Non-bossy Single Object Auctions *

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Abstract

We study deterministic single object auctions in the private values environment. We show that an allocation rule is implementable (in dominant strategies) and non-bossy if and only if it is a strongly rationalizable allocation rule. With a mild continuity condition, we show that an allocation rule is implementable and non-bossy if and only if it is a simple utility maximizer (with appropriate tie-breaking). All our characterizations extend the seminal result of Roberts (1979) from the unrestricted domain to the restricted domain of single object auctions.

JEL Classification Codes: D44

Keywords: single object auction; implementation in dominant strategies; rationalizability; non-bossiness.

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1 Introduction

We study single object auctions in the private values model. We restrict attention to deterministic single object auctions, i.e., auctions where the probability of allocating the object to any agent is either zero or one. An allocation rule for single object auction is implementable if we can find payments such that truth-telling is a dominant strategy for every agent. A central result in mechanism design is that the efficient allocation rule in the single object auction private values model is implementable using the Vickrey auction (Vickrey, 1961; Clarke, 1971; Groves, 1973). On the other hand, a revenue maximizing auction in the independent private values model maximizes the *virtual valuations* of the agents (Myerson, 1981). English auction with a reserve price is popular in practice (seen on EBay and other Internet sites) and in theory, for instance, in designing approximately optimal auctions (Hartline and Roughgarden, 2009; Dhangwatnotai et al., 2010). Such an auction implements a *constrained* efficient allocation rule with a reserve price - it does not allocate the object if the valuation of each bidder is less than the reserve price, but when it allocates the object it does so to the highest bidder.

While the set of implementable allocation rules is quite rich, we encounter only these particular simple class of implementable allocation rules in theory and practice. Hence, it is important to understand how these allocation rules distinguish themselves from the remaining implementable allocation rules. A primary motivation of this paper is to carry out a systematic analysis of this question axiomatically.

Common features of all these auctions are that the allocation rules are deterministic, dominant strategy implementable, and involve maximization of some form. If ties in these maximizations are broken carefully, then the allocation rules mentioned above satisfy another appealing property - *non-bossiness*. Non-bossiness is the following requirement. Suppose agent i is not winning the object at a particular valuation profile (v_i, v_{-i}) and we go to another valuation profile (v'_i, v_{-i}), where the valuation of only agent i changes, such that agent i still does not win the object. Then, the agent who was winning the object at the valuation profile (v_i, v_{-i}) continues to win the object at (v'_i, v_{-i}). In other words, if an agent cannot change his own outcome, then it cannot change the outcome of any other agent. [1]

We provide a complete characterization of implementable and non-bossy allocation rules. For this characterization, we introduce a novel notion of rationalizability in the single object allocation model, and use it to define a class of allocation rules that we call the *strongly rationalizable* allocation rules. Our characterization says that an allocation rule is implementable and non-bossy if and only if it is a strongly rationalizable allocation rule.

Under a mild continuity condition, we sharpen our characterization. We define the notion of a *simple utility function*, which is any non-decreasing function that maps the set of possible

[1] The use of non-bossiness axiom in social choice theory with private good allocations, specially matching problems, is extensive - it was first used by Satterthwaite and Sonnenschein (1981), and subsequently in matching problems (Svensson, 1999; Papai, 2000; Ehlers, 2002; Hatfield, 2009) and cost sharing problem (Mutuswami, 2005).

valuations of an agent to the set of real numbers. A simple utility maximizer is an allocation rule that chooses a simple utility function for every agent. Then, at every valuation profile (a) it does not allocate the object if every agent has negative simple utility and (b) if at least one agent has positive simple utility, then it allocates the object to an agent with the highest simple utility. We show that if an allocation rule satisfies a mild continuity condition, then it is implementable and *non-bossy* if and only if it is a simple utility maximizer allocation rule (supplemented with an appropriate tie-breaking rule).

All the commonly used allocation rules in single object auctions (e.g., efficient allocation rule, efficient allocation rule with a reserve price, the optimal auction allocation rule in Myerson (1981)) are simple utility maximizer allocation rules. Hence, our results provide an axiomatic foundation for a rich class of commonly used allocation rules. Although we characterize implementable and non-bossy allocation rules, using revenue equivalence (Myerson, 1981), we can pin down the payments that will implement these allocation rules. Thus, we get a complete characterization of "mechanisms" that use non-bossy allocation rules.

Our characterizations have a common feature - implementability and non-bossiness is equivalent to some form of maximization by the seller at every valuation profile. These results relate to two fundamental results in mechanism design and auction theory. A benchmark result in private value mechanism design in quasi-linear environments is the Roberts' affine maximizer theorem (Roberts, 1979). It considers general multidimensional type spaces with finite set of alternatives. A type of an agent in such models is a vector in $\mathbb{R}^{|A|}$, where A is the set of alternatives. Roberts (1979) showed that if there are at least three alternatives and the type space is *unrestricted* (i.e., $\mathbb{R}^{|A|}$), then every onto implementable allocation rule is an affine maximizer. It can be shown that every affine maximizer is implementable. [2] An affine maximizer can be thought to be a linear simple utility function. The single object auction model has a restricted type space. As a result, Roberts' result does not apply. Our characterizations can be thought as extension of Roberts' affine maximizer result to the single object auction model.

Further, in a seminal result, **?** showed that the interim allocation probability obtained by every Bayesian and randomized allocation rule can be obtained by taking convex combination of certain dominant strategy implementable allocation rules that he called *hierarchical allocation rules* - see also Manelli and Vincent (2010); **?**. As we discuss later, a hierarchical allocation rule can be written as a convex combination of simple utility maximizer allocation rules that we identify (which are deterministic, dominant strategy implementable, and non-bossy allocation rules). Hence, the set of dominant strategy implementable and non-bossy deterministic allocation rules occupy a pivotal role in the set of all randomized and Bayesian implementable allocation rules.

[2] Carbajal et al. (2012) show that if there are at least three alternatives and the type space of every agent is unrestricted, then an onto allocation rule is implementable if and only if it is a *lexicographic* affine maximizer. Lexicographic affine maximizers contain a particular class of affine maximizers where ties are broken carefully.

Finally, we extend our idea of simple utility maximizer allocation rule to define an even larger class of allocation rules that we call *generalized utility maximizer* allocation rules. We show that implementability is equivalent to these allocation rules. While this result is also in the spirit of Roberts' affine maximizer theorem, the proof is a simple consequence of Myerson's monotonicity characterization of implementable allocation rule, which we discuss below. Generalized utility maximizers are more complex allocation rules than simple utility maximizers. This shows how a natural condition like non-bossiness helps us to separate complex auction rules from simple and commonly used auction rules.

1.1 Relationship with Literature

Myerson (1981) shows that implementability is equivalent to a monotonicity property of the allocation rules. [3] The monotonicity property is equivalent to requiring that for every agent i and for every valuation profile of other agents, there is a cutoff valuation of agent i below which he does not get the object and above which he gets the object. [4]

The relationship between our results and the monotonicity characterization can be best illustrated by reference to parallel results in the strategic voting literature. **?** show that Maskin monotonicity, the counterpart of monotonicity in the strategic voting models, is necessary for dominant strategy implementation, and if the domain is unrestricted then it is also sufficient. However, the seminal results of Gibbard (1973) and Satterthwaite (1975) show that dictatorship is the only dominant strategy implementable voting rule satisfying unanimity.

In the quasi-linear private values models, Roberts' theorem can be thought of as the counterpart of the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975). After the result of Gibbard (1973) and Satterthwaite (1975), a vast literature in social choice theory has pursued the characterization of implementable allocation rules in restricted "voting" domains, e.g., the median voting rule and its generalizations characterize implementable allocation rules in single-peaked domains (Moulin, 1980; Barbera et al., 1993). Indeed, these characterizations of implementable allocation rules are all in the spirit of Roberts' theorem - they describe the precise *parameters* that are required to design an implementable allocation rule. In this spirit, our results give explicit characterization of implementable allocation rules for the single object auction model.

There have been extensions of Roberts' theorem to certain environments. For instance, Mishra and Sen (2012) show that Roberts' theorem holds in certain bounded but full dimensional type spaces under an additional condition of *neutrality*. Their neutrality condition

[3] See also extensions of this characterization to the multidimensional private values models in Bikhchandani et al. (2006); Saks and Yu (2005); Ashlagi et al. (2010); Cuff et al. (2012); Mishra and Roy (2012).

[4] The results in Myerson (1981) are more general. In particular, he considers implementation in Bayes-Nash equilibrium and allows for randomization. But the expected revenue maximizing allocation rule he identifies is a deterministic and dominant strategy implementable allocation rule.

is vacuous in the single object auction model. Moreover, the type space in the single object auction model is not full dimensional. Carbajal et al. (2012) extend Roberts' theorem to certain restricted type spaces which satisfy some technical conditions. Though it covers many interesting models, including those with infinite set of alternatives, the single object auction model does not satisfy their technical conditions. Marchant and Mishra (2012) extend Roberts' theorem to the case of two alternatives. Since the number of alternatives in the single object auction model is more than two, their results do not hold in our model.

Jehiel et al. (2008) show that a version of the Roberts' theorem holds even in the interdependent values model (they require implementation in ex-post equilibrium). They also require the complete domain assumption like Roberts (1979), and remark that their result does not hold in restricted one-dimensional settings like the single object auction.

Two related work in computer science literature deserve special mention. Lavi et al. (2003) focus on a particular restricted domain, which they call order-based domains (this includes some auction domains). Under various additional restrictions on the allocation rule (which includes an independence condition), they show that every implementable allocation rule must be an "almost" affine maximizer - roughly, almost affine maximizers are affine maximizers for large enough values of types of agents.

Next, **?** consider the single object auction model and show that if the *object is always allocated* then the only implementable allocation rules satisfying non-bossiness and three more additional conditions are *min function* allocation rules. [5] Min function allocation rules are simple utility maximizer allocation rules, but with some additional limiting and continuity properties. Though our characterization of simple utility maximizer is related to their result, it has several important differences. First, their result requires that we *always* sell the good. This rules out any allocation rule with a reserve price, such as Myerson's revenue maximizing allocation rule. Further, our proof shows that allowing the object to be not sold adds several non-trivial complications in deriving our results. Second, they seem to require different types of range and tie-breaking conditions than our continuity requirement. On the other hand, our characterization of simple utility maximizer makes it explicit the way ties need to be broken. Finally, they have no analogue of our other characterizations.

There have been many simplifications of the original proof of Roberts (Jehiel et al., 2008; Lavi, 2007; Dobzinski and Nisan, 2009; Vohra, 2011; Mishra and Sen, 2012). But none of these proofs show how Roberts' theorem can be extended to a restricted domain like the single object auction model. Unlike most of the literature, our goal is not to characterize "affine maximizers" - indeed, all our characterizations capture a larger class of implementable allocation rules than affine maximizers.

An alternate approach is to characterize the set of dominant strategy mechanisms directly

[5] **?** consider a more general environment than ours in which a planner needs to select a path in a graph, where each edge represents an agent. Informally, their three additional conditions are various range and tie-breaking conditions, and called *edge autonomy, path autonomy,* and *sensitivity*. The non-bossy condition is called *independence* by them.

by imposing conditions on mechanisms rather than just on allocation rules. A contribution along this line is Ashlagi and Serizawa (2011). They show that any mechanism which always allocates the object, satisfies individual rationality, non-negativity of payments, *anonymity in net utility*, and dominant strategy incentive compatibility must be the Vickrey auction. This result is further strengthened by Mukherjee (2012), who shows that any strategy-proof and anonymous (in net utility) mechanism which always allocates the object must use the efficient allocation rule. Further, Sakai (2012) characterizes the Vickrey auction with a reserve price using various axioms on the *mechanism* (this includes an axiom on the allocation rule which requires a weak version of efficiency). By placing minimal axioms on *allocation rules*, we are able to characterize a broader class of mechanisms (using revenue equivalence) than these papers.

2 The Single Object Auction Model

A seller is selling an indivisible object to n potential agents (buyers). The set of agents is denoted by $N := \{1, \ldots, n\}$. The private value of agent i for the object is denoted by $v_i \in \mathbb{R}_{++}$. The set of all possible private values of agent i is $V_i \subseteq \mathbb{R}_{++}$ - note that we do not allow zero valuations. We will use the usual notations v_{-i} and V_{-i} to denote a profile of valuations without agent i and the set of all profiles of valuations without agent i respectively. Let $V := V_1 \times V_2 \times \ldots \times V_n$.

The set of alternatives is denoted by $A := \{e^0, e^1, \ldots, e^n\}$, where each e^i is a vector in $\mathbb{R}^n$. In particular, e^0 is the zero vector in $\mathbb{R}^n$ and e^i is the unit vector in $\mathbb{R}^n$ with i-th component 1 and all other components zero. The j-th component of the vector e^i will be denoted by e^i_j. The alternative e^0 is the alternative where the seller keeps the object and for every $i \in N$, e^i is the alternative where agent i gets the object. Notice that our model focuses on deterministic alternatives. Every agent $i \in N$ gets zero value from any alternative where he does not get the object. An allocation rule is a mapping $f : V \to A$. For every $v \in V$ and for every $i \in N$, the notation $f_i(v) \in \{0, 1\}$ will denote if agent i gets the object ($f_i(v) = 1$) or not ($f_i(v) = 0$) at valuation profile v in allocation rule f.

