No-Envy, Strategyproofness and Cost-Monotonicity in Sequencing Problems

Parikshit De[a], Devwrat Dube[b]

[a]*Department of Economic Sciences, IISER Bhopal, 507, Therm, IISER Bhopal, Bhauri, Bhopal, 462066, M.P, India*
[b]*Department of Economic Sciences, IISER Bhopal, 105-B, Plasma, IISER Bhopal, Bhauri, Bhopal, 462066, M.P, India*

Abstract

We investigate the implications of *no-envy* as introduced by Foley [1] within the context of sequencing problems. We identify the complete set of sequencing rules that admit envy-free allocations. For these sequencing rules, we identify the class of *strategyproof* mechanisms. For the case of two agents, we give necessary and sufficient conditions for mechanisms to be strategyproof and satisfy no-envy. We show that envyfreeness amongst all adjacent pairs of agents in the sequence is sufficient to ensure *no-envy*, while it is necessary by definition of *no-envy*. We extend the result and identify *strategyproof envy-free* mechanisms for the n-agent case. Maniquet [3] introduced the notion of *Negative Cost Monotonicity* in the queuing problems. We extend the same notion to sequencing problems. We characterise the class of *strategyproof* mechanisms satisfying *no-envy* and *negative cost monotonicity* and prove its equivalence with the class of strategyproof mechanisms satisfying no envy and *independence of preceding cost*. By changing the tie-breaking rule, our results hold for the entire set of mechanisms. In this sense, the results hold for the *Pareto indifferent* correspondence pair of a set of sequencing rules and a corresponding set of transfer rules. Regarding queueing problems, Chun [2] discuss the importance of imposing Pareto indifference to characterise more than one mechanism. We make similar observations for sequencing problems. Yengin & Chun [11] relax the *budget balancedness* condition and characterise the sub-classes of VCG Mechanisms satisfying either *negative* or *positive cost monotonicity* in queueing problems. Our characterisation result adds to this literature. Further, we show that *Strategyproof* mechanisms satisfying *no-envy* and *positive cost monotonicity* may exist only for a subclass of sequencing problems, the queueing problem. We establish the non-existence of such mechanisms for the general sequencing problem.

Keywords: Sequencing, Solidarity, Strategyproofness, No-envy, Pareto indifference
JEL: Classification: C72, D63, D71, D82

1. Introduction

A sequencing problem refers to those situations where multiple but finite agents need a service that may vary with agents. Agents must be served in a facility that can cater to one agent at a time, and all the agents incur some waiting costs while waiting to be served. Any interruption of service by the facility is not allowed. Agents' waiting costs per unit of time are private information. Monetary transfers are allowed to compensate the agents waiting in the sequence (queue). Agents' utility structure follows a quasi-linear framework. When all the agents ask for the same service, the context is known as the queueing problem.
The definition of *allocation* plays a critical role while one introduces *no-envy* in the sequencing problem;

Email addresses: `ParikshitDe@iiserb.ac.in` (Parikshit De), `Devwrat19@iiserb.ac.in` (Devwrat Dube)

however, that is exactly not the case under queueing problems. Since the job processing times of agents in queueing problems are the same for all the agents, the job completion time depends only on the agent's position in the queue. So identifying a bundle or *allocation* either by (queue position, transfer) or by (job completion time, transfer) essentially has the same implication. However, the difference is obvious in the case of the sequencing problem. We identify an *allocation* with the following: (job completion time, transfer). The *no-envy* condition restricts the sequencing rule: assigning completion times to agents so that agents are served in non-increasing order of their reported unit waiting costs. We find that *no-envy* is incompatible with the *efficient allocation rule*, which requires that agents be served in non-increasing order of the ratio of their unit waiting costs to their service times (Smith [23]). Further, the difference in monetary transfers between any two agents is also bounded above and below. We show that envyfreeness amongst all adjacent pairs of agents in the sequence is sufficient to ensure *no-envy*, while it is necessary by definition of *no-envy*. The transfer rule typically depends on the overall sequence. Along with the fairness criterion, we study the strategic aspect of implementing envy-free allocations in dominant strategies. For the case of two agents, we completely characterise strategyproof envy-free mechanisms in Sequencing problems. We then extend this characterisation to the general case. After investigating the fairness and strategic aspects, we use axioms of solidarity to search for strategyproof and envy-free mechanisms satisfying either positive or negative cost monotonicity. We establish a non-existence result for the sequencing problem for positive cost monotonicity but characterise strategyproof envy-free mechanisms satisfying negative cost monotonicity. We show that this class can also be characterised as that of strategyproof envy-free mechanisms satisfying independence of preceding costs.

