. (2)\end{aligned}$$

Under H_0,

$S_1(t)/S(t) = \phi = pr(\delta = 1)$.

This implies that $\Delta_3(S_1, S) = 0$.

Under H_2, $S_1(t) > \phi S(t)$

and hence $\Delta_3(S_1, S) \geq 0$.

Define the symmetric kernel

$$\psi_3(T_i, \delta_i, T_j, \delta_j) = \begin{cases} 1 & \text{if } T_j > T_i,\ \delta_j = 1 \\ & \text{or if } T_i > T_j,\ \delta_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

which is equivalent to

$$\psi_3(T_i, \delta_i, T_j, \delta_j) = \begin{cases} 1 & \text{if } T_j > T_i,\ \delta_j = 1, \delta_i = 0 \\ & \text{if } T_i > T_j,\ \delta_i = 1, \delta_j = 0 \\ & \text{if } \delta_i = \delta_j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the U-statistic corresponding to $\Delta_3(S_1, S)$ is given by

$$U_3 = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \psi_3(T_i, \delta_i, T_j, \delta_j).$$

Note that $E(U_3) = 2\Delta_3(S_1, S) + \phi$.

Under H_0, $E(U_3) = \phi$.

Under H_2 $E(U_3) \geq \phi$.

Note that the statistic U_3 has earlier been proposed by Bagai *et al.* (1989) for testing the equality of failure rates of two independent competing risks. Then,

$$\binom{n}{2} U_3 = \sum_{i=1}^{n} (R_i - 1)\delta_i = \sum_{i=1}^{n} (i-1)W_i. \quad (3)$$

Under H_0, the moment generating function is given by

$$M(t) = \prod_{j=1}^{n} [(1-\phi) + \phi exp\{t(j-1)\}].$$

When $\phi = 1/2$, $M(t)$ is same as that of Wilcoxon signed rank statistics with n replaced by $(n+1)$.

Theorem 6 *As n tends to ∞, under H_0, $n^{1/2}\{U_3 - E(U_3)\}$ converges in distribution to $N(0, \sigma_3^2)$, where $\sigma_3^2 = (4/3)\phi(1-\phi)$.*

A consistent estimator of variance is

$$\widehat{\sigma_3^2} = (4/3)\widehat{\phi_0}(1 - \widehat{\phi_0}).$$

We reject the null hypothesis for large values of $Z = n^{1/2}(U_3 - \widehat{\phi})/\widehat{\sigma_2}$.

Test based on $\phi_0^*(t)$

H_2 is also equivalent to

$$H_2 : \Phi_0^*(t) \geq \phi_0 \text{ for all } t$$

with strict inequality for some t.

Exactly as in the earlier section, we have

Theorem 7 *As n tends to ∞, $n^{1/2}\{U_3^* - E(U_3^*)\}$ converges in distribution to $N(0, {\sigma_3^*}^2)$, where*

$$\binom{n}{2} U_3^* = n(n-1)/2 - \sum_{i=1}^{n} (n-i) W_i \qquad (4)$$

and ${\sigma_{3^*}}^2 = (4/3)\phi_0(1-\phi_0)$.

A consistent estimator of variance is

$$\widehat{\sigma_3^2} = (4/3)\widehat{\phi}(1-\widehat{\phi}).$$

We reject the null hypothesis for large values of $Z = n^{1/2}(U_3^* - \widehat{\phi_0})/\widehat{\sigma_3^*}$.

Note that

$$U_k = U_3 + U_3^* - 1.$$

Testing independence against $RTI(\delta \mid T)$

Here, we consider testing H_0 againstthe alternative that δ is RTI in $T/$. This is equivalent to

$$H_3 : \Phi_1(t) \uparrow t, t > 0.$$

A. Test I - U_1

$\Phi_1(t) \uparrow t$ is equivalent to

$\Phi_1(t_1) \leq \Phi_1(t_2)$, whenever $t_1 \leq t_2$.

This gives

$\delta(t_1, t_2) = S_1(t_2)S(t_1) - S_1(t_1)S(t_2) \geq 0, t_1 \leq t_2$ with strict inequality for some (t_1, t_2).