Payments are allowed and agents have quasi-linear utility functions over payments. A payment rule of agent $i \in N$ is a mapping $p_i : V \to \mathbb{R}$.

Definition 1 *An allocation rule f is* **implementable** *(in dominant strategies) if there exist payment rules $(p_1, \ldots, p_n)$ such that for every agent $i \in N$ and for every $v_{-i} \in V_{-i}$*

$$v_i f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i f_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i}) \qquad \forall\ v_i, v'_i \in V_i.$$

In this case, we say $(p_1, \ldots, p_n)$ implement f and the mechanism $(f, p_1, \ldots, p_n)$ is **incentive compatible**.

Notice that we focus on deterministic dominant strategy implementation.

Myerson (1981) showed that the following notion of monotonicity is equivalent to implementability - see also **?** for a similar characterization.

Definition 2 *An allocation rule f is* **monotone** *if for every $i \in N$, for every $v_{-i} \in V_{-i}$, and for every $v_i, v'_i \in V_i$ with $v_i < v'_i$ and $f_i(v_i, v_{-i}) = 1$, we have $f_i(v'_i, v_{-i}) = 1$.*

Myerson (1981) shows that an allocation rule is implementable if and only if it is monotone - this result does not require any restriction on the space of valuations (see Vohra (2011), for instance). Throughout the paper, our results will be driven by the monotonicity condition.

3 Implementation, Non-Bossiness, and Rationalizability

We now provide the main results of this paper. We will define the notion of a *non-bossy* allocation rule. Then, we will provide a complete characterization of non-bossy and implementable allocation rules. Finally, we will add a mild continuity-like condition to sharpen this characterization even further.

The backbone of this result is a notion of rationalizability in our model, and this reveals an elegant structure of implementable and non-bossy allocation rules. We introduce this idea of rationalizability in the single object auctions next.

3.1 Rationalizability

To define rationalizability in our context, we view the mechanism designer as a decision maker who is making choices using his allocation rule. Notice that at every profile of valuations, by choosing an alternative, the mechanism designer assigns values to each agent - zero to all agents who do not get the object but positive value to the agent who gets the object. Denote by $\mathbf{1}_{v_i}$ the vector of valuations in $\mathbb{R}^n_+$, where all the components except agent i has zero and the component corresponding to agent i has v_i. Further, denote by $\mathbf{1}_0$ the n-dimensional zero vector. For convenience, we will write $\mathbf{1}_0$ as $\mathbf{1}_{v_0}$ at any valuation profile.

Using this notation, at a valuation profile $(v_1, \ldots, v_n)$, a mechanism designer's choice of an alternative in A can lead to the selection of one of the following $(n+1)$ vectors in $\mathbb{R}^n_+$ to be chosen - $\mathbf{1}_{v_0}, \mathbf{1}_{v_1}, \ldots, \mathbf{1}_{v_n}$. We will refer to these vectors as *utility vectors*. Any allocation rule f can alternatively thought of choosing utility vectors at every valuation profile. The domain of valuations V_i of agent i gives rise to a set of feasible utility vectors where only agent i gets positive value. In particular define for every $i \in N$, $D_i := \{\mathbf{1}_{v_i} : v_i \in V_i\}$. Further, let $D_0 := \{\mathbf{1}_{v_0}\}$ and $V_0 = \{0\}$. Denote by $D := D_0 \cup D_1 \cup D_2 \cup \ldots \cup D_n$ the set of all utility vectors consistent with the domain of profile of valuations V.

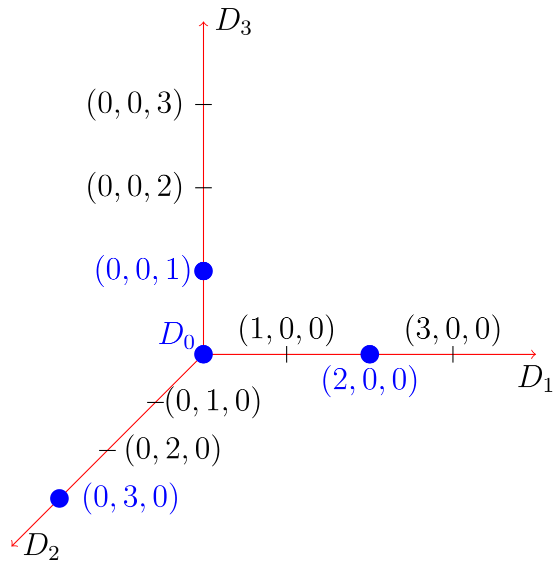
To define the notion of a rational allocation rule, we will use orderings (reflexive, complete, and transitive binary relation) on the set of utility vectors D. For any ordering $\succeq$ on D, let $\succ$ be the asymmetric component of $\succeq$ and $\sim$ be the symmetric component of $\succeq$. A strict linear ordering is an anit-symmetric ordering with no symmetric component. An ordering $\succeq$ on D is monotone if for every $i \in N$, for every $v_i, v'_i \in V_i$ with $v_i > v'_i$, we have $\mathbf{1}_{v_i} \succ \mathbf{1}_{v'_i}$. Our notion of rational allocation requires that at every profile of valuations it must choose a maximal element among the utility vectors at that valuation profile, where the maximal element is defined using a monotone ordering on D.

An example with three agents will clarify some of the concepts.

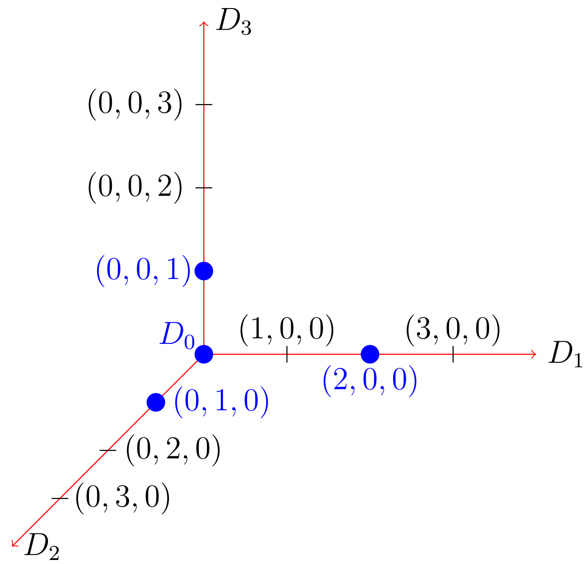
Example 1

Let $N = \{1, 2, 3\}$. So, the set of alternatives is $A = \{e^0, e^1, e^2, e^3\}$. Let $V_1 = V_2 = V_3 = \{1, 2, 3\}$. In that case, the utility vectors are vectors in $\mathbb{R}^3_+$. In particular, D_0 contains the origin, $D_1 = \{(1, 0, 0), (2, 0, 0), (3, 0, 0)\}$, $D_2 = \{(0, 1, 0), (0, 2, 0), (0, 3, 0)\}$, and $D_3 = \{(0, 0, 1), (0, 0, 2), (0, 0, 3)\}$. Figures 1(a) and 1(b) show D_0, D_1, D_2, D_3 with two valuation profiles (shown in dark circles in each figure). A valuation profile corresponds to four points in $D \equiv (D_0 \cup D_1 \cup D_2 \cup D_3)$. The valuation profile (v_1, v_2, v_3) corresponding to Figure 1(a) is $(2, 3, 1)$ (the corresponding utility vectors are shown in dark blue dots in the figure) and that corresponding to Figure 1(b) is $(2, 1, 1)$.

Now, consider the following ordering $\succeq$ defined on D: $(0, 0, 3) \succ (0, 3, 0) \succ (0, 2, 0) \succ (3, 0, 0) \succ (0, 0, 0) \sim (0, 0, 2) \succ (0, 1, 0) \sim (2, 0, 0) \succ (1, 0, 0) \succ (0, 0, 1)$. Note that $\succeq$ is monotone. Consider an allocation rule f, which chooses the $\succeq$-maximal utility vector at every valuation profile. For instance, consider the utility vectors corresponding to valuation profile $(2, 3, 1)$ (shown in Figure 1(a)). The $\succeq$-maximal utility vector at this valuation profile is $(0, 3, 0)$ and hence, f allocates the object to agent 2. Similarly, consider the utility vectors corresponding to valuation profile $(2, 1, 1)$ (shown in Figure 1(b)). The $\succeq$-maximal utility vector at this valuation profile is $(0, 0, 0)$ and hence, f does not allocate the object to any agent. We call such allocation rules *rationalizable* allocation rules.



(a)



(b)

Figure 1: Illustration of rationalizable allocation rule

We now formally define a rationalizable allocation rule. For every allocation rule f, let $G^f : V \to D$ be a *social welfare function* induced by f, i.e., for all $v \in V$, $G^f(v) = \mathbf{1}_{v_j}$ if $f(v) = e^j$ for any $j \in \{0, 1, \ldots, n\}$.

DEFINITION **3** *An allocation rule f is* **rationalizable** *if there exists a monotone ordering $\succeq$ on D such that for all $v \in V$, $G^f(v) \succeq \mathbf{1}_{v_j}$ for all $j \in \{0, 1, \ldots, n\}$. In this case, we say $\succeq$ rationalizes f.*

An allocation rule f is **strongly rationalizable** *if there exists a monotone strict linear ordering $\succ$ on D such that for all $v \in V$, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ for all $j \in \{0, 1, \ldots, n\} \setminus \{i\}$, where $G^f(v) = \mathbf{1}_{v_i}$. In this case, we say $\succ$ strongly rationalizes f.*

We will investigate the relationship between (strongly) rationalizable allocation rules and implementable allocation rules. The following lemma establishes that a rational allocation rule is implementable.

LEMMA **1** *Every rationalizable allocation rule is implementable.*

Proof: Consider a rationalizable allocation rule f and let $\succeq$ be the corresponding ordering on D. Fix an agent i and valuation profile v_{-i}. Consider two valuations of agent i: v_i and v'_i with $v_i < v'_i$ with $f(v_i, v_{-i}) = e^i$. By definition of $\succeq$, $\mathbf{1}_{v_i} \succeq \mathbf{1}_{v_j}$ for all $j \in (N \cup \{0\}) \setminus \{i\}$. Since $\succeq$ is monotone, $\mathbf{1}_{v'_i} \succ \mathbf{1}_{v_i}$. By transitivity, $\mathbf{1}_{v'_i} \succ \mathbf{1}_{v_j}$ for all $j \in (N \cup \{0\}) \setminus \{i\}$. Then, by the definition of $\succeq$, $f(v'_i, v_{-i}) = e^i$. Hence, f is monotone, which further implies that it is implementable (Myerson, 1981). ■

The converse of Lemma 1 is not true. The following example establishes that.

EXAMPLE **2**

Suppose there are two agents: $N = \{1, 2\}$. Suppose $V_1 = V_2 = \mathbb{R}_{++}$. Consider an allocation rule f defined as follows. At any valuation profile (v_1, v_2), if $\max(v_1 - 2v_2, v_2 - v_1) < 0$, then $f(v_1, v_2) = e^0$. Else, if $v_1 - 2v_2 < v_2 - v_1$, then $f(v_1, v_2) = e^2$ and if $v_1 - 2v_2 \geq v_2 - v_1$, then $f(v_1, v_2) = e^1$. It is easy to verify that f is monotone, and hence, implementable.

We argue that f is not a rationalizable allocation rule. Assume for contradiction that f is a rationalizable allocation rule and $\succeq$ is the corresponding monotone ordering. Consider the profile of valuation (v_1, v_2), where $v_1 = 1$ and $v_2 = 2$. For $\epsilon > 0$ but arbitrarily close to zero, $f(v_1, v_2 - \epsilon) = e^2$. Hence, $\mathbf{1}_{v_2 - \epsilon} \succeq \mathbf{1}_{v_0}$. By monotonicity, $\mathbf{1}_{v_2} \succ \mathbf{1}_{v_0}$. Now, consider the profile of valuations (v'_1, v_2), where $v'_1 = 2 + \epsilon$ and $v_2 = 2$. Note that $f(v'_1, v_2) = e^0$. Hence, $\mathbf{1}_{v_0} \succeq \mathbf{1}_{v_2}$. This is a contradiction.

A feature of this example is that at valuation profile (v_1, v_2), the allocation rule was choosing e^2. But when valuation of agent 1 changed to v'_1, it chose e^0 at valuation profile (v'_1, v_2). Hence, agent 1 could change the outcome without changing his own outcome. As we show next, such allocation rules are incompatible with rationalizability.

3.2 Non-bossy Single Object Auctions

In this section, we will show that the set of implementable and *non-bossy* allocation rules are characterized by strongly rationalizable allocation rules.