The paper is organized as follows. In Section 2, we provide the framework of sequencing problems and introduce a few necessary definitions. Section 2 ends by discussing the axiom of *Pareto indifference* and motivates the need for *Essentially Single Valuedness (ESV)* as an additional axiom that needs to be satisfied by envyfree mechanisms if they are to be *Pareto Indifferent*. In Section 3, we investigate the notion of No-Envy (*NE* for short), which derives from the idea of No-Envy introduced by Foley [1] applying it to our definition of allocation. We find the *NE* condition (Condition 6) that any envy-free mechanism must satisfy in this setting. We introduce two weaker notions of fairness :(*No Domination across agents* and *Equal treatment of Equals*), which are always satisfied by a mechanism that satisfies no envy. Subsection 3.1 discusses the consequence of the no-envy condition on sequencing rules. In subsection 3.2, we show that no-envy amongst all the adjacent agents in a sequence is *necessary and sufficient* to establish no-envy throughout the sequence. Subsection 3.3 discusses the consequence of *NE condition* on transfer rules. Implementability in dominant strategies (strategyproofness) is our main strategic axiom, without which the private information setting of sequencing problems renders the outcome manipulable. Section 4 identifies the mechanisms (ICJ*) that *Implement* the allocations satisfying no-envy. The strategyproof and no-envy condition (condition 14) is derived in subsection 4.1. We identify *Pareto Indifferent* mechanisms, recommending all allocations under which the utility of each agent remains the same. In subsection 4.2, We present a complete characterisation of strategyproof mechanisms satisfying no-envy for the two agent cases and motivate the idea behind the characterisation. In subsection 4.3, we extend the characterisation to the general ($|N|$-agent) case. In section 5, we discuss axioms of *Solidarity* and *Independence*. We show that the presence of strategyproofness and no-envy, *negative cost monotonicity* holds if and only if *independence of preceding costs* holds. We characterise these mechanisms. We obtain an impossibility result in establishing strategyproofness, no-envy, and *positive cost monotonicity* together. The only subclass of problems where a possibility exists is the queueing problems, where Chun & Yengin [26] have already identified the required subclass. Section 6 concludes.

1.1. Related Literature

The incentive perspectives of sequencing or queueing problems have been extensively studied so far (Suijs [5], Mitra [6], [7], De & Mitra [4]) and a variety of normative perspectives has been discussed and analyzed in queueing problems (Chun [2], Kayi & Ramaekers [8], Chun et al. [9], Chun & Yengin [10], Yengin & Chun [11]). However, sequencing problems have so far received only limited attention from normative points of view (see De [12], Banerjee, De & Mitra [13]).

In this paper, we have tried to address this gap. Envyfreeness is a well-recognized fairness norm and has been studied extensively for the general class of allocation problems. The notion of *no-envy* as introduced by Foley [1] requires that no agent should end up with a higher utility (than their own) by consuming the *bundle* allocated to any other agent. Chun [2] and Yengin & Chun [11] extensively studied the implication of *no-envy* in queueing problem, and they came up with a positive result in this context.

2. The framework

Let $I \equiv \{1, 2, \ldots\}$ be an (infinite) universe of "potential" agents, and $\mathcal{N}$ be the family of non-empty finite subsets of I. The agents $i \in I$ are characterised by a constant but agent-specific (per-unit-time) *waiting cost* $\theta_i \in \Theta := \mathbb{R}_+$[1] which measures their value of time, and a job processing time $s_i \in \mathbb{R}_{++}$[2]. Given $N \in \mathcal{N} : |N| = n$ where[3] n is finite, each of the agents indexed by $i, j, \ldots \in N = (1, 2, \ldots, n)$ requires a job to be processed by the serving *facility*. There is a single *facility* serving all the agents. Only one job can be processed at a time, and a job, once started, cannot be interrupted until finished. The waiting cost θ_i is private information of agent-i. A *profile* is the list $\theta = (\theta_i, \theta_{-i})$[4] $= (\theta_1, \theta_2, \ldots, \theta_n) \in \Theta^{|N|} \subseteq \mathbb{R}_+{}^{|N|}$. The *vector of processing times* is $s = (s_1, \ldots, s_n) \in \mathcal{S} := \mathbb{R}^{|N|}_{++}$. The *vector of processing times* s is public information. We denote the class of sequencing problems by Ω. The class of sequencing problems with a given s is denoted by Ω^s. A sequencing problem with the given set of agents N (implies a given *profile*) and the vector of processing times s is denoted by Ω^s_N[5]. Some (or all) agents may report equal unit waiting costs, and tie-breaking may be necessary. A *tie-breaking rule* is any exogenous preference ordering over the agents. Let $TB = (TB_1, \ldots, TB_v, \ldots)$ be the set of tie-breaking rules. We denote by Σ the set of all possible orders over N, thus $|\Sigma| = |N|!$.