Define

$$\begin{aligned} \Delta_1(S_1, S) &= \int\int_{t_1 \leq t_2} \delta(t_1, t_2) dF_1(t_1) dF_1(t_2) \\ &= \int_0^\infty [S_1^2(t) - \phi^2/2] S(t) dF_1(t). \end{aligned}$$

Under H_0,

$$S_1(t)/S(t) = \phi, \Delta_1(S_1, S) = 0.$$

Under H_3,

$$\Delta_1(S_1, S) \geq 0.$$

Define the kernel $\psi_1^*((T_i, \delta_i, T_j, \delta_j, T_k, \delta_k, T_l, \delta_l) =$ ψ_1^* (say)

$$\psi_1^* = \begin{cases} 1 & \text{if } T_k > T_j > T_l > T_i, \\ & \delta_i = \delta_j = \delta_k = 1, \delta_l = 0 \\ -1 & \text{if } T_l > T_j > T_k > T_i, \\ & \delta_i = \delta_j = \delta_k = 1, \delta_l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the U-statistic corresponding to $\Delta_1(S_1, S)$ is U_1

$$\frac{1}{\binom{n}{4}} \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \psi_1(T_{i_1}, \delta_{i_1}, T_{i_2}, \delta_{i_2}, T_{i_3}, \delta_{i_3}, T_{i_4}, \delta_{i_4}),$$

where ψ_1 is the symmetric version corresponding to ψ_1^*.

$E(U_1) = 24\Delta_1(S_1, S)$.

Under $H_0, E(U_1) = 0$

Under H_3, $E(U_1) \geq 0$.

Let $T's$ corresponding to $1's$ be called $X's$ $[n_1]$

those corresponding to $0's$ be called $Y's$ $[n_2]$.

Let $R_{(i)}(S_{(j)})$ be the rank of $X_{(i)}(Y_{(j)})$ be the $ith(jth)$ ordered statistic in the $X(Y)$ sample in the combined arrangement of $n_1 X's$ and $n_2 Y's$ (in fact $nT's$).

Hence

$$\binom{n}{4} U_1 = \sum_{j=1}^{n_2} (S_{(j)} - j)\binom{n_1 + j - S_{(j)}}{2} - \sum_{j=1}^{n_2} \binom{S_{(j)} - j}{3}.$$

In terms of $X's$ and $Y's$ the above statistic is the same as that proposed by Kochar (1979) for testing equality of failure rates, the only difference being that the number of $X's$ and $Y's$ is random.

Theorem 8 *As n tends to ∞, under H_0, $n^{1/2}\{U_1 - E(U_1)\}$ converges in distribution to $N(0, \sigma_1^2)$, where $\sigma_1^2 = (96/35)\phi^5(1-\phi)$.*

The null hypothesis is rejected for large values of $n^{1/2}U_1/\widehat{\sigma_1}$ where $\widehat{\sigma_1^2} = (96/35)\widehat{\phi^5}(1-\widehat{\phi})$.

B. Test II - U_2

As mentioned earlier, H_3 is equivalent to $S_i(t)$ being TP_2.

Under TP_2,

$$S_1(t_2)S_0(t_1) - S_1(t_1)S_0(t_2) > 0, t_1 < t_2.$$

Consider

$$\Delta_2(S_1, S) = \int_{t_1<t_2} [S_1(t_2)S_0(t_1) - S_1(t_1)S_0(t_2)]$$

$$d[F_1(t_1)F_0(t_2) + F_1(t_2)F_0(t_1)].$$

Under H_0 $\Delta_2(S_1, S) = 0$ Under H_3 $\Delta_2(S_1, S) \geq 0$.