Definition 4 *An allocation rule f is* **non-bossy** *if for every $i \in N$, for every $v_{-i} \in V_{-i}$ and for every $v_i, v_i' \in V_i$ with $f_i(v_i, v_{-i}) = f_i(v_i', v_{-i})$, we have $f(v_i, v_{-i}) = f(v_i', v_{-i})$.*

Non-bossiness requires that if an agent does not change his own allocation (i.e., whether he is getting the object or not) by changing his valuation, then he should not be able to change the allocation of anyone. It was first proposed by Satterthwaite and Sonnenschein (1981). As discussed in the introduction, it is a plausible condition to impose in private good allocation problems and has been extensively used in the strategic social choice theory literature.

We give an example of a bossy and a non-bossy allocation rule in Figure 2(a) and Figure 2(b) respectively. These figures indicate a scenario with two agents. The possible outcomes of the allocation rules at different valuation profiles are depicted in the Figures. In Figure 2(a), the allocation rule is bossy since if we start from a region where alternative e^2 is chosen and agent 1 increases his value, then we can come to a region where alternative e^0 is chosen (i.e., agent 1 can change the outcome without changing his own outcome). However, such a problem is absent for the allocation rule in Figure 2(b).

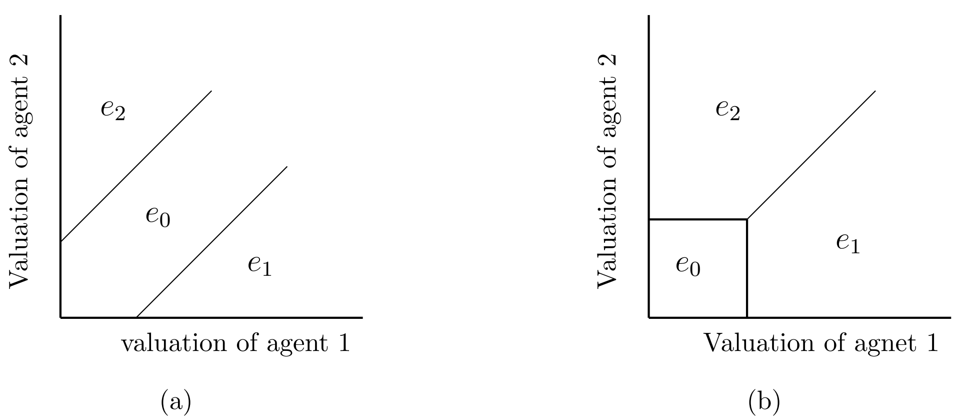


Figure 2: Bossy and non-bossy allocation rules

Lemma 2 *A strongly rationalizable allocation rule is non-bossy.*

Proof: Let f be a strongly rationalizable allocation rule with $\succ$ being the corresponding ordering on D. Fix an agent i and $v_{-i} \in V_{-i}$. Consider $v_i, v_i' \in V_i$ such that $f(v_i, v_{-i}) = e^j \neq e^i$. By definition, $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_k}$ for all $k \in (N \cup \{0\}) \setminus \{j\}$. Suppose $f(v_i', v_{-i}) = e^l \neq e^i$. By definition, $\mathbf{1}_{v_l} \succ \mathbf{1}_{v_k}$ for all $k \in (N \cup \{0\}) \setminus \{l\}$. Assume for contradiction $e^l \neq e^j$. Then, we get that $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_l}$ and $\mathbf{1}_{v_l} \succ \mathbf{1}_{v_j}$, which is a contradiction. ■

This leads to the formal connection between implementability and rationalizability.

Theorem 1 *An allocation rule is implementable and non-bossy if and only if it is strongly rationalizable.*

The proof of Theorem 1 is in the appendix. Theorem 1 reveals a surprising connection between rationalizability and single object auction design. Such a connection of rationalizability and mechanism design was first established in Mishra and Sen (2012). They consider general quasi-linear environments with private values. They show that if the type space is a *multidimensional open interval*, then every implementable and *neutral* allocation rule is rationalizable. Note that rationalizability is weaker than strong rationalizability in the sense that it does not require the underlying ordering to be a strict linear ordering. Our results depart from those in Mishra and Sen (2012) in many ways. First, as discussed earlier, their domain condition is not satisfied in our model, and neutrality is vacuous in the single object auction models. Second, we show that implementability and non-bossiness is *equivalent to strong rationalizability*. Mishra and Sen (2012) do not provide any such equivalence. Indeed, the non-bossiness that we use, is a condition that is specific to private good allocation problems, and cannot be used in general mechanism design problems.

Notice that Theorem 1 does not require any restriction on V_i. If the strict linear ordering we constructed in the proof of Theorem 1 can be represented using a utility function, then the characterization will be even more direct. If for every agent $i \in N$, V_i is finite, then it is possible. But, as the next example illustrates, this is not always possible.

Example 3

Suppose $N = \{1, 2\}$ and $V_1 = V_2 = \mathbb{R}_{++}$. Consider the allocation rule f such that for all valuation profiles (v_1, v_2), $f(v_1, v_2) = e^1$ if $v_1 \geq 1$, $f(v_1, v_2) = e^2$ if $v_1 < 1$ and $v_2 \geq 1$, and $f(v_1, v_2) = e^0$ otherwise. It can be verified that f is implementable (monotone) and non-bossy. By Theorem 1, f is strongly rationalizable. Now, consider the strict linear order defined in the proof of Theorem 1 that strongly rationalizes f - denote it by $\succ^f$. If $v_1 = v_2 = 1$, we have $f(v_1, v_2) = e^1$. Hence, $\mathbf{1}_{v_1} \succ^f \mathbf{1}_{v_2}$.

Now, consider the following definition.

Definition 5 *An ordering $\succeq$ on the set D is separable if there exists a countable set $Z \subseteq D$ such that for every $x, y \in D$ with $x \succ y$, there exists $z \in Z$ such that $x \succeq z \succeq y$.*

It is well known that an ordering on D has a utility representation if and only if it is separable - the result goes back to at least Debreu (1954) (see also **?** for details). We show that $\succ^f$ is not separable. Consider $v_1 = v_2 = 1$. By definition of f, $\mathbf{1}_{v_1} \succ^f \mathbf{1}_{v_2} \succ^f \mathbf{1}_{v_0}$. Note that since $\succ^f$ is monotone, any utility vector between $\mathbf{1}_{v_1}$ and $\mathbf{1}_{v_2}$ (according to $\succ^f$) will be of the form $\mathbf{1}_{v_2+\epsilon}$ or $\mathbf{1}_{v_1-\epsilon}$ for some $\epsilon > 0$. But, $f(v_1, v_2 + \epsilon) = e^2$ implies that $\mathbf{1}_{v_2+\epsilon} \succ^f \mathbf{1}_{v_1}$ for all $\epsilon > 0$. Also, $f(v_1 - \epsilon, v_2) = e^2$ implies that $\mathbf{1}_{v_2} \succ^f \mathbf{1}_{v_1-\epsilon}$ for all $\epsilon > 0$. Hence, there cannot exist $z \in D$ such that $\mathbf{1}_{v_1} \succ^f z \succ^f \mathbf{1}_{v_2}$.

3.3 Simple Utility Maximization

We saw that the strict linear ordering that strongly rationalizes an allocation rule may not have a utility representation. The aim of this section is to explore minimal conditions that allow us to define a new ordering for any implementable and non-bossy allocation rule which has a utility representation. This allows us to sharpen our characterization, and relate it to a seminal result of **?**. Our extra condition is a *continuity* condition.

Definition **6** *An allocation rule f satisfies* **Condition** $\mathcal{C}^*$ *if for every $i, j \in N$ ($i \neq j$) and for every v_{-ij}, for every $\epsilon > 0$, there exists a $\delta_{\epsilon, v_{-ij}} > 0$ such that for every v_i, v_j with $f(v_i, v_j, v_{-ij}) = e^i$, we have $f(v_i + \epsilon, v_j + \delta_{\epsilon, v_{-ij}}, v_{-ij}) = e^i$.*

Condition $\mathcal{C}^*$ requires some version of continuity of the allocation rule. It says that if some agent i is winning the object at a valuation profile, for every increase in value of agent i, there exists some increase in value of agent j such that agent i continues to win the object. Later, we provide an example to show that Condition $\mathcal{C}^*$ and non-bossiness do not imply implementability.

If f is monotone (implementable) and non-bossy, then Condition $\mathcal{C}^*$ implies that for every $i, j \in N$ ($i \neq j$) and for every v_{-ij}, for every $\epsilon > 0$, there exists a $\delta_{\epsilon, v_{-ij}} > 0$ such that for every v_i, v_j with $f(v_i, v_j, v_{-ij}) = e^i$, we have $f(v_i + \epsilon, v_j + \delta, v_{-ij}) = e^i$ for all $0 < \delta < \delta_{\epsilon, v_{-ij}}$. To see this, choose some $\delta \in (0, \delta_{\epsilon, v_{-ij}})$ and assume for contradiction, $f(v_i + \epsilon, v_j + \delta, v_{-ij}) = e^k$ for some $k \neq i$. If $k = j$, then by monotonicity, $f(v_i + \epsilon, v_j + \delta_{\epsilon, v_{-ij}}, v_{-ij}) = e^j$, which is a contradiction to Condition $\mathcal{C}^*$. If $k \neq \{i, j\}$, then by non-bossiness, $f(v_i + \epsilon, v_j + \delta_{\epsilon, v_{-ij}}, v_{-ij}) \in \{e^j, e^k\}$, again a contradiction to Condition $\mathcal{C}^*$. Since we will use Condition $\mathcal{C}^*$ along with implementability and non-bossiness, we can freely make use of this implication.

We will now introduce a new class of allocation rules.

Definition **7** *An allocation rule f is a* **simple utility maximizer (SUM)** *if there exists a non-decreasing function $U_i : V_i \to \mathbb{R}$ for every $i \in N \cup \{0\}$, where $U_0(0) = 0$, such that for every valuation profile $v \in V$, $f(v) = e^j$ implies that $j \in \arg\max_{i \in N \cup \{0\}} U_i(v_i)$.*

Notice that an SUM allocation rule is simpler to state and, hence, more suitable for practical use than a strongly rationalizable allocation rule. The aim of this section is to show that the SUM allocation rules are not much different from the strongly rationalizable allocation rules.

It can be easily seen that not every SUM allocation rule is non-bossy. For instance, consider the efficient allocation rule that allocates the good to an agent with the highest value. Suppose there are three agents with valuations $10, 10, 8$ respectively and suppose that the efficient allocation rule allocates the object to agent 1. Consider the valuation profile $(10, 10, 9)$ and suppose that the efficient allocation rule now allocates the object to agent 2. This violates non-bossiness. As we will show that such violations can happen in case of ties (as was the case here with ties between agents 1 and 2), and when ties are broken carefully, an SUM allocation rule becomes non-bossy.

Similarly, not every SUM allocation rule is implementable. For instance, consider an example with two agents $\{1, 2\}$ with $V_1 = V_2 = \mathbb{R}_{++}$. Let $U_1(v_1) = 1$ and $U_1(v_2) = v_2$. Now, suppose we pick agent 1 as the winner of the object at valuation profile $(1, 1)$ but pick agent 2 as the winner of the object at valuation profile $(2, 1)$. Note that this is consistent with simple utility maximization but violates monotonicity, and hence, not implementable.

Now, consider the following modification of the SUM allocation rule.

DEFINITION **8** *An allocation rule f is a* **simple utility maximizer (SUM) with order-based tie-breaking** *if there exists a non-decreasing function $U_i : V_i \to \mathbb{R}$ for every $i \in N \cup \{0\}$, where $U_0(0) = 0$, and a monotone strict linear ordering $\succ$ on D such that for every valuation profile $v \in V$, $f(v) = e^j$ implies that $j \in \arg\max_{i \in N \cup \{0\}} U_i(v_i)$ and $\boldsymbol{1}_{v_j} \succ \boldsymbol{1}_{v_k}$ for all $k \neq j$ and $k \in \arg\max_{i \in N \cup \{0\}} U_i(v_i)$, i.e., $\boldsymbol{1}_{v_j}$ is the unique simple utility maximizer according to $\succ$.*

The tie-breaking rule that we specified is very general. It covers some intuitive tie-breaking rules such as having an ordering over $N \cup \{0\}$ and breaking the tie in simple utility maximization using this ordering.