A *sequencing rule* $\sigma(\theta, s, TB(J*)) \in \boldsymbol{\sigma}(\theta, s)$ is a function[6] $\sigma(\theta, s, TB(J*)) : \Theta^{|N|} \times \mathcal{S} \to \Sigma$ that specifies for each reported profile $\theta \in \Theta^{|N|}$ and for each vector of processing times $s \in \mathcal{S}$ (and subject to a fixed tie-breaking rule $TB(J*)$), a unique sequence $\sigma(\theta, s, TB(J*)) = (\sigma_1(\theta, s, TB(J*)), \ldots, \sigma_n(\theta, s, TB(J*))) \in \Sigma$. Agent-$i$'s position in the sequence $\sigma(\theta, s, TB(J*))$ is written as $\sigma_i(\theta, s, TB(J*))$. The sequence position is an indivisible good; therefore, $\sigma_j(\theta, s, TB(J*)) \neq \sigma_i(\theta, s, TB(J*)), \forall j \neq i \in N$.[7] The set $P_i(\sigma(\theta, s, TB(J*))) :=$

[1] $\mathbb{R}_+$ denotes the non-negative orthant of the real line.

[2] $\mathbb{R}_{++}$ denotes the positive orthant of the real line.

[3] For any set A, $|A|$ denotes the cardinality of A.

[4] θ_{-i} is the reduced profile obtained from profile θ by deleting the entry θ_i. Hence $\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$. We use the notation $(\hat{\theta}_i, \theta_{-i})$ to denote the profile where θ_i is replaced by $\hat{\theta}_i$ in the profile θ and all other entries remain unchanged.

[5] Hence $\Omega^s_N \subset \Omega^s \subset \Omega$.

[6] We are using a general notation. The sequencing rule is a function and not a correspondence. Hence, tie-braking may be necessary. Tie-breaking is not required for profiles with untied reports. Some sequences only consider the profile θ and do not depend on s (See Just Sequence in De & Mitra [4]). We invoke set-theoretic solutions in this paper and hence need correspondence. Therefore, we do not omit tie-breaking rules in notation. We denote a sequencing rule as $\sigma(\theta, s, TB(J*))$ and sequencing correspondence as $\boldsymbol{\sigma}(\theta, s)$ to denote the set of all sequencing rules that differ only in the tie-breaking rules. Specifically, $\boldsymbol{\sigma}(\theta, s) = \{\sigma(\theta, s, TB(J*)) | TB(J*) \in TB\}$.

[7] This restriction does not remain if there are multiple servers. See, for instance, Mitra & Mutuswami [24].

$\{j \in N | \sigma_j(\theta, s, TB(J*)) < \sigma_i(\theta, s, TB(J*))\}$ is the set of agents served before (predecessors of) agent-i in the sequence $\sigma(\theta, s, TB(J*))$, and the set $F_i(\sigma(\theta, s, TB(J*))) := \{j \in N | \sigma_j(\theta, s, TB(J*)) > \sigma_i(\theta, s, TB(J*))\}$ is the set of agents served after (followers of) agent-i in the sequence $\sigma(\theta, s, TB(J*))$. A *transfer rule* $\tau(\theta, s, TB(J*)) \in \boldsymbol{\tau}(\theta, s)$ is a function $\tau : \Theta^{|N|} \times \mathcal{S} \rightarrow \mathbb{R}^{|N|}$ that specifies for each reported profile $\theta \in \Theta^{|N|}$ and vector of processing times $s \in \mathcal{S}$ (and subject to a fixed tie-breaking rule $TB(J*)$), a transfer vector $\tau(\theta, s, TB(J*)) = (\tau_1(\theta, s, TB(J*)), \ldots, \tau_n(\theta, s, TB(J*))) \in \mathbb{R}^{|N|}$. A positive transfer to an agent means the agent gets that amount, while a negative transfer means the agent pays the amount. For a given sequencing problem Ω_N^s with reported profile θ, a (direct revelation) mechanism is the pair $\mu(\theta, s, TB(J*)) = (\sigma(\theta, s, TB(J*)), \tau(\theta, s, TB(J*)))$ comprising of a *sequencing rule* $\sigma(\theta, s, TB(J*))$ and a *transfer rule* $\tau(\theta, s, TB(J*))$. We denote by $\boldsymbol{\mu}(\boldsymbol{\sigma}(\theta, s), \boldsymbol{\tau}(\theta, s))$, the set of all mechanisms such that $\mu(\theta, s, TB(J*)) = (\sigma(\theta, s, TB(J*)), \tau(\theta, s, TB(J*))) \in \boldsymbol{\mu}(\boldsymbol{\sigma}(\theta, s), \boldsymbol{\tau}(\theta, s))$, $\forall TB(J*) \in TB$.
An *agent's bundle* comprises of two things. First, the total time (hereafter *completion time*) the agent spends in the facility, including the job processing time of their predecessors in the sequence *plus* their own job processing time and second, the net monetary transfer to them according to the transfer rule in the mechanism.

Definition 1. ***Agent's bundle:*** *For a given sequencing problem Ω_N^s, a bundle $z_i(\theta, s, TB(J*))$ allocated to any agent-$i \in N$ is defined as:*

$$z_i(\theta, s, TB(J*)) = \Big(S_i(\sigma(\theta, s, TB(J*))), \tau_i(\theta, s, TB(J*))\Big) \in \mathbb{R}_{++} \times \mathbb{R}$$

where $S_i(\sigma(\theta, s, TB(J*))) = \sum_{k \in P_i(\sigma(\theta, s, TB(J*))) \cup \{i\}} s_k = \sum_{k \in P_i(\sigma(\theta, s, TB(J*)))} s_k + s_i$ is the job completion time over which disutility accrues to agent-i.