Define the kernel ψ_2^* as

$$= \begin{cases} 1 & \text{if } T_k > T_j > T_l > T_i,\ \delta_i = \delta_k = 1, \delta_j = \delta_l = 0 \\ & \text{if } T_k > T_i > T_l > T_j,\ \delta_i = \delta_k = 1, \delta_j = \delta_l = 0 \\ -1 & \text{if } T_l > T_j > T_k > T_i,\ \delta_i = \delta_k = 1, \delta_j = \delta_l = 0 \\ & \text{or } T_l > T_i > T_k > T_j,\ \delta_i = \delta_k = 1, \delta_j = \delta_l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding U-statistic U_2 is given by

$$\frac{1}{\binom{n}{4}} \sum_{1\le i_1<i_2<i_3<i_4\le n} \psi_2(T_{i_1},\delta_{i_1},T_{i_2},\delta_{i_2},T_{i_3},\delta_{i_3},T_{i_4},\delta_{i_4}),$$

ψ_2 is the symmetric version of ψ_2^*.

$$\begin{aligned}
E(U_2) &= 24\Delta_2(S_1,S) \\
&= 24[\phi^2(1-\phi)^2/4 \\
&\quad -\phi(1-\phi)\int_0^\infty S_0(t)dF_1(t) \\
&\quad +\int_0^\infty S_1(t)S_0^2(t)dF_1(t)].
\end{aligned}$$

$$\begin{aligned}
\binom{n}{4}U_2 &= \sum_{i=1}^{n_1}(n_1-i)\binom{R_{(i)}-i}{2} \\
&- \sum_{i=1}^{n_1}(n_1-i)(R_{(i)}-i)(n_2-R_{(i)}+i) \\
&+ \sum_{j=1}^{n_2}(S_{(j)}-j)(n_1-S_{(j)}+j)(j-1) \\
&- \sum_{j=1}^{n_2}(n_2-j)\binom{(S_{(j)}-j}{2}.
\end{aligned}$$

In terms of $X's$ and $Y's$, the above statistic is the same as another one proposed by Kochar (1979) to test for equality of failure rates with n_1 and n_2 fixed.

Theorem 9 *As n tends to ∞, under H_0, $n^{1/2}\{U_2 - E(U_2)\}$ converges in distribution to $N(0, \sigma_2^2)$, where $\sigma_2^2 = (384/35)\phi^3(1-\phi)^3$.*

We reject the null hypothesis for large value of $n^{1/2}U_2/\widehat{\sigma_2}$ where $\widehat{\sigma_2^2} = (384/35)\widehat{\phi}^3(1-\widehat{\phi})^3$.

Tests proposed in this section will help in discriminating between the constant or proportional warning-constant inspection and random sign censoring models and also to determine whether the corresponding mode of failure becomes more likely with increasing age.

Testing independence against $LTD(\delta \mid T)$

Here, we consider testing H_0 against the alternative H_4 that δ is LTD in T where H_4 can equivalently be stated as

$$H_4 : \Phi_0^*(t) \downarrow t, t > 0.$$

A. Test I - U_1^*

$\Phi_0^*(t) \downarrow t$ is equivalent to

$\Phi_0^*(t_1) \geq \Phi_0^*(t_2)$, whenever $t_1 \leq t_2$.

$\delta(t_1, t_2) = F_0(t_1)F(t_2) - F_0(t_2)F(t_1) \geq 0,$

$t_1 \leq t_2$ with strict inequality for some (t_1, t_2).

Define

$$\begin{aligned} \Delta_1(F_0, F) &= \int\int_{t_1 \leq t_2} \delta(t_1, t_2) dF_0(t_1) dF_0(t_2) \\ &= \int_0^\infty [F_0^2(t) - \phi_0^2/2] F(t) dF_0(t). \end{aligned}$$

Under H_0,

$F_0(t)/F(t) = \phi_0, \;\; \Delta_1(F_0, F) = 0.$

Under H_4, $\Delta_1(F_0, F) \geq 0$.

Define the kernel ψ_1^*

$$\begin{cases} 1 & \text{if } T_k < T_j < T_l < T_i,\ \delta_i = \delta_j = \delta_k = 0, \delta_l = 1 \\ -1 & \text{if } T_l < T_j < T_k < T_i,\ \delta_i = \delta_j = \delta_k = 0, \delta_l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then U_1^*, the corresponding U-statistic is

$$\frac{1}{\binom{n}{4}} \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \psi_1(T_{i_1}, \delta_{i_1}, T_{i_2}, \delta_{i_2}, T_{i_3}, \delta_{i_3}, T_{i_4}, \delta_{i_4}),$$

where ψ_1 is the symmetric version corresponding to ψ_1^*.