LEMMA **3** *An SUM allocation rule with order-based tie-breaking is implementable.*

Proof: Suppose f is an SUM allocation rule with order-based tie-breaking. Let the corresponding simple utility functions be $U_0, U_1, \ldots, U_n$ and $\succ$ be the ordering used to break ties. At any valuation profile v, let

$$W(v) = \{j \in N \cup \{0\} : U_j(v_j) \geq U_k(v_k) \ \forall \ k \in N \cup \{0\}\}.$$

Fix an agent i and the valuation profile of other agents at v_{-i}. Consider v_i, v'_i such that $v_i < v'_i$ and $f(v_i, v_{-i}) = e^i$. Then, by SUM maximization, $i \in W(v_i, v_{-i})$. Further, by order-based tie-breaking $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ for all $j \in W(v_i, v_{-i})$. Since U_i is non-decreasing, $U_i(v'_i) \geq U_j(v_j)$ for all $j \in (N \cup \{0\}) \setminus \{i\}$. Hence, $i \in W(v'_i, v_{-i})$. Again, by order-based tie-breaking, $\mathbf{1}_{v'_i} \succ \mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ for all $j \in W(v'_i, v_{-i})$. This implies that $f(v'_i, v_{-i}) = e^i$. So, f is monotone, and hence, implementable. ■

An SUM allocation rule with order-based tie-breaking is also non-bossy.

LEMMA **4** *An SUM allocation rule with order-based tie-breaking is non-bossy.*

Proof: Let f be an SUM allocation rule with order-based tie-breaking and v be a valuation profile such that $f(v) \neq e^j$ for some $j \in N$. Suppose $f(v'_j, v_{-j}) \neq e^j$. Then, by definition, the unique simple utility maximizer of f remains the same in (v_j, v_{-j}) and (v'_j, v_{-j}). So, $f(v_j, v_{-j}) = f(v'_j, v_{-j})$, and hence, f is non-bossy. ■

We are now ready to state the main result of this section.

Theorem 2 *Suppose $V_i = (0, \beta_i)$, where $\beta_i \in \mathbb{R}_{++} \cup \{\infty\}$, for all $i \in N$ and f is an allocation rule satisfying Condition $\mathcal{C}^*$. Then, the following statements are equivalent.*

1. *f is an implementable and non-bossy allocation rule.*

2. *f is a simple utility maximizer allocation rule with order-based tie-breaking.*

The proof of Theorem 2 is given in the Appendix. The non-trivial part of the proof is to establish that under Condition $\mathcal{C}^*$, implementability and non-bossiness imply simple utility maximization. This part of the proof is long and tedious, but reveals beautiful structure of implementable and non-bossy allocation rules. Once this is established, we use Theorem 1 to conclude how the ties must be broken. As we discussed earlier, the strict linear ordering induced by an implementable and non-bossy allocation rule on the set of utility vectors D may not have a utility representation. Hence, we cannot invoke Theorem 1 directly to show Theorem 2. The proof of Theorem 2 constructs another ordering (which is not a linear order) and shows that this has a utility representation under Condition $\mathcal{C}^*$. We provide some remarks on Theorem 2 below.

Some simple utility maximizers. An efficient allocation rule is also an SUM allocation rule, where $U_i(v_i) = v_i$ for all $i \in N$ and for all $v_i \in V_i$. Similarly, we can define for every $i \in N$ and for every $v_i \in V_i$, $U_i(v_i) = \lambda_i v_i + \kappa_i$ for some $\lambda_i \geq 0$ and $\kappa_i \in \mathbb{R}$, and this SUM will correspond to the affine maximizer allocation rules of Roberts (1979). The simple utility function in Myerson (1981) takes the form $U_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$, where F_i and f_i are respectively the cumulative density function and density function of the distribution of valuation of agent i.

Payments. It is well known that revenue equivalence (Myerson, 1981) implies that for any implementable allocation rule, the payments are determined uniquely up to an additive constant. Suppose V_i is an interval for all $i \in N$. For any implementable allocation rule f, define the cutoff for agent i and valuation profile v_{-i} as $\kappa_i^f(v_{-i}) = \inf\{\alpha \in V_i : f(\alpha, v_{-i}) = e^i\}$, where $\kappa_i^f(v_{-i}) = 0$ if $f(\alpha, v_{-i}) \neq e^i$ for all $\alpha \in V_i$. It is well known that for every $i \in N$ and for every $(v_i, v_{-i}) \in V$, $p_i^f(v_i, v_{-i}) = \kappa_i^f(v_{-i})$ if $f(v_i, v_{-i}) = e^i$ and $p_i^f(v_i, v_{-i}) = 0$ if $f(v_i, v_{-i}) \neq e^i$ is a payment rule which implements f. Further, by revenue equivalence, any payment rule p which implements f must satisfy for every $i \in N$ and for every (v_i, v_{-i}), $p_i(v_i, v_{-i}) = p_i^f(v_i, v_{-i}) + h_i(v_{-i})$, where $h_i : V_{-i} \to \mathbb{R}$ is any function. Thus, by characterizing implementable allocation rules, we characterize the class of dominant strategy incentive compatible mechanisms.

Other versions of non-bossiness. Another version of non-bossiness, which seems appealing is the *utility non-bossiness*. Utility non-bossiness is a condition on *mechanisms* rather than on allocation rules only. In particular, an incentive compatible mechanism (f, p) satisfies **utility non-bossiness** if for every $i \in N$, for every v_{-i}, and for every $v_i, v_i' \in V_i$, such that $v_i f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) = v'_i f_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i})$, we have $v_j f_j(v_i, v_{-i}) - p_j(v_i, v_{-i}) = v_j f_j(v'_i, v_{-i}) - p_j(v'_i, v_{-i})$ for all $j \in N$. In words, if an agent changes his valuation such that his net utility does not change, then the net utility of every agent must remain unchanged.

We do not impose such version of utility non-bossiness because this is a condition on *mechanisms*, and we are interested in conditions on *allocation rules*. Further, utility non-bossiness is not satisfied by many canonical mechanisms. For instance, the second-price Vickrey auction is not utility non-bossy. To see this, consider an example with two agents with valuations 10 and 7 respectively. Note that the allocation rule in a second-price Vickrey auction is an efficient allocation rule. The net utilities of agents 1 and 2 in the second-price Vickrey auction are 3 and 0 respectively. Now, consider the valuation profile $(10, 8)$. At this valuation profile, agent 2 continues to get zero net utility in the second price Vickrey auction, but the net utility of agent 1 is reduced to 2. This shows that the second-price Vickrey auction is not utility non-bossy. On the other hand, the efficient allocation rule with order-based tie-breaking is a non-bossy allocation rule.

CONDITION $\mathcal{C}^*$. We give an example of an allocation rule which is non-bossy and satisfies Condition $\mathcal{C}^*$ but not implementable. The example illustrates that Condition $\mathcal{C}^*$ and non-bossiness do not make implementability a redundant condition. In other words, these two conditions together are not stronger than monotonicity.

EXAMPLE **4**

Let $N = \{1, 2\}$. Suppose $V_1 = V_2 = \mathbb{R}_{++}$. Let $U_1(v_1) = -v_1$ and $U_2(v_2) = -v_2$. The allocation rule f is defined as follows. It chooses e^0 (not allocating the object) if $U_1(v_1)$ and $U_2(v_2)$ are less than -1. Else, it allocates the object to the agent with the highest $U_i(v_i)$, breaking ties in favor of agent 1.

Formally, if $\max(U_1(v_1), U_2(v_2)) \leq -1$, then $f(v_1, v_2) = e^0$. Else, if $U_1(v_1) \geq \max(U_2(v_2), -1)$, then $f(v_1, v_2) = e^1$ and if $U_2(v_2) > U_1(v_1)$ and $U_2(v_2) \geq -1$, then $f(v_1, v_2) = e^2$. Clearly, this allocation rule is not monotone, and hence, not implementable. However, it is non-bossy and satisfies Condition $\mathcal{C}^*$.

3.4 Randomization and Bayesian Implementation via Border's Hierarchical Allocation Rules

We relate our results to Border's *hierarchical allocation rules* (**?**). [6] Border considered allocation rules which are not necessarily deterministic and Bayesian implementable. To describe his results, we consider randomized allocation rules in this section. A randomized allocation rule is a map $f : V \to \Delta A$, where ΔA denotes the convex hull of the $(n+1)$ vectors $\{e^0, e^1, \ldots, e^n\}$ in $\mathbb{R}^n$. Hence, $f_i(v)$ will now denote the probability of agent i getting the

[6] I am grateful to Mallesh Pai for motivating the contents of this section.

object at valuation profile v. **?** considers independent private values setting. Each bidder i has a probability distribution G_i using which it draws its value from V_i. Denote by $G_{-i}(v_{-i}) \equiv \times_{j \neq i} G_j(v_j)$. The interim allocation probability of an allocation rule f for agent i is

$$a_i^f(v_i) = \int_{V_{-i}} f_i(v_i, v_{-i}) dG_{-i}(v_{-i}).$$

Border also considers Bayesian implementation. An allocation rule f is Bayesian implementable if there exists a payment rules $(p_1, \ldots, p_n)$ such that for every $i \in N$, for every $v_i, v_i' \in V_i$

$$v_i a_i^f(v_i) - \int_{V_{-i}} p_i(v_i, v_{-i}) dG_{-i}(v_{-i}) \geq v_i a_i^f(v_i') - \int_{V_{-i}} p_i(v_i', v_{-i}) dG_{-i}(v_{-i}).$$

DEFINITION **9** *An allocation rule f^h is a* **hierarchical allocation rule** *if there exists non-decreasing functions $I_i : V_i \to \mathbb{R}$ for all $i \in N$ such that at every valuation profile $v \in V$*

$$f_i^h(v) = \begin{cases} \frac{1}{|\{j \in N : I_i(v_i) = I_j(v_j)\}|} & \text{if } I_i(v_i) \geq 0 \text{ and } I_i(v_i) \geq I_j(v_j) \text{ for all } j \in N \\ 0 & \text{otherwise} \end{cases}$$

In a seminal result, Border showed that for every Bayesian implementable allocation rule f, there exist a set of hierarchical allocation rules whose randomization gives the same interim allocation probability as f - see also Manelli and Vincent (2010); **?**); **?**. [7]

Now, notice that a hierarchical allocation rule is a randomization over simple utility maximizers (which are deterministic allocation rules). To see this, we define $(n+1)!$ order based tie-breaking rules. Take any strict linear ordering P of the set of alternatives in A. Define an ordering $\succ$ on the set of utility vectors D as follows. For any $i \in N$, if $\mathbf{1}_{v_i}, \mathbf{1}_{v_i'} \in D_i$ with $v_i > v_i'$, then $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_i'}$. If $e^i P e^j$, then for every $\mathbf{1}_{v_i} \in D_i$ and every $\mathbf{1}_{v_j} \in D_j$, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$. Note that $\succ$ can be defined exactly $(n+1)!$ ways, one for each P. Let $\mathcal{P}$ be the set of all such orderings of D. Now, given a hierarchical allocation rule with $(I_1, \ldots, I_n)$, we can construct $(n+1)!$ simple utility maximizers with $U_i = I_i$ for all $i \in N$ and taking as tie-breaking rule one of the orderings in $\mathcal{P}$. Clearly, uniform randomization over these simple utility maximizers produce the hierarchical allocation rule. Hence, randomization over the hierarchical allocation rules is equivalent to randomization over simple utility maximizers.

Thus, simple utility maximizers occupy a central role in the theory of private value single object auctions. By characterizing simple utility maximizers, Theorem 2 indirectly provides an axiomatic foundation for Border's hierarchical allocation rules. In particular, the interim allocation probability of any implementable allocation rule can be obtained by randomizing over the set of implementable and non-bossy allocation rules satisfying Condition $\mathcal{C}^*$.

[7] Although **?** does not consider incentive constraints, it is clear how his results can be modified in the presence of incentive constraints.

3.5 Extension of Roberts' Theorem

Consider a general mechanism design set up with private values and quasi-linear utility. Let A be a finite set of alternatives. Suppose $|A| \geq 3$. The type of agent i is denoted as $v_i \in \mathbb{R}^{|A|}$ and $v_i(a)$ denotes the valuation of agent i for alternative a. Roberts (1979) shows that if type space of every agent is $\mathbb{R}^{|A|}$, then for every onto and implementable allocation rule f, there exists $\lambda_1, \ldots, \lambda_n \geq 0$, not all of them equal to zero, and $\kappa : A \to \mathbb{R}$ such that at every valuation profile v,

$$f(v) \in \arg\max_{a \in A}[\sum_{i \in N} \lambda_i v_i(a) + \kappa(a)].$$

Such allocation rules are called affine maximizer allocation rules. Theorems 1 and 2 can be thought of as the analogue of Roberts' affine maximizer theorem in the single object auction model (under non-bossiness). It shows how much the set of implementable allocation rule expands in a restricted domain like the single object auction domain.

4 The Complete Characterization

Theorems 1 and 2 characterize implementable allocation rules under additional assumptions. In this section, we drop these additional assumptions and provide a complete characterization of implementable allocation rules. These characterizations are in the spirit of extending the Roberts' affine maximizer theorem. In particular, we show that an implementable allocation rule is equivalent to a *generalized utility maximizer* allocation rule.

A **generalized utility function (GUF)** of agent $i \in N$ is a function $u_i : V \to \mathbb{R}$. Notice that the generalized utility of an agent may be negative also. Further, a simple utility function is a GUF. We will need the following version of single crossing property.