An *allocation* is the vector of bundles allocated to the agents. Definition 2 formalises the notion of allocation in sequencing problems literature (see Mitra [7], De [12], De & Mitra [4] and Bannerjee, De & Mitra [13]).

Definition 2. ***Allocation:*** *An allocation is the vector $z(\theta, s, TB(J*)) = (z_i(\theta, s, TB(J*)))_{i \in N}$.*

We assume that the agents have a quasi-linear utility of the form:

$$U_i(z_i(\theta, s, TB(J*))) = U_i(S_i(\sigma(\theta, s, TB(J*))), \tau_i(\theta, s, TB(J*))) = -\theta_i \cdot S_i(\sigma(\theta, s, TB(J*))) + \tau_i(\theta, s, TB(J*))$$

Specifically, given any mechanism, if (θ_i', θ_{-i}) is the reported profile when the true waiting cost of agent-i is θ_i and the vector of processing times is s, then utility of agent-i is:

$$U_i(S_i(\sigma((\theta_i', \theta_{-i}), s, TB(J*))), \tau_i((\theta_i', \theta_{-i}), s, TB(J*)); \theta_i) = -\theta_i S_i(\sigma((\theta_i', \theta_{-i}), s, TB(J*))) + \tau_i((\theta_i', \theta_{-i}), s, TB(J*)).$$

Gevers [31] introduced an axiom, which he called *non-discrimination between Pareto indifferent allocations*. *Pareto Indifference*, as it has come to be known, is one of the desirable properties of mechanisms that are not single-valued. It demands that if the mechanism recommends an allocation and another allocation exists such that each agent's utility under both allocations is the same, the mechanism does not distinguish between allocations. Recently, Svensson *et al* [28] have employed this notion in characterising social choice correspondences. Welfarism should be based on individuals' welfare. This notion is already employed in d'Aspremont & Gevers [29], and Hammond [30]. Chakravorty *et al.* [32] employ the same notion.

Definition 3. ***Pareto Indifference:*** *For a given sequencing problem Ω_N^s, an allocation rule satisfies Pareto indifference if, whenever $z(\theta, s) = \mu(\sigma(\theta, s, TB(J*)), \tau(\theta, s, TB(J*)))$ and $z'(\theta, s) = \mu(\sigma(\theta, s, TB_t), \tau(\theta, s, TB_t))$ are such that $U_i(z_i(\theta, s)) = U_i(z_i'(\theta, s)), \forall i \in N$, then $z \in M(\sigma, \tau)$, where $M(\sigma, \tau)$ is the set of allocations recommended by the mechanism.*

A set of mechanisms $\boldsymbol{\mu}(\boldsymbol{\sigma}(\theta, s), \boldsymbol{\tau}(\theta, s))$ is called *Essentially Single Valued (ESV)* if for every sequencing problem $\Omega_N^s \in \Omega$, the utility of each agent is the same under the allocations recommended by each mechanism in the set. Under the sequencing setup, this implies that when there are ties, the utility obtained by every agent is the same, irrespective of the tie-breaking rule used. In a general model of indivisible good allocation, Sonmez [34] establishes the ESV of a non-empty core. In contrast, Takamiya [33] uses the ESV property of the core over conditionally rich preference domains to establish that all core allocations are strategyproof. Ehlers [35] consider general allocation problems with indivisibilities where agents' preferences possibly exhibit externalities. They also establish that when the core is non-empty, allocation rules satisfying Individual rationality, Strategy-proofness, and efficiency must select an allocation from the core and that it is ESV. Bogomolnaia *et al.* [36] have used ESV, Pareto indifference, strategyproofness and varying degrees of non-bossiness for characterisation results. Fleurbaey & Maniquet [37], Pérez-Castrillo & Sun [38], and Tadenuma & Thomson [39] are some other examples of works employing the idea of ESV.

Definition 4. ***Essential Single Valuedness (ESV):*** *For any sequencing problem $\Omega_N^s \in \Omega$, an allocation rule satisfies Essential Single-valuedness only if $U_i(z_i(\theta, s)) = U_i(z_i'(\theta, s)), \forall i \in N, s \in \mathcal{S}, \forall z(\theta, s), z'(\theta, s) \in M(\sigma, \tau)$ and for all $\theta \in \Theta^{|N|}$.*