$E(U_1^*) = 24\Delta_1(F_0, F)$.

Under H_0, $E(U_1^*) = 0$

Under H_4, $E(U_1^*) \geq 0$.

A rank representation of U_1^* is

$$\binom{n}{4} U_1^* = \sum_{j=1}^{n_1} \binom{R_{(j)} - j}{2} (n_2 + j - R_{(j)})$$

$$- \sum_{j=1}^{n_1} \binom{n_2 - R_{(j)} + j}{3}.$$

Theorem 10 *As n tends to ∞, under H_0, $n^{1/2}\{U_1^* - E(U_1^*)\}$ converges in distribution to $N(0, {\sigma_1^*}^2)$, where ${\sigma_1^*}^2 = (96/35)\phi_0^5(1-\phi_0) = (96/35)\phi(1-\phi)^5$.*

We reject the null hypothesis for large values of $n^{1/2}U_1^*/\widehat{\sigma_1^*}$, where $\widehat{\sigma_1^*}^2 = (96/35)\widehat{\phi}_0^5(1 - \widehat{\phi}_0) = (96/35)\widehat{\phi}(1 - \widehat{\phi})^5$.

B. Test II - U_2^*

Test for H_0 against H_4 using the TP_2 property of the subdistribution functions of (T, δ).

H_4 is equivalent to $F_i(t)$ being TP_2. Under TP_2,

$$F_1(t_2)F_0(t_1) - F_1(t_1)F_0(t_2) > 0$$

for $t_1 < t_2$. Consider

$$\Delta_2(F_0, F) = \int_{t_1<t_2} [F_1(t_2)F_0(t_1)F_1(t_1)F_0(t_2)]$$

$$[dF_1(t_1)dF_0(t_2) + dF_1(t_2)dF_0(t_1)].$$

Under H_0 $\Delta_2(F_0, F) = 0$,

Under H_4, $\Delta_2(F_0, F) \geq 0$.

Define the kernel ψ_2^*

$$\begin{cases} 1 & \text{if } T_k < T_j < T_l < T_i,\ \delta_i = \delta_k = 0, \delta_j = \delta_l = 1 \\ & \text{if } T_k < T_i < T_l < T_j,\ \delta_i = \delta_k = 0, \delta_j = \delta_l = 1 \\ -1 & \text{if } T_l < T_j < T_k < T_i,\ \delta_i = \delta_k = 0, \delta_j = \delta_l = 1 \\ & \text{or } T_l < T_i < T_k < T_j,\ \delta_i = \delta_k = 0, \delta_j = \delta_l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding U-statistic U_2^* is given by

$$\frac{1}{\binom{n}{4}} \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \psi_2(T_{i_1}, \delta_{i_1}, T_{i_2}, \delta_{i_2}, T_{i_3}, \delta_{i_3}, T_{i_4}, \delta_{i_4}),$$

ψ_2 is the symmetric version of ψ_2^*.

$$\begin{aligned} E(U_2^*) &= \Delta_2(F_0, F) \\ &= \phi_0^2(1-\phi_0)^2/4 - \phi_0(1-\phi_0)\int_0^\infty F_1(t)dF_0(t) \\ &\quad + \int_0^\infty F_0(t)F_1^2(t)dF_0(t). \end{aligned}$$

$$\begin{aligned} \binom{n}{4}U_2^* &= \sum_{i=1}^{n_1}(n_1 - i)\binom{R_{(i)} - i}{2} \\ &+ \sum_{i=1}^{n_1}(n_1 - i)(R_{(i)} - i)(n_2 - R_{(i)} + i) \\ &- \sum_{j=1}^{n_2}(S_{(j)} - j)(n_1 - S_{(j)} + j)(j-1) \\ &- \sum_{j=1}^{n_2}(n_2 - j)\binom{(S_{(j)} - j}{2}. \end{aligned}$$

Theorem 11 *As* n *tends to* ∞, *under* H_0, $n^{1/2}\{U_2^* - E(U_2^*)\}$ *converges in distribution to* $N(0, {\sigma_2^*}^2)$, *where* ${\sigma_2^*}^2 = (384/35)\phi_0^3(1-\phi_0)^3 = (384/35)\phi^3(1-\phi)^3$.