Definition **10** *The GUFs* $(u_1, \ldots, u_n)$ *satisfy* **top single crossing** *if for every* $i \in N$*, for every* $v_{-i} \in V_{-i}$*, and for every* $v_i, v'_i \in V_i$ *with* $v_i > v'_i$ *and* $u_i(v'_i, v_{-i}) \geq \max(0, \max_{k \in N \setminus \{i\}} u_k(v'_i, v_{-i}))$*, we have* $u_i(v_i, v_{-i}) > \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i, v_{-i}))$.

The top single crossing condition is a very general inter-agent crossing condition. Such crossing conditions are extensively used in the literature of interdependent value auctions - see for instance, **?**Maskin (1992); **?**); **?**. For the finite type space, **?** use conditions similar to our top single crossing to establish implementation (in ex post equilibrium) of the efficient allocation rule in the interdependent values model.

The standard definition of a "single crossing" property, which implies top single crossing, is the following.

Definition **11** *GUFs* $(u_1, \ldots, u_n)$ *satisfy* **single crossing** *if for every* $i, j \in N$*, for every* $v_{-i} \in V_{-i}$*, for every* $v'_i, v_i \in V_i$ *with* $v_i > v'_i$*, we have* $u_i(v_i, v_{-i}) - u_i(v'_i, v_{-i}) > u_j(v_i, v_{-i}) - u_j(v'_i, v_{-i})$.

A GUF u_i is **increasing** if for every $v_{-i} \in V_{-i}$ and for every $v_i, v_i' \in V_i$ with $v_i > v_i'$ we have $u_i(v_i, v_{-i}) > u_i(v_i', v_{-i})$.

Lemma **5** *If GUFs $(u_1, \ldots, u_n)$ satisfy single crossing and u_i is increasing for every $i \in N$, then they satisfy top single crossing.*

Proof: Consider $i \in N$ and $v_{-i} \in V_{-i}$. Let $v_i, v_i' \in V_i$ such that $v_i > v_i'$ and $u_i(v_i', v_{-i}) \geq \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i', v_{-i}))$. Since u_i is increasing, $u_i(v_i, v_{-i}) > u_i(v_i', v_{-i}) \geq 0$. Further, by single crossing, $u_i(v_i, v_{-i}) - u_i(v_i', v_{-i}) > u_j(v_i, v_{-i}) - u_j(v_i', v_{-i})$ for all $j \neq i$. Using the fact that $u_i(v_i', v_{-i}) \geq u_j(v_i', v_{-i})$ for all $j \neq i$, we get that $u_i(v_i, v_{-i}) > u_j(v_i, v_{-i})$ for all $j \neq i$. Hence, $u_i(v_i, v_{-i}) > \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i, v_{-i}))$. ■

We are now ready to introduce a new class of implementable allocation rules.

Definition **12** *An allocation rule f is a* **generalized utility maximizer** *if there exist GUFs $(u_1, \ldots, u_n)$ satisfying top single crossing such that for every $v \in V$, $f(v) = e^i$ implies that $i \in \arg\max_{i \in N \cup \{0\}} u_i(v)$, where $u_0(v) = 0$.*

Generalized utility maximizers are implementable. The proof is similar to the proof in **?**, who establish implementation (in ex post equilibrium) of efficient allocation rule in an interdependent values model.

Lemma **6** *If f is a generalized utility maximizer, then it is implementable.*

Proof: Fix a generalized utility maximizer f, and let $(u_1, \ldots, u_n)$ be the corresponding GUFs satisfying top single crossing. Consider agent i and $v_{-i} \in V_{-i}$. Also, consider any $v_i, v_i' \in V_i$ with $v_i > v_i'$ and $f(v_i', v_{-i}) = e^i$. By definition, $u_i(v_i', v_{-i}) \geq \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i', v_{-i}))$. By top single crossing, $u_i(v_i, v_{-i}) > \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i, v_{-i}))$. Hence, $f(v_i, v_{-i}) = e^i$. So, f is monotone, and hence, implementable. ■

This leads to the main result of this section.

Theorem **3** *Suppose $V_i \subseteq \mathbb{R}_{++}$ is bounded for every $i \in N$. Then, f is implementable if and only if it is a generalized utility maximizer.*

Proof: Lemma 6 showed that every GUF maximizer is implementable. Now, for the converse, suppose f is implementable. Fix an agent $i \in N$ and $v_{-i} \in V_{-i}$. If $f(v_i, v_{-i}) \neq e^i$ for all $v_i \in V_i$, then define $\kappa_i^f(v_{-i}) = \sup\{v_i : v_i \in V_i\}$. Else, define $\kappa_i^f(v_{-i}) = \inf\{v_i \in V_i : f(v_i, v_{-i}) = e^i\}$. Since V_i is bounded, $\kappa_i^f(v_{-i})$ is well defined. Further, since f is monotone, for every agent $i \in N$, for every v_{-i}, and for every $v_i \in V_i$, if $v_i > \kappa_i^f(v_{-i})$, we have $f(v_i, v_{-i}) = e^i$ and for every $v_i < \kappa_i^f(v_{-i})$ we have $f(v_i, v_{-i}) \neq e^i$. Define for every $i \in N$ and for every (v_i, v_{-i}), $u_i(v_i, v_{-i}) := v_i - \kappa_i^f(v_{-i})$. By definition, if $f(v) = e^i$, then $v_i - \kappa_i^f(v_{-i}) \geq 0$ and $v_j - \kappa_j^f(v_{-j}) \leq 0$ for all $j \neq i$. Hence, $i \in \arg\max_{k \in N \cup \{0\}} u_k(v)$, where $u_0(v) = 0$.

To show that $(u_1, \ldots, u_n)$ satisfy top single crossing, consider $i \in N$ and $v_{-i} \in V_{-i}$. Let $v_i, v_i' \in V_i$ such that $v_i > v_i'$ and $u_i(v_i', v_{-i}) \geq \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i', v_{-i}))$. Notice that $u_i(v_i, v_{-i}) > u_i(v_i', v_{-i}) \geq 0$. By definition of $u_1, \ldots, u_n$, if $u_i(v_i, v_{-i}) > 0$, then $v_i > \kappa_i^f(v_{-i})$, and hence, $f(v_i, v_{-i}) = e^i$. But, this implies that $u_k(v_i, v_{-i}) = v_k - \kappa_k^f(v_{-k}) \leq 0$ for all $k \neq i$. Hence, $u_i(v_i, v_{-i}) > \max(0, \max_{k \in N \setminus \{i\}} u_k(v_i, v_{-i}))$. ■

Our characterization of implementability shows that implementability is equivalent to maximizing generalized utilities. Generalized utilities transform the original valuation of an agent to a new utility, which depends on the valuations of *all* the agents. In contrast to simple utility functions, generalized utility functions are much harder to construct. This illustrates how a natural axiom like non-bossiness helps to simplify the class of implementable allocation rules.

Generalized utility maximizers are similar to implementing the *efficient* allocation rule in an interdependent values model with the qualification that we allow generalized utilities to be negative, which is precluded in the standard interdependent value model. It is well known that the efficient allocation rule is not generally implementable in the interdependent values single object auction unless some inter agent crossing condition holds (**?**Maskin, 1992; **?**; **?**; **?**). Our top single crossing condition is similar to these conditions in the interdependent values literature. Our result reveals a surprising and interesting connection between these seemingly different models.

5 Discussions

We conclude by discussing some of the open questions that remain.

Randomization and Bayesian Implementation. Although we focus on deterministic dominant strategy implementation, randomization is a natural extension of our model. Indeed, the monotonicity characterization of Myerson (1981) extends to single object auctions with randomization. Extending characterizations of deterministic allocation rules to randomized allocation rules present several challenges. A natural way to think of randomization is that of domain restriction - the utility from a lottery alternative is restricted to be the expected utility from the deterministic alternatives in its support. Thus, the challenges of going from deterministic to randomized allocation rules is similar to that of going from a larger domain to a restricted domain. For instance, a counterpart of Roberts' seminal result with randomization is still not known in the unrestricted domain.

However, we provided a relationship of our simple utility maximizer and Border's hierarchical allocation rules that can be used to obtain interim allocation probability of every Bayesian and randomized allocation rule. Hence, our characterizations can be used in an indirect way to characterize interim allocation probabilities of Bayesian implementable randomized allocation rules. However, the direct characterization remains an open question.

Optimizing payments. A popular research theme in auction theory and mechanism design is to "optimize" over the set of incentive compatible mechanisms. This usually involves optimizing over payments and assumes some prior distribution over valuations of agents by the mechanism designer. The implications of such optimizations in the single object auctions is fairly well understood.

Clearly, our results do not contribute to this literature. Our characterizations are more tailored towards understanding the inherent structure of deterministic single object auctions in private values set up. They completely describe the set of "options" available to a mechanism designer (without bothering about the distributional assumptions) in the single object auctions. Our main characterizations provide axiomatic foundations to various commonly used auctions.

We also believe that this opens a door for carrying out similar exercises in multidimensional mechanism design models, including the multi-object auction model. The problem of finding an expected revenue maximizing mechanism in such models is considered a difficult problem **??**. Perhaps, understanding the structure of incentive compatible mechanisms will allow us to simplify these problems.

Appendix: Omitted Proofs

Proof of Theorem 1

By virtue of Lemmas 1 and 2, we only need to show that if an allocation rule f is implementable and non-bossy then it is strongly rationalizable. We do the proof in several steps.

Step 1. For any $i, j \in N \cup \{0\}$ with $i \neq j$, consider $\mathbf{1}_{v_i}$ and $\mathbf{1}_{v_j}$ for some $v_i \in V_i$ and $v_j \in V_j$. Suppose for some v_{-ij}, we have $f(v_i, v_j, v_{-ij}) = e^i$. We will show that if f is non-bossy, then $f(v_i, v_j, v'_{-ij}) \neq e^j$ for all v'_{-ij}. Consider any $k \notin \{i, j\}$ and the profile $(v_i, v_j, v'_k, v_{-ijk})$. By non-bossiness, $f(v_i, v_j, v'_k, v_{-ijk}) \in \{e^i, e^k\}$. Repeating this argument for all $k \notin \{i, j\}$, we get $f(v_i, v_j, v'_{-ij}) \neq e^j$.

Step 2. We will first define a binary relations $\succ$ on $D \times D$ [8] using f as follows. For every $i, j \in N \cup \{0\}$ with $i \neq j$, $\mathbf{1}_{v_i} \in D_i$ and $\mathbf{1}_{v_j} \in D_j$, define

$$\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$$

if there is some v_{-ij} such that $f(v_i, v_j, v_{-ij}) = e^i$. Further, for every $i \in N$ and every $v_i \in V_i$, define

$$\mathbf{1}_{v_i+\epsilon} \succ \mathbf{1}_{v_i}$$

[8] To remind, D is the set of all utility vectors given the type space.

for all $\epsilon > 0$ such that $(v_i + \epsilon) \in V_i$. Using Step 1, if $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$, then $\mathbf{1}_{v_j} \not\succ \mathbf{1}_{v_i}$. Hence, $\succ$ is anti-symmetric. Further, $\succ$ is irreflexive by definition.

STEP 3. Let $D^f := \{x \in D : G^f(v) = x \text{ for some } v \in V\}$. We now show that $\succ$ satisfies the following conditions:

1 for every $x, y \in D^f$, either $x \succ y$ or $y \succ x$ (but not both), where $D^f = \{x \in D : G^f(v) = x \text{ for some } v \in V\}$,

2 for every $x \in D^f$ and for every $y \notin D^f$, $x \succ y$,

3 for all $v \in V$, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ for all $j \in \{0, 1, \ldots, n\} \setminus \{i\}$, where $G^f(v) = \mathbf{1}_{v_i}$.

- PROOF OF (1). Pick $x, y \in D^f$. By definition, there is $v \in V$, such that $G^f(v) = x$. If $x = \mathbf{1}_{v_i}$, then $f(v) = e^i$. Suppose $y = \mathbf{1}_{v'_i}$. Then, by definition, either $x \succ y$ or $y \succ x$. Hence, suppose $y = \mathbf{1}_{v'_j}$ for some $j \neq i$. Then, by monotonicity and non-bossiness, $f(v_i, v'_j, v_{-ij}) \in \{e^i, e^j\}$. Hence, either $x \succ y$ or $y \succ x$. Since $\succ$ is anti-symmetric, either $x \succ y$ or $y \succ x$ but not both.

- PROOF OF (2). Pick $x \in D^f$ but $y \notin D^f$. By definition, there is $v \in V$, such that $G^f(v) = x$. If $x = \mathbf{1}_{v_i}$, then $f(v) = e^i$. Suppose $y = \mathbf{1}_{v'_i}$. Then, if $v'_i > v_i$, we have $f(v'_i, v_{-i}) = e^i$ by monotonicity, and this contradicts the fact that $y \notin D^f$. Hence, $v'_i < v_i$, and by definition, $x \succ y$.