In the queueing and sequencing literature, a mechanism is a pair of two functions, not correspondences. The mechanisms studied so far in Chun [2], Maniquet [3], De & Mitra, [4], and others are all ESV, and hence the choice of the tie-breaking rule is of no significance with regard to any agent's welfare. Thus, any arbitrary choice of tie-breaking rule leads to all agents getting the same utility as they would under any other tie-breaking rule. However, this does not take into consideration other normative axioms of interest. For example, if each agent can be made to have equal utilities across allocations, but the budget required for these allocations differs, then the menu of mechanisms available to the designer ought to be explored. As shown in Appendix B, some mechanisms resort to the ESV idea to reduce the set of mechanisms to a mechanism employing an arbitrary tie-breaking rule; such mechanisms do not recommend all allocations satisfying ESV. In this sense, they are not *Pareto indifferent* whereas our notational approach handles such a problem. Moreover, although the utilities obtained by agents under *Pareto indifferent* allocations may be the same, sometimes, the mechanism designer may be interested in the variety of possible allocations and perhaps at other times desire to compare allocations only based on monetary transfers or waiting costs incurred by agents. Such comparisons are facilitated when the set of mechanisms is also considered. Appendix B shows such a comparison.

The framework of sequencing problems is one involving private information. The utility of the agents depends on their self-reports. This makes *Dominant strategy Incentive Compatibility (DSIC) implementability* one of the most important strategic considerations of the mechanism.

Definition 5. ***DSIC Implementability:*** *A mechanism (σ, τ) implements the sequencing rule $\sigma(\theta, s, TB(J*))$ in dominant strategies if $\forall i \in N$, any $\theta_i, \theta_i' \in \Theta$,[8] any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$ and a fixed tie-breaking rule $TB(J*)$, the transfer rule $\tau : \Theta^{|N|} \times \mathcal{S} \rightarrow \mathbb{R}^{|N|}$ is such that:*

$$U_i(S_i(\sigma((\theta_i, \theta_{-i}), s, TB(J*))), \tau_i((\theta_i, \theta_{-i}), s, TB(J*)); \theta_i) \geq U_i(S_i(\sigma((\theta_i', \theta_{-i}), s, TB(J*))), \tau_i((\theta_i', \theta_{-i}), s, TB(J*); \theta_i). \quad (1)$$

De [27] define a subclass of sequencing rules called *Non-Increasing* sequencing rules.

[8] θ_{-i} is an ordered tuple of $|N \setminus \{i\}|$ individual waiting costs, whereas θ is an ordered tuple of $|N|$ entries.

Definition 6. ***NI Sequencing rules:*** *A sequencing rule σ is non-increasing (or NI) if for any $i \in N$ and any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, the chosen order $\sigma(\theta_i, \theta_{-i}, s, TB(J*))$ for each $\theta_i \in \Theta$ is such that the completion time $S_i(\sigma(\theta_i, \theta_{-i}), s, TB(J*)) = s_i + \sum_{k \in P_i(\sigma(\theta_i, \theta_{-i}, s, TB(J*)))} s_k$ is non-increasing in θ_i.*

De [27] show that a sequencing rule is implementable if and only if it is an NI sequencing rule. We denote the set of all NI sequencing rules by $\boldsymbol{\sigma}^{NI}$ and $\boldsymbol{\sigma}^{NI}(\theta, s)$ denotes the set of all NI Sequencing rules for a given sequencing problem. Further, NI sequencing rules are implementable only if the mechanism is *cut-off* based mechanism (see De [27]).

Definition 7. ***Cut off based Mechanism:*** *Consider any $\sigma \in \boldsymbol{\sigma}^{NI}(\theta, s)$ and a mechanism (σ, τ) with transfer rule $\tau : \Theta^{|N|} \to \mathbb{R}^{|N|}$. The mechanism is "Cut-off" based if the transfer rule τ is obtained from the following procedure. For each $i \in N$, we first select any function $h_i(\theta_{-i}) : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$ and then, given any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, we consider the waiting cost cut off vector[9] $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$ where $0 := \theta_i^{(0)} < \theta_i^{(1)}(\theta_{-i}) < \ldots < \theta_i^{(T-1)}(\theta_{-i}) < \theta_i^{(T)} =: \infty$ such that for any $t \in \{1, 2, \ldots, T-1\}$, $S_i(\sigma(\theta_i^t, \theta_{-i})) := \bar{S}(t, \theta_{-i})$ for all $\theta_i^t \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i}))$. Define $D_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - \bar{S}(t, \theta_{-i})$ and $\bar{D}_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - S_i(\sigma((\theta_i^{(t)}, \theta_{-i}), s, TB(J*))$ for any $t \in \{1, 2, \ldots, T-1\}$. Given the selected function $h_i(\theta_{-i}) : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$, for any profile θ_{-i} of all but agent-i and the associated cut off vector $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$, the transfer of agent-i is the following:*
(PI1) For any $\theta_i \in \Theta \setminus \{\theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i})\}$, $\tau_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i, \theta_{-i})$ where

$$I_i(\theta_i, \theta_{-i}) = \begin{cases} 0 & \text{if } \theta_i \in (\theta_i^{(T)}, \theta_i^{T-1}(\theta_{-i})), \\ \sum_{r=t}^{T-1} \theta_i^{(r)}(\theta_{-i}) D_r(\theta_{-i}) & \text{if } \theta_i \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i})), t = \{1, 2, \ldots, T-1\} \end{cases} \tag{2}$$

(PI2) For $T \geq 2$, any $t \in \{1, 2, \ldots, T-1\}$ and cut off point $\theta_i^{(t)}(\theta_{-i})$, $\tau_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i})$ where the incentive payment $I_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i}) = I_i(\theta_i^t, \theta_{-i}) - \theta_i^{(t)}(\theta_{-i})\bar{D}_t(\theta_{-i}) + \theta_i^{(t)}(\theta_{-i})D_t(\theta_{-i})$ and $\theta_i^t \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i}))$.