We reject the null hypothesis for large values of $n^{1/2}U_2^*/\widehat{\sigma_2^*}$, where $\widehat{{\sigma_2^*}^2} = (384/35)\widehat{\phi_0^3}(1-\widehat{\phi_0})^3 = (384/35)\widehat{\phi^3}(1-\widehat{\phi})^3$.

Asymptotic relative efficiency

To compare alternative tests proposed in this paper for testing H_0 against H_2, H_0 against H_3 and H_0 against H_4, we compute asymptotic relative efficiency of the tests within a semiparametric family of distributions proposed in Deshpande (1990). The semiparametric family considered here is

$F_1(t) = pF^a(t)$,

$F_0(t) = F(t) - pF^a(t)$, where $1 \leq a \leq 2$, $0 \leq p \leq 0.5$ and $F(t)$ is a proper distribution function.

$\phi = p$

$$\Phi_1(t) = \frac{p(1 - F^a(t))}{1 - F(t)}$$

which is an increasing function of t.

$$\Phi_0^*(t) = 1 - pF^{a-1}(t)$$

which is a decreasing function of t.

H_0 corresponds to $a = 1$, and other alternative hypotheses correspond to $1 < a \leq 2$.

By the limiting theorem of U-statistics, all the U-statistics proposed here have asymptotic normal distribution under both null and the alternative hypothesis.

The asymptotic relative efficiency of test U_1 with respect to test U_2 is then defined as

$$eff(U_1, U_2) = e(U_2)/e(U_1)$$

$$e(U) = \mu'^2(1)/var(U \mid H_0)$$

$\mu'(1)$ is the derivative of expected value of U with respect to a evaluated at $a = 1$, and $var(U \mid H_0)$ is the asymptotic variance of $n^{1/2}U$ under H_0.

Tests U_1 and U_2 are equally efficient and the same is true for tests U_1^* and U_2^*. Tests U_3 and U_3^* are equally efficient but the general test U_k is four times more efficient compared to these tests. This indicates the superiority of U_k as it is consistent for the alternative H_2.

For this particular family of distributions, the other alternative tests are equally efficient. But this need not be true in general.

Illustrations

In case of Nair's data, the value of Z is 2.70 and hence the hypothesis of independence of the failure time and the type of failure is rejected. The nonlinearity of the plot in Figure 1 supports this conclusion.

In case of Hoel's data, the value of Z is 1.86 and the Figure 2 shows a flat curve. Hence, the time of death and the cause of death are independent.

The following tables gives the value of the test statistics and the conclusions reached for the two data sets.

Table 1: Values of the test statistics for Nair's data (1993)

U-statistics	Expectation	Variance	Z	Conclusion
$U_k = 0.26$	0	0.33	2.70	Reject H_0
$U_1 = 0.04$	0	0.03	1.45	Accept H_0
$U_2 = 0.15$	0	0.17	2.26	Reject H_0
$U_1^* = 0.06$	0	0.06	2.29	Reject H_0
$U_2^* = 0.15$	0	0.17	2.18	Reject H_0
$U_3 = 0.59$	0.46	0.33	1.35	Accept H_0
$U_3^* = 0.67$	0.54	0.33	1.35	Accept H_0

Table 2: Values of the test statistics for Hoel's data (1972)

U-statistics	Expectation	Variance	Z	Conclusion
$U_k = 0.11$	0	0.32	1.86	Accept H_0
$U_1 = 0.04$	0	0.09	1.50	Accept H_0
$U_2 = 0.06$	0	0.15	1.63	Accept H_0
$U_1^* = 0.01$	0	0.02	1.14	Accept H_0
$U_2^* = 0.05$	0	0.15	1.38	Accept H_0
$U_3 = 0.66$	0.61	0.32	0.93	Accept H_0
$U_3^* = 0.45$	0.39	0.32	0.53	Accept H_0

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