  Suppose $y = \mathbf{1}_{v'_j}$ for some $j \neq i$. Then, by monotonicity and non-bossiness, $f(v_i, v'_j, v_{-ij}) \in \{e^i, e^j\}$. Using the fact that $y \notin D^f$, we get that $f(v_i, v'_j, v_{-ij}) = e^i$. Hence, $x \succ y$.

- PROOF OF (3). At any valuation profile $(v_1, \ldots, v_n)$, if $f(v_1, \ldots, v_n) = e^i$, then, by definition, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ for all $j \neq i$.

STEP 4. We show that $\succ$ is transitive. Suppose for some $i \in N$, $\mathbf{1}_{v_i+\epsilon} \succ \mathbf{1}_{v_i}$ for some $\epsilon > 0$ such that $v_i + \epsilon \in V_i$. Also, for some $j \neq i$, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$. Then, by definition, for some v_{-ij}, $f(v_i, v_j, v_{-ij}) = e^i$. By monotonicity, $f(v_i + \epsilon, v_j, v_{-ij}) = e^i$. Hence, $\mathbf{1}_{v_i+\epsilon} \succ \mathbf{1}_{v_j}$.

We also know that for some $i \in N$ and for some $\epsilon > 0, \delta > 0$, if $\mathbf{1}_{v_i+\epsilon+\delta} \succ \mathbf{1}_{v_i+\epsilon}$ and $\mathbf{1}_{v_i+\epsilon} \succ \mathbf{1}_{v_i}$, then $\mathbf{1}_{v_i+\epsilon+\delta} \succ \mathbf{1}_{v_i}$.

Finally, pick $v_i \in V_i, v_j \in V_j$ and $v_k \in V_k$ such that $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_j}$ and $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_k}$, where i, j, k are distinct. This means, $f(v_i, v_j, v'_{-ij}) = e^i$ for some v'_{-ij}. By monotonicity and non-bossiness, $f(v_i, v_j, v_k, v'_{-ijk}) \in \{e^i, e^k\}$. But, $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_k}$ implies that $f(v_i, v_j, v_k, v'_{-ijk}) \neq e^k$. Hence, $f(v_i, v_j, v_k, v'_{-ijk}) = e^i$. Hence, $\mathbf{1}_{v_i} \succ \mathbf{1}_{v_k}$. This shows that $\succ$ is transitive.

STEP 5. We show that f is strongly rationalizable. Since $\succ$ is an anti-symmetric, irreflexive and transitive binary relation on $D \times D$, we can extend it to an anti-symmetric, irreflexive, complete, and transitive binary relation $\succ'$ on $D \times D$ due to Szpilrajn's extension theorem

- see **?** for instance. By definition of $\succ'$ and Step 3, at any valuation profile $(v_1, \ldots, v_n)$, if $f(v_1, \ldots, v_n) = e^i$, then, $\mathbf{1}_{v_i} \succ' \mathbf{1}_{v_j}$ for all $j \neq i$. By definition, $\succ'$ is monotone. Hence, f is strongly rationalizable.

Proof of Theorem 2

By Lemmas 3 and 4, an SUM allocation rule with order-based tie-breaking is implementable and non-bossy. We show that every implementable and non-bossy allocation rule satisfying Condition $\mathcal{C}^*$ is an SUM allocation rule with order-based tie-breaking. We do the proof in various steps. Throughout we assume that $V_i = (0, \beta_i)$, where $\beta_i \in \mathbb{R}_{++} \cup \{\infty\}$, for all $i \in N$.

Step 1. In this step, we show that if f is implementable and non-bossy allocation rule satisfying Condition $\mathcal{C}^*$, then there is an ordering $\succeq^f$ on D which rationalizes f. We construct this specific $\succeq^f$ in this step. [9]

Suppose f is an implementable and non-bossy allocation rule satisfying Condition $\mathcal{C}^*$. We first define the notion of a **winning set**. The winning set of allocation rule f at a valuation profile v is denoted as $W^f(v)$, and defined as follows. For any $i \in N$, we say $e^i \in W^f(v)$ if for all $\epsilon > 0$, we have $f(v_i + \epsilon, v_{-i}) = e^i$, where $(v_i + \epsilon) \in V_i$. We say that $e^0 \in W^f(v)$ if for all $\epsilon > 0$, we have $f(\{v_j - \epsilon\}_{j \in N}) = e^0$, where $(v_j - \epsilon) \in V_j$ for all $j \in N$. The first claim is that $W^f(v)$ is non-empty for all valuation profiles v.

Lemma 7 *If f is implementable and non-bossy, then for every value profile v, $f(v) \in W^f(v)$.*

Proof: Consider an implementable and non-bossy allocation rule f and a valuation profile v. If $f(v) = e^j \neq e^0$, then by monotonicity $f(v_j + \epsilon, v_{-j}) = e^j$ for all $\epsilon > 0$. Hence, $f(v) \in W^f(v)$.

If $f(v) = e^0$, then consider any $\epsilon > 0$ and a valuation profile v' such that $v'_i - \epsilon > 0$ for all $i \in N$. We argue that $f(v') = e^0$, and hence, $e^0 = f(v) \in W^f(v)$. Assume for contradiction that $f(v') = e^j \neq e^0$. Now, we go from v' to v by increasing the valuation of one agent at a time. By monotonicity, $f(v_j, v'_{-j}) = e^j$. Now, pick any $k \in N \setminus \{j\}$. Then, either $f(v_j, v_k, v'_{-jk}) = e^k$ or by non-bossiness $f(v_j, v_k, v'_{-jk}) = e^j$. In both cases, we see that $f(v_j, v_k, v'_{-jk}) \neq e^0$. Continuing in this manner, we will reach the valuation profile v and get that $f(v) \neq e^0$, a contradiction. ■

Step 1.1. In this step, we show that an implementable and non-bossy allocation rule satisfying Condition $\mathcal{C}^*$ satisfies a form of independence property.

[9] Notice that by Theorem 1, if f is implementable and non-bossy, then it is a strongly rationalizable allocation rule, and hence, a rationalizable allocation rule. The novelty of this step of the proof is to be able to construct a *specific* ordering which rationalizes f.

DEFINITION **13** *An allocation rule f satisfies* **binary independence** *if for any pair of alternatives $e^j, e^k \in A$ and any pair of valuation profiles v, v' such that $\mathbf{1}_{v_j} = \mathbf{1}_{v'_j}$ and $\mathbf{1}_{v_k} = \mathbf{1}_{v'_k}$, the following conditions hold.*

1. *if $e^k \in W^f(v)$ and $e^j \in W^f(v')$, then $e^k \in W^f(v')$,*
2. *if $e^j \in W^f(v)$ and $e^k \notin W^f(v)$, then $e^k \notin W^f(v')$.*

Intuitively, the binary independence property says that the comparison of any pair of utility vectors is independent of what the other utility vectors are.

PROPOSITION **1** *An implementable and non-bossy allocation rule satisfying Condition $\mathcal{C}^*$ satisfies binary independence.*

Proof: We will use the following lemma to prove the proposition.

LEMMA **8** *Suppose v and v' are two distinct valuation profiles such that $v_i \geq v'_i$ for all $i \in N$. Let $B(v, v') = \{e^i \in A : v_i > v'_i\}$. If f is an implementable and non-bossy allocation rule, then $W^f(v) \setminus B(v, v') \subseteq W^f(v')$.*

Proof: Let f be an implementable and non-bossy allocation rule and v and v' be two distinct valuation profiles with $v_i \geq v'_i$ for all $i \in N$. We will go from v to v' by lowering one agent's value at a time. Pick any $e^j \in B(v, v')$. Consider a new type profile v'' such that the value of every agent $i \neq j$ remains v_i and the value of agent j is v'_j, which is strictly less than v_j. Pick any $e^k \in W^f(v)$ such that $e^k \neq e^j$. Then, we consider two cases.

CASE 1: $e^k \neq e^0$. We argue that $e^k \in W^f(v'')$. Assume for contradiction that $e^k \notin W^f(v'')$. Then, for some $\epsilon > 0$, we have $f(v_k + \epsilon, v'_j, v_{-kj}) \neq e^k$. If $f(v_k + \epsilon, v'_j, v_{-kj}) = e^j$, then by monotonicity, we have $f(v_k + \epsilon, v_j, v_{-kj}) = e^j$. This is a contradiction since $e^k \in W^f(v)$. If $f(v_k + \epsilon, v'_j, v_{-kj}) = e^l \notin \{e^j, e^k\}$, then monotonicity and non-bossiness implies that $f(v_k + \epsilon, v_j, v_{-kj}) \in \{e^l, e^j\}$. But this contradicts $e^k \in W^f(v)$.

CASE 2: $e^k = e^0$. Since $e^0 \in W^f(v)$, for any $\epsilon > 0$ such that $\bar{v}_i := v_i - \epsilon > 0$ for all $i \in N$, we have $f(\bar{v}_1, \ldots, \bar{v}_n) = e^0$. Note that $v'_i - \epsilon = v_i - \epsilon = \bar{v}_i$ for all $i \neq j$ for any ϵ. Now, fix any $\epsilon > 0$ such that $v'_j - \epsilon > 0$. Consider the valuation profile $(\bar{v}_{-j}, v'_j - \epsilon)$. Since $f(\bar{v}_1, \ldots, \bar{v}_n) = e^0$ and $\bar{v}_j = v_j - \epsilon > v'_j - \epsilon$, by monotonicity and non-bossiness, we have $f(v'_j - \epsilon, \bar{v}_{-j}) = e^0$. Hence, $e^0 \in W^f(v'')$.

This establishes that $e^k \in W^f(v'')$ for any $e^k \neq e^j$. Hence, $W^f(v) \setminus \{e^j\} \subseteq W^f(v'')$. Repeating this argument for other elements of $B(v, v')$ one by one, we conclude that $W^f(v) \setminus B(v, v') \subseteq W^f(v')$. ■

Now, let f be an implementable and non-bossy allocation rule satisfying Condition $\mathcal{C}^*$. Pick any pair of alternatives $e^j, e^k \in A$ and any pair of valuation profiles v, v' such that $\mathbf{1}_{v_j} = \mathbf{1}_{v'_j}$ and $\mathbf{1}_{v_k} = \mathbf{1}_{v'_k}$. We will show that f satisfies both (1) and (2) of Definition 13.

1. Suppose $e^k \in W^f(v)$ and $e^j \in W^f(v')$. We will show that $e^k \in W^f(v')$. Construct a new type profile v'' such that $v''_i = \min(v_i, v'_i)$ for all $i \in N$. Note that $\mathbf{1}_{v''_j} = \mathbf{1}_{v_j} = \mathbf{1}_{v'_j}$ and $\mathbf{1}_{v''_k} = \mathbf{1}_{v_k} = \mathbf{1}_{v'_k}$. By Lemma 8, $e^j, e^k \in W^f(v'')$. Now, assume for contradiction that $e^k \notin W^f(v')$. We now consider various cases.

   Case 1: $e^j, e^k \in A \setminus \{e^0\}$. Since $e^k \notin W^f(v')$, there exists $\epsilon > 0$ such that $f(v'_k + \epsilon, v'_{-k}) \neq e^k$. By monotonicity and non-bossiness, for all $\epsilon' > 0$ we have $f(v'_j + \epsilon', v'_k + \epsilon, v'_{-jk}) \neq e^k$. Further, we show that $f(v'_j + \epsilon', v'_k + \epsilon, v'_{-jk}) = e^j$ for all $\epsilon' > 0$. To see this, suppose $f(v'_j + \epsilon', v'_k + \epsilon, v'_{-jk}) = e^l$ for some $e^l \notin \{e^j, e^k\}$. Then, by monotonicity and non-bossiness, we get $f(v'_j + \epsilon', v'_k, v'_{-jk}) = e^l$, and this contradicts $e^j \in W^f(v')$. Hence, $f(v'_j + \epsilon', v'_k + \epsilon, v'_{-jk}) = e^j$ for all $\epsilon' > 0$. Now, applying monotonicity and non-bossiness again, for all $\epsilon' > 0$, we have

$$f(v'_j + \epsilon', v'_k + \epsilon, v''_{-jk}) = e^j. \tag{1}$$

   Since $e^k \in W^f(v'')$, we have $f(v'_j, v'_k + \frac{\epsilon}{2}, v''_{-jk}) = e^k$. By Condition $\mathcal{C}^*$, there is an $\epsilon' > 0$ such that $f(v'_j + \epsilon', v'_k + \epsilon, v''_{-jk}) = e^k$. This contradicts Equation 1.

   Case 2: $e^j = e^0$. We have to show that $e^0 \in W^f(v')$ implies $e^k \in W^f(v')$. Assume for contradiction that $e^k \notin W^f(v')$ but $e^0 \in W^f(v')$. For this, we first show that there is some $\epsilon_i > 0$ for every $i \in N$ such that $f(v'_k + \epsilon_k, \{v'_i - \epsilon_i\}_{i \neq k}) = e^0$.