De & Mitra [4] introduce Rawlsian sequencing and the Just sequencing rule. Rawlsian rule does not guarantee state contingent unique order selection (See Example 1 in De & Mitra [4]).

Definition 8. *A sequencing rule σ^R is Rawlsian if for each $\theta \in \Theta^n$, and $s \in \mathcal{S}$*

$$\sigma^R \in \min_{\sigma \in \Sigma} \max_{j \in N} S_j(\sigma(\theta, s, TB(J*))) \cdot \theta_j$$

Definition 9. *A sequencing rule σ^J is called the Just sequencing rule if for any vector of processing times $s \in \mathcal{S}$, and any profile $\theta \in \Theta^n$ the chosen order $\sigma^J(\theta, s, TB(J))$ satisfies the following property: for any $i, j \in N$ $\theta_i > \theta_j \implies \sigma_i^J(s, \theta) < \sigma_j^J(s, \theta)$.*
We use the following tie-breaking rule for the just sequencing rule σ^J.
TB(J): *There is a linear order $\succ_r$ on N such that if $\theta_i = \theta_j$ and either $\frac{\theta_i}{s_i} > \frac{\theta_j}{s_j}$ or $\frac{\theta_i}{s_i} = \frac{\theta_j}{s_j}$ and $i \succ_r j$ then $\sigma_i^J(s, \theta) < \sigma_j^J(s, \theta)$.*

Just sequencing is always Rawlsian. The Just sequencing rule is implementable if and only if the mechanism (σ^J, τ) that implements it is an ICJ Mechanism introduced in De & Mitra [4].

[9] T denotes the total number of different job completion time that agent-i faces depending upon the sequencing rule under consideration and given θ_{-i}.

Definition 10. *For the just sequencing rule σ^J, a mechanism (σ^J, τ) is an Incentive Compatible Just mechanism (ICJ mechanism) if the transfer rule is such that for all $\theta \in \Theta^n$, $s \in \mathcal{S}$, and all $i \in N$,*

$$\tau_i(\theta, s) = h_i(\theta_{-i}) - \sum_{k \in F_i(\sigma^J(\theta, s, TB(J)))} s_k \theta_k \tag{3}$$

where the function $h_i(\theta_{-i}) : \Theta^{|N/\{i\}|} \to \mathbb{R}$ is arbitrary. Also, $\sum_{k \in A} s_k \theta_k = 0$ whenever $A = \phi$.

We define *Just** Sequencing rules.

Definition 11. ***Just* Sequencing rule:*** *A sequencing rule $\sigma^{J*} : \Theta^{|N|} \to \Sigma$ is called the Just* sequencing rule if for any profile $\theta \in \Theta^{|N|}$ the chosen order $\sigma^{J*}(s, \theta)$ satisfies the following property: for any $i, j \in N$ $\theta_i > \theta_j \implies \sigma_i^{J*}(s, \theta) < \sigma_j^{J*}(s, \theta)$.*
We use the following tie-breaking rule for the just sequencing rule σ^{J*}.*
TB(J*): *There is a linear order $>_r$ on N such that if $\theta_i = \theta_j$ and $i >_r j$ then $\sigma_i^{J*}(s, \theta) < \sigma_j^{J*}(s, \theta)$.*

The *Just** sequencing rule serves the agents in non-increasing order of their reported unit waiting costs; like the *Just* sequencing rule, they only differ in the tie-breaking rules employed. The set of all such sequencing rules serving agents in non-increasing order of their waiting costs is $:= \boldsymbol{\sigma}^{J*} \subset \boldsymbol{\sigma}^{NI} \subset \Sigma$. It is trivial to verify that $\sigma^{J*}, \sigma^J \in \boldsymbol{\sigma}^{J*} \subset \boldsymbol{\sigma}^{NI}$. Since $\boldsymbol{\sigma}^{J*} \subset \boldsymbol{\sigma}^{NI}$, therefore following De [27], $\boldsymbol{\sigma}^{J*}$ can be implemented only if the transfer rules are as per *Cut off based Mechanisms* in definition 7. A cut off vector for our discussion relating to *ICJ* mechanism* is $0 := \theta_i^{(0)} < \theta_i^{(1)}(\theta_{-i}) < \ldots < \theta_i^{(T-1)}(\theta_{-i}) < \theta_i^{(T)} =: \infty$, where $\theta_i^{(t)}(\theta_{-i}) \in \theta_{-i}$, meaning that there is an agent (other than agent-i) who reports waiting cost $\theta_i^{(t)}(\theta_{-i})$. Whether or not $D_t(\theta_{-i})$ equals $\bar{D}_t(\theta_{-i})$ depends on the second term in their definition and hence the tie-breaking rule employed[10]. If there are no ties, the two always coincide. Note: for $\theta_i \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i})), t = \{1, 2, \ldots, T-1\}$, no ties occur because no other agent reports waiting cost in the open interval $(\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i}))$. The ICJ Mechanism implements the *Just* Sequencing rule (see De & Mitra [4]), and the ICJ* mechanism implements the *Just** sequencing rule (see Lemma 1, and Appendix A for proof). Both results can be derived from the cut-off-based mechanisms.