   To see this, suppose $f(v'_k + \epsilon_k, \{v'_i - \epsilon_i\}_{i \neq k}) = e^k$ for all $\{\epsilon_i\}_{i \in N}$. Fix any $l \neq k$. Then, by Condition $\mathcal{C}^*$, for every ϵ there is a δ such that, $f(v'_k + \epsilon_k + \epsilon, (v'_l - \epsilon_l + \delta), \{v'_i - \epsilon_i\}_{i \neq k,l}) = e^k$ for all $\{\epsilon_i\}_{i \in N}$. Fix some $\epsilon > 0$. By, Condition $\mathcal{C}^*$, we can choose $\epsilon_l = \delta$. Also, let $\epsilon_k = \epsilon$. Hence, we get $f(v'_k + 2\epsilon, v'_l, \{v'_i - \epsilon_i\}_{i \neq k,l}) = e^k$. Repeating this, we reach $f(v'_k + (n-1)\epsilon, v'_{-k}) = e^k$. Since $n > 1$, we get that $e^k \in W^f(v')$. But this contradicts the fact that $e^k \notin W^f(v')$.

   Similarly, suppose $f(v'_k + \epsilon_k, \{v'_i - \epsilon_i\}_{i \neq k}) = e^l$ for some $l \neq 0, k$. Then, by monotonicity and non-bossiness, we get that $f(\{v'_i - \epsilon_i\}_{i \in N}) = e^l$. This means $f(\{v'_i - \epsilon_i\}_{i \in N}) \neq e^0$. Now, choose $\epsilon' < \min_{i \in N} \epsilon_i$. Then, consider the profile $\{v'_i - \epsilon'\}_{i \in N}$. By repeated application of monotonicity and non-bossiness, $f(\{v'_i - \epsilon'\}_{i \in N}) \neq e^0$. This contradicts $e^0 \in W^f(v')$.

   This shows that there is some $\epsilon_i > 0$ for all $i \in N$ such that $f(v'_k + \epsilon_k, \{v'_i - \epsilon_i\}_{i \neq k}) = e^0$. By monotonicity and non-bossiness, $f(v'_k + \epsilon_k, \{v''_i - \epsilon_i\}_{i \neq k}) = e^0$. But $e^k \in W^f(v'')$ implies that $f(v''_k + \epsilon_k, v''_{-k}) = e^k$ (to remind, $v'_k = v''_k$). But monotonicity and non-bossiness implies that $f(v'_k + \epsilon_k, \{v''_i - \epsilon_i\}_{i \neq k}) = e^k$. This gives us a contradiction.

   Case 3: $e^k = e^0$. We have to show that if $e^j \in W^f(v')$ then $e^0 \in W^f(v')$. Assume for contradiction $e^0 \notin W^f(v')$. We first show that for some $\epsilon > 0$ and $\epsilon' > 0$, $f(v'_j - \epsilon, \{v'_i - \epsilon'\}_{i \neq j}) = e^j$.

To see this, suppose that $f(v'_j - \epsilon, \{v'_i - \epsilon'\}_{i \neq j}) = e^0$ for all ϵ, ϵ'. Then, by monotonicity and non-bossiness, we see that $f(\{v'_i - \min(\epsilon, \epsilon')\}_{i \in N}) = e^0$ for all ϵ, ϵ'. But this contradicts $e^0 \notin W^f(v')$.

Similarly, suppose that $f(v'_j - \epsilon, \{v'_i - \epsilon'\}_{i \neq j}) = e^l$ for some $l \in N \setminus \{j\}$ and for all ϵ, ϵ'. By Condition $\mathcal{C}^*$, there is some $\delta := \delta_{\epsilon', v'_{-lj}} < \epsilon'$ such that $f(v'_j - \epsilon + \delta, v'_l, \{v'_i - \epsilon'\}_{i \neq j,l}) = e^l$ for all ϵ, ϵ'. Since δ is independent of ϵ, we can choose $\epsilon = \frac{\delta}{2}$ for every ϵ'. Hence, we have $f(v'_j + \frac{\delta}{2}, v'_l, \{v'_i - \epsilon'\}_{i \neq j,l}) = e^l$ for every ϵ'. Further, since $e^j \in W^f(v')$, we know that $f(v'_j + \frac{\delta}{2}, v'_{-j}) = e^j$ for all ϵ'. By repeatedly applying monotonicity and non-bossiness, we get that $f(v'_j + \frac{\delta}{2}, v'_l, \{v'_i - \epsilon'\}_{i \neq j,l}) = e^j$ for every ϵ'. This gives us a contradiction.

This shows that $f(v'_j - \epsilon, \{v'_i - \epsilon'\}_{i \neq j}) = e^j$ for some $\epsilon > 0$ and $\epsilon' > 0$. By repeatedly applying monotonicity and non-bossiness, we get that $f(v'_j - \epsilon, \{v''_i - \epsilon'\}_{i \neq j}) = e^j$ for some $\epsilon > 0$ and $\epsilon' > 0$. Since $e^0 \in W^f(v'')$, we know that $f(\{v'_i - \min(\epsilon, \epsilon')\}_{i \in N}) = e^0$. By repeatedly applying monotonicity and non-bossiness, we get that $f(v'_j - \epsilon, \{v''_i - \epsilon'\}_{i \neq j}) = e^0$. This is a contradiction.

This concludes the proof of Property (1) in Definition 13.

2. Property (2) in Definition 13 follows by applying Property (1). To see this, pick any $e^j, e^k \in A$ and v, v' as in Definition 13. Suppose $e^j \in W^f(v)$ but $e^k \notin W^f(v')$. We need to show that $e^k \notin W^f(v')$. Assume for contradiction $e^k \in W^f(v')$. Then, by changing the role of v and v' in (1), we get that $e^k \in W^f(v)$, which is a contradiction.

■

Step 1.2. In this step, we define an ordering on the set of utility vectors D. We denote this ordering as $\succeq^f$. The anti-symmetric part of this ordering is denoted as $\succ^f$ and the symmetric part is denoted as $\sim^f$. For any $i \in N$ and for any $v_i, v'_i \in V_i$ with $v_i > v'_i$, we define $\mathbf{1}_{v_i} \succ^f \mathbf{1}_{v'_i}$. Further, for every $i \in N$ and every $v_i \in V_i$, we define $\mathbf{1}_{v_i} \sim^f \mathbf{1}_{v_i}$ (reflexive). For any $i, j \in N \cup \{0\}$ (with $i \neq j$) and any $v_i \in V_i$ and $v_j \in V_j$, we define

1. $\mathbf{1}_{v_i} \succ^f \mathbf{1}_{v_j}$, if there exists a valuation profile v' such that $\mathbf{1}_{v'_i} = \mathbf{1}_{v_i}$, $\mathbf{1}_{v'_j} = \mathbf{1}_{v_j}$, and $e^i \in W^f(v')$ but $e^j \notin W^f(v')$;

2. $\mathbf{1}_{v_i} \sim^f \mathbf{1}_{v_j}$, if (a) there exists a valuation profile v' such that $\mathbf{1}_{v'_i} = \mathbf{1}_{v_i}$, $\mathbf{1}_{v'_j} = \mathbf{1}_{v_j}$, and $e^i, e^j \in W^f(v')$ or (b) at every valuation profile v' such that $\mathbf{1}_{v'_i} = \mathbf{1}_{v_i}$, and $\mathbf{1}_{v'_j} = \mathbf{1}_{v_j}$, we have $e^i, e^j \notin W^f(v')$.

We show that the binary relation $\succeq$ is well defined.

Lemma **9** *Suppose f is implementable, non-bossy, and satisfies Condition $\mathcal{C}^*$. Then, $\succeq^f$ is well-defined.*

Proof: Fix some $x, y \in D$. If $x, y \in D_i$ for some $i \in N$, and $x = \mathbf{1}_{v_i}$ and $y = \mathbf{1}_{v'_i}$ with $v_i > v'_i$ then, by definition, $x \succ^f y$. Similarly, if $x \in D_i$ and $y \in D_j$ for some $i \neq j$, and for every valuation profile v with $\mathbf{1}_{v_i} = x$ and $\mathbf{1}_{v_j} = y$ we have $e^i, e^j \notin W^f(v)$, then, by definition, $x \sim^f y$.

So, we just need to consider the case where $x \in D_i$ and $y \in D_j$ for some $i \neq j$, and there is a valuation profile v with $\mathbf{1}_{v_i} = x$ and $\mathbf{1}_{v_j} = y$ with either e^i or e^j or both are in $W^f(v)$. We consider two cases.

Case 1. Suppose $e^i, e^j \in W^f(v)$. Now, consider any other valuation profile v' such that $\mathbf{1}_{v_i} = \mathbf{1}_{v'_i} = x$ and $\mathbf{1}_{v_j} = \mathbf{1}_{v'_j} = y$. By Proposition 1, $e^i \in W^f(v')$ if and only if $e^j \in W^f(v')$. This means that the relation $x \sim^f y$ is well-defined.

Case 2. Suppose $e^i \in W^f(v)$ but $e^j \notin W^f(v)$. Now, consider any other valuation profile v' such that $\mathbf{1}_{v_i} = \mathbf{1}_{v'_i} = x$ and $\mathbf{1}_{v_j} = \mathbf{1}_{v'_j} = y$. By Proposition 1, $e^j \notin W^f(v')$. This means that the relation $x \succ^f y$ is well-defined. ■

Step 1.3. In this step, we show that $\succeq^f$ is an ordering, i.e., the binary relation is reflexive, complete, and transitive. The fact that $\succeq^f$ is reflexive and complete is clear. We show that $\succeq^f$ is transitive.

Proposition **2** *If f is an implementable and non-bossy allocation rule satisfying Condition $\mathcal{C}^*$, then $\succeq^f$ is transitive.*

Proof: For this, we will show that $\succ^f$ and $\sim^f$ are transitive, and this in turn will imply that $\succeq^f$ is transitive. Pick any $x, y, z \in D$ such that $x \neq y \neq z$. We consider three cases.

Case 1. Suppose $x, y, z \in D_i$ for some $i \in N$ and $x = \mathbf{1}_{v_i}, y = \mathbf{1}_{v'_i}, z = \mathbf{1}_{v''_i}$. Suppose $x \succ^f y$ and $y \succ^f z$. Then, it must be $v_i > v'_i > v''_i$. By definition, we have $x \succ^f z$.

Case 2. $x, y \in D_i$ but $z \in D_j$ for some i, j where $i \neq j$. Suppose $x = \mathbf{1}_{v_i}, y = \mathbf{1}_{v'_i}$, and $z = \mathbf{1}_{v_j}$. Suppose $x \succ^f y$ and $y \succ^f z$. Note that $x \succ^f y$ implies $v_i > v'_i$. We consider two subcases.

Case 2a. Suppose $j \neq 0$. Since $y \succ^f z$, there is a valuation profile v'' such that $v''_i = v'_i$, $v''_j = v_j$, and $e^i \in W^f(v'')$ but $e^j \notin W^f(v'')$. Now consider the type profile $\bar{v}$, where $\bar{v}_k = v''_k$ if $k \neq i$ and $\bar{v}_i = v_i$. We show that $e^i \in W^f(\bar{v})$ and $e^j \notin W^f(\bar{v})$, and this will show that $x \succ^f z$. Since $e^i \in W^f(v'')$, we know that $f(v'_i + \epsilon, v_j, v''_{-ij}) = e^i$ for all $\epsilon > 0$. By monotonicity, $f(v_i + \epsilon, v_j, v''_{-ij}) = e^i$ for all $\epsilon > 0$. Hence, $e^i \in W^f(\bar{v})$. Since $e^j \notin W^f(v'')$, there is some $\epsilon > 0$ such that $f(v'_i, v_j + \epsilon, v''_{-ij}) \neq e^j$. By monotonicity and non-bossiness, $f(v_i, v_j + \epsilon, v''_{-ij}) \neq e^j$. Hence, $e^j \notin W^f(\bar{v})$.

Case 2b. Suppose $j = 0$. So, z is the n-dimensional zero vector. Since $y \succ^f z$, there is a valuation profile $\bar{v}$ with $\mathbf{1}_{\bar{v}_i} = \mathbf{1}_{v'_i} = y$ and $e^i \in W^f(\bar{v})$ but $e^0 \notin W^f(\bar{v})$. Now, consider the valuation profile $v'' \equiv (v_i, \bar{v}_{-i})$. Since $v_i > v'_i$, by monotonicity, we have $e^i \in W^f(v'')$.

Since $e^0 \notin W^f(\bar{v})$, there is some $\epsilon > 0$ such that $f(\{\bar{v}_k - \epsilon\}_{k \in N}) \neq e^0$. Now, since $v_i > v'_i$, by monotonicity and non-bossiness, $f(v_i - \epsilon, \{\bar{v}_k - \epsilon\}_{k \neq i}) \neq e^0$. Hence, $e^0 \notin W^f(v'')$.

This completes the proof of Case 2.

Case 3. $x \in D_i$, $y \in D_j$, $z \in D_k$, where i, j, k are distinct. Suppose $x = \mathbf{1}_{v_i}, y = \mathbf{1}_{v_j}$, and $z = \mathbf{1}_{v_k}$. Here, we will consider transitivity of both $\succ^f$ and $\sim^f$.