Definition 12. *For the sequencing rule $\sigma^{J*} \in \boldsymbol{\sigma}^{J*}$, a mechanism (σ^{J*}, τ^{J*}) is an Incentive Compatible Just* mechanism (ICJ* mechanism) if the transfer rule is such that for all $\theta \in \Theta^n$, $s \in \mathcal{S}$, any $TB(J*) \in TB$ and every $i \in N$,*

$$\tau_i^{J*}(\theta, s, TB(J*)) = h_i(\theta_{-i}, s) - \sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))} s_k \theta_k \tag{4}$$

where the function $h_i(\theta_{-i}, s) : \Theta^{|N/\{i\}|} \times \mathbb{R}_{++}^{|N|} \to \mathbb{R}$ is arbitrary. Also, $\sum_{k \in A} s_k \theta_k = 0$ whenever $A = \phi$.

Both the *ICJ mechanism* and the *ICJ* mechanism* coincide for profiles with no ties. However, if any agents are tied in their unit waiting costs, then the *Just* sequencing rule serves such tied agents in non-decreasing order of the agents' own job processing times. Amongst the set of queues serving agents in non-increasing order of their unit waiting costs, *ICJ mechanism* chooses the sequence minimising aggregate waiting costs of all agents. However, if agents are tied in their waiting costs as well as processing times, then *ICJ mechanism* is indifferent to the choice of tie-breaking rule employed since its recommendations are *Essentially Single Valued*. In contrast, *ICJ* mechanism* does not explicitly consider the job-processing time of the agents. If the agents are tied in their reported unit waiting costs, *ICJ* mechanism* is indifferent to the choice of

[10]Throughout this paper, we denote the tie-breaking rule by $TB(J*) \in TB$.

tie-breaking rule employed since its recommendations are *Essentially Single Valued*. If agents are tied in their waiting costs as well as processing times, then *ICJ* mechanism* coincides with the *ICJ mechanism* if the linear order used in TB(J) and TB(J*) is the same. The set of allocations obtainable under *ICJ mechanism* by varying the linear order used in tie-breaking rule TB(J) for agents tied on both parameters is a subset of the allocations which are obtainable under *ICJ* mechanism* by varying the tie-breaking rule TB(J*). The *superiority* of *ICJ mechanism* allocations over the other allocations in *ICJ* mechanism* is that *ICJ mechanism's* allocation is always aggregate cost minimising amongst all the obtainable allocations under *ICJ* mechanism*. Since *ICJ* mechanism* is *ESV*, *ICJ mechanism* does not satisfy *Pareto indifference*. Whether aggregate cost minimisation can be traded for *Pareto indifference* is another debate we don't go into. All the results obtained for *ICJ* mechanism* in this paper apply to *ICJ mechanism*. But this seems an interesting direction to further inquire into the significance of *Pareto indifference*. Is it only a normative axiom, or does richness in prescribed allocations have perhaps a practical significance outweighing the aggregate cost savings achievable by compromising *Pareto indifference*? This remains an open question.
In section 3, we define *No-Envy (NE)* as in Foley [1] and study its implications for Sequencing problems. We obtain the *NE* condition for sequencing problems and discuss how envyfreeness restricts the sequencing and transfer rules. We show that to ensure envyfreeness, it is necessary and sufficient to ensure envyfreeness amongst all $(|N| - 1)$ pairs of adjacent agents only.

3. No Envy

Our notion of fairness is *No-envy (NE, in short)*, introduced in Foley [1]. It is perhaps the central and well-studied fairness concept. *No-Envy* requires that each agent finds their own bundle at least as desirable as any other agent's. Hence, given the opportunity to choose among all the bundles comprising an allocation, an agent should never reject their assigned bundle in favour of any other agent's bundle. Agent-i does not envy another agent-j's allocation if the consumption of $z_j(\theta, s, TB(J*))$ does not give to agent-i a higher utility than what agent-i obtains by consuming the allocation $z_i(\theta, s, TB(J*))$. This requires $\forall i, j \in N, \forall \theta \in \Theta^{|N|}, \forall s \in \mathcal{S}$, $U_i(z_i(\theta, s, TB(J*))) \geq U_i(z_j(\theta, s, TB(J*)))$. From the definition of allocations in definition 1, we get definition 13.