Case 3a - Transitivity of $\succ^f$. Suppose $x \succ^f y$ and $y \succ^f z$. Since $x \succ^f y$, there is some valuation profile v'' where $\mathbf{1}_{v''_i} = x, \mathbf{1}_{v''_j} = y$, and $e^i \in W^f(v'')$ but $e^j \notin W^f(v'')$.

First, note that $i \neq 0$. To see this, since $y \succ^f z$ there is a valuation profile v' where $\mathbf{1}_{v'_j} = y, \mathbf{1}_{v'_k} = z$, and $e^j \in W^f(v')$ but $e^k \notin W^f(v')$. But $\mathbf{1}_{v'_i} = x$ implies that $y \succeq^f x$, which contradicts $x \succ^f y$. Hence, $i \neq 0$.

Suppose $v''_k < v_k$. Since $e^i \in W^f(v'')$, for every $\epsilon > 0$, $f(v''_i + \epsilon, v''_j, v''_k, v''_{-ijk}) = e^i$. By monotonicity and non-bossiness, $f(v''_i + \epsilon, v''_j, v_k, v''_{-ijk}) \in \{e^i, e^k\}$ for every $\epsilon > 0$. But $f(v''_i + \epsilon, v''_j, v_k, v''_{-ijk}) = e^k$ for any $\epsilon > 0$ will imply that $z \succeq^f y$, and this will contradict $y \succ^f z$. Hence, $f(v''_i + \epsilon, v''_j, v_k, v''_{-ijk}) = e^i$ for every $\epsilon > 0$. So, $e^i \in W^f(v''_i, v''_j, v_k, v''_{-ijk})$. Since $y \succ^f z$, $e^k \notin W^f(v''_i, v''_j, v_k, v''_{-ijk})$. Hence, $x \succ^f z$.

Suppose $v''_k \geq v_k$. As before, since $e^i \in W^f(v'')$, for every $\epsilon > 0$, $f(v''_i + \epsilon, v''_j, v''_k, v''_{-ijk}) = e^i$. By monotonicity and non-bossiness, $f(v''_i + \epsilon, v''_j, v_k, v''_{-ijk}) = e^i$ for every $\epsilon > 0$. Hence, $e^i \in W^f(v''_i, v''_j, v_k, v''_{-ijk})$. Since $y \succ^f z$, $e^k \notin W^f(v''_i, v''_j, v_k, v''_{-ijk})$. Hence, $x \succ^f z$.

Case 3b - Transitivity of $\sim^f$. Suppose $x \sim^f y$ and $y \sim^f z$. Suppose for every valuation profile v' such that $\mathbf{1}_{v'_i} = x$ and $\mathbf{1}_{v'_j} = y$, we have $e^i, e^j \notin W^f(v')$. Further, suppose for every valuation profile $\bar{v}$ with $\mathbf{1}_{\bar{v}_j} = y$ and $\mathbf{1}_{\bar{v}_k} = z$, we have $e^j, e^k \notin W^f(\bar{v})$. Consider any valuation profile v'' such that $\mathbf{1}_{v''_i} = x$ and $\mathbf{1}_{v''_k} = z$. Assume for contradiction $e^i \in W^f(v'')$. Consider the valuation profile $\hat{v}$ such that $\mathbf{1}_{\hat{v}_j} = y$ and $\hat{v}_l = v''_l$ for all $l \neq j$. Since $\mathbf{1}_{\hat{v}_k} = z$, by definition $e^j, e^k \notin W^f(\hat{v})$. By monotonicity and non-bossiness, $e^i \in W^f(\hat{v})$. But, this is not possible since $\mathbf{1}_{\hat{v}_i} = x$ implies that $e^i, e^j \notin W^f(\hat{v})$. This means that at every valuation profile v'' with $\mathbf{1}_{v''_i} = x$ and $\mathbf{1}_{v''_k} = z$ we must have $e^i, e^k \notin W^f(v'')$. Hence, $x \sim^f z$.

Now, consider the case where $y \sim^f z$ and there is some valuation profile v' such that $\mathbf{1}_{v'_j} = y$, $\mathbf{1}_{v'_k} = z$, and $e^j, e^k \in W^f(v')$. If $x = \mathbf{1}_{v_0}$, then by Proposition 1, $e^i \in W^f(v')$, and this immediately implies that $x \sim^f z$. Suppose $x = \mathbf{1}_{v_i}$ and $i \neq 0$. Then, either $j \neq 0$ or $k \neq 0$. We consider the case of $j \neq 0$ - the proof for $k \neq 0$ is similar. Since $e^j \in W^f(v')$, $f(v'_j + \epsilon, v'_{-j}) = e^j$ for all $\epsilon > 0$. By monotonicity and non-bossiness, $f(v'_j + \epsilon, v_i, v'_{-ij}) \in \{e^i, e^j\}$ for all $\epsilon > 0$. If $f(v'_j + \epsilon, v_i, v'_{-ij}) = e^i$, then by monotonicity and non-bossiness, $e^i \in W^f(v'_j, v_i, v'_{-ij})$. Since, $x \succ^f y$ and $y \succ^f z$, by repeated application of Proposition 1, we get that $e^j, e^k \in W^f(v'_j, v_i, v'_{-ij})$. This implies that $x \succ^f z$. Similarly, if $f(v'_j+\epsilon, v_i, v'_{-ij}) = e^j$, then $e^j \in W^f(v'_j, v_i, v'_{-ij})$. Again, using the fact that $x \succ^f y$ and $y \succ^f z$, by repeated application of Proposition 1, we get that $e^i, e^k \in W^f(v'_j, v_i, v'_{-ij})$. So, $x \succ^f z$. ■

STEP 1.4. We conclude Step 1 by showing that f is a rationalizable allocation rule and $\succeq^f$ rationalizes f. Note that the ordering $\succeq^f$, defined in Steps 1.2 and 1.3, is a monotone ordering. By Lemma 7, for every valuation profile v, $f(v) \in W^f(v)$. Hence, by definition of $\succeq^f$, $G^f(v) \succeq^f \mathbf{1}_{v_i}$ for all $i \in N \cup \{0\}$. This shows that f is a rationalizable allocation rule and $\succeq^f$ rationalizes f.

STEP 2. In this step, we show that if f is a non-bossy allocation rule satisfying Condition $\mathcal{C}^*$, then it is implementable if and only if it is an SUM allocation rule. By Lemma 3, an SUM allocation rule is implementable. Suppose f is an implementable and non-bossy allocation rule satisfying Condition $\mathcal{C}^*$. By Step 1, f can be rationalized by the monotone ordering $\succeq^f$, defined as in Step 1.2. We say that $\succeq^f$ has a **utility representation** if there exists a utility function $U : D \to \mathbb{R}$ such that for all $x, y \in D$ we have $U(x) > U(y)$ if and only if $x \succ^f y$.

STEP 2.1. In this step, we will show that $\succeq^f$ is separable in the sense of Definition 5. Let $Z := \{x \in D : x = \mathbf{1}_{v_i}$ for some $i \in N \cup \{0\}$ and v_i is rational$\}$. Note that since the set of rational numbers is countable, Z is a countable subset of D. Now, pick $x, y \in D$ such that $x \succ^f y$. If $x, y \in D_i$ for some $i \in N$, then let $x = \mathbf{1}_{v_i}$ and $y = \mathbf{1}_{v'_i}$. By definition, $v_i > v'_i$. Then, we can find a rational v''_i such that $v_i > v''_i > v'_i$ (this is because the set of rational numbers is a dense set). Let $z = \mathbf{1}_{v''_i}$. By definition, $z \in Z$ and $x \succ^f z \succ^f y$. Now, assume that $x = \mathbf{1}_{v_i}$ and $y = \mathbf{1}_{v_j}$ for some $i, j \in N \cup \{0\}$ with $i \neq j$. We consider various cases.

CASE A. Suppose $i \neq 0$ and $j \neq 0$. Since $x \succ^f y$, there is a valuation profile $v \equiv (v_i, v_j, v_{-ij})$ such that $e^i \in W^f(v)$ but $e^j \notin W^f(v)$. Since $e^j \notin W^f(v)$, there is some $\epsilon > 0$ such that $f(v_i, v_j + \epsilon, v_{-ij}) \neq e^j$. This means that $e^j \notin W^f(v_i, v_j + \frac{\epsilon}{2}, v_{-ij})$. Consider any $\delta > 0$. Since $f(v_i, v_j + \frac{\epsilon}{2}, v_{-ij}) \neq e^j$, by monotonicity and non-bossiness, $f(v_i + \delta, v_j + \frac{\epsilon}{2}, v_{-ij}) \neq e^j$. Since $e^i \in W^f(v)$, $f(v_i+\delta, v_j, v_{-ij}) = e^i$. By monotonicity and non-bossiness, $f(v_i+\delta, v_j+\frac{\epsilon}{2}, v_{-ij}) \in \{e^i, e^j\}$. This implies that $f(v_i + \delta, v_j + \frac{\epsilon}{2}, v_{-ij}) = e^i$. Hence, $e^i \in W^f(v_i, v_j + \frac{\epsilon}{2}, v_{-ij})$. Then, $x = \mathbf{1}_{v_i} \succ \mathbf{1}_{v_j+\frac{\epsilon}{2}} \succ \mathbf{1}_{v_j} = y$. Since the set of rational numbers is dense, we can find a $z \in Z$ arbitrarily close to $\mathbf{1}_{v_j+\frac{\epsilon}{2}}$ such that $x \succ^f z \succ^f y$.

CASE B. Suppose $i \neq 0$ and $j = 0$. Since $x \succ^f y$, there is a valuation profile (v_i, v_{-i}) such that $e^i \in W^f(v_i, v_{-i})$ but $e^0 \notin W^f(v_i, v_{-i})$. This means for some $\delta > 0$, we have $f(\{v_j - \delta\}_{j \in N}) \neq e^0$. Suppose $f(\{v_j - \delta\}_{j \in N}) = e^k$ for some $k \neq 0$. Then, $\mathbf{1}_{v_k - \delta} \succeq^f y$. Since $e^i \in W^f(v_i, v_{-i})$, we get that $x = \mathbf{1}_{v_i} \succeq^f \mathbf{1}_{v_k} \succ^f \mathbf{1}_{v_k-\delta}$. Hence, $x \succ^f \mathbf{1}_{v_k-\delta} \succeq^f y$. Since the set of rational numbers is dense, we can choose a $z \in Z$ arbitrarily close to $\mathbf{1}_{v_k-\delta}$ such that $x \succ^f z \succeq^f y$.

Case C. Suppose $i = 0$ and $j \neq 0$. Since $x \succ^f y$, there is a valuation profile (v_j, v_{-j}) such that $e^j \notin W^f(v_j, v_{-j})$ but $e^0 \in W^f(v_j, v_{-j})$. Then, for some $\epsilon > 0$, we have $f(v_j+\epsilon, v_{-j}) = e^k$, where $k \neq j$. This implies that $\mathbf{1}_{v_k} \succeq^f \mathbf{1}_{v_j+\epsilon} \succ^f \mathbf{1}_{v_j} = y$. But $e^0 \in W^f(v_j, v_{-j})$ implies that $x \succeq^f \mathbf{1}_{v_k}$. Hence, $x \succeq^f \mathbf{1}_{v_j+\epsilon} \succ^f y$. Since the set of rational numbers is dense, we can find $z \in Z$ arbitrarily close to $\mathbf{1}_{v_j+\epsilon}$ such that $x \succeq^f z \succ^f y$.

This shows that $\succeq^f$ is separable. Using Debreu (1954), $\succeq^f$ has a utility representation. Let $U : D \to \mathbb{R}$ be a utility function representing $\succeq^f$. Without loss of generality, we can assume $U(\mathbf{1}_{v_0}) = 0$. Now, for every $i \in N \cup \{0\}$, define $U_i : V_i \to \mathbb{R}$ as follows: $U_i(v_i) = U(\mathbf{1}_{v_i})$ for all $v_i \in V_i$. Note that by the definition of $\succeq^f$, each U_i is well-defined and increasing.

Since U represents $\succeq^f$ and f is a rationalizable allocation rule with $\succeq^f$ being the corresponding ordering, we get that for all valuation profiles v, $f(v) \in \arg\max_{i \in N \cup \{0\}} U_i(v_i)$. Hence, f is an SUM allocation rule.

By Theorem 1, f is a strongly rationalizable allocation rule. Let $\succ$ be the strict linear ordering that strongly rationalizes f. By definition, for all $x \in D^f$ and for all $y \notin D^f$, $x \succ y$. Further, for all $v \in V$ if $f(v) = e^j$, then $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_i}$ for all $i \neq j$. In that case, $\mathbf{1}_{v_j} \succ \mathbf{1}_{v_k}$ for all $k \neq j$ and $k \in \arg\max_{i \in N \cup \{0\}} U_i(v_i)$. Hence, f is an SUM allocation rule with order-based tie-breaking.

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