Definition 13. ***No-Envy (NE):*** *A Mechanism* (σ, τ) *satisfies No-Envy if the mechanism is such that:*

$$\forall i, j \in N, \forall \theta \in \Theta^{|N|}, \forall s \in \mathcal{S}, U_i(S_i(\sigma(\theta, s, TB(J*))), \tau_i(\theta, s, TB(J*))) \geq U_i(S_j(\sigma(\theta, s, TB(J*))), \tau_j(\theta, s, TB(J*))) \quad (5)$$

Applying the NE definition 5 to any two arbitrary agents $i, j \in N$, we obtain the *No-Envy Condition* 6.

$$\theta_i \cdot (S_j(\theta, s, TB(J*)) - S_i(\theta, s, TB(J*))) \geq \tau_j(\theta, s, TB(J*)) - \tau_i(\theta, s, TB(J*)) \geq \theta_j \cdot (S_j(\theta, s, TB(J*)) - S_i(\theta, s, TB(J*))) \quad (6)$$

We now introduce two notions of fairness weaker than *no-envy*. *Equal treatment of Equals (ETE, in short)* is a desideratum for fairness and requires that two agents having identical preferences be assigned consumption bundles to which they are indifferent. However, unlike queueing problems (where all agents have equal job processing times), sequencing problems allow two different notions of *equality* of agents. Should two agents be considered equal under the stricter requirement that they have the same unit waiting costs *and* job processing times, or the weaker definition where agents with the same waiting costs are considered equal irrespective of processing times? The weaker definition leads to stricter demands on the mechanism. In the scenario where only waiting costs are private information, and because we define allocations as (completion time, transfer) and not as (queue position, transfer), agents evaluate other agents' allocation with the other agents' processing time and not their own. It seems more reasonable to consider the weaker definition, encompassing all the cases under the stricter definition. After defining *Equality of agents*, we define *Equal treatment of Equals (ETE, in short)*. The definition is consistent with the same notion in queueing problems (see Chun [17], [19], Chun et al. [25], [18], [21], [20]).

Definition 14. ***Equality of Agents:*** *Two agents $i, j \in N$ are said to be equal if their unit waiting costs are equal, i.e. $\theta_i = \theta_j$.*

Definition 15. ***Equal treatment of Equals (ETE):*** *A mechanism $\mu = (\sigma, \tau)$ satisfies equal treatment of equals if: $\forall i, j \in N, \forall \theta \in \Theta^{|N|}, \forall s \in \mathcal{S}$,*

$$\theta_i = \theta_j \implies U_i(S_i(\sigma(\theta, s, TB(J*))), \tau_i(\theta, s, TB(J*))) = U_j(S_j(\sigma(\theta, s, TB(J*))), \tau_j(\theta, s, TB(J*))) \quad (7)$$

Kayi & Ramaekars [8] and Mitra & Mutuswami [24] have already pointed out that in queueing problems, *no-envy* implies *equal treatment of equals*. In sequencing problems, using the weaker definition of equality of agents (definition 14), it can be shown *no-envy* still implies *equal treatment of equals* in this broader class[11]. Letting $\theta_i = \theta_j$ in condition 6, the difference of transfers has to match with the difference in waiting costs exactly, and therefore, in our framework, NE implies ETE.
An allocation satisfies *No Domination Across Agents (NDAA, in short)* if no agent receives at least as much of all goods as, and more of at least one good than, some other agent (See Thomson [40]). In sequencing problems, NDAA requires that no agent, in comparison with any other agent, should have a better allocation in terms of both offerings, completion time, and transfer.

Definition 16. ***No Domination Across Agents (NDAA):*** *A mechanism (σ, τ) satisfies no-domination across agents if:*

$$\forall \theta \in \Theta^{|N|}, \forall s \in \mathcal{S}, \forall i, j \in N,\ S_i(\sigma(\theta, s, TB(J*))) \geq S_j(\sigma(\theta, s, TB(J*))) \implies \tau_i(\theta, s, TB(J*)) \geq \tau_i(\theta, s, TB(J*)) \quad (8)$$

If any agent gets more monetary transfer than any of their followers, they are envied by such followers. If any agent gets less monetary transfer than any of their predecessors, they envy all such predecessors. Hence, in our sequencing framework, NDAA is a necessary but not sufficient condition for NE. In other words, NE implies NDAA.
In subsection 3.1, we discuss the implication of *No-envy* condition for the sequencing rule and identify the subclass of sequences that may be envy-free subject to appropriate transfers.

3.1. Envy-free sequencing rules

Given that $\forall i \in N, s_i \in \mathbb{R}_{++}$, the completion time(s) are increasing in sequence position. Hence $(S_j(\theta, s, TB(J*)) - S_i(\theta, s, TB(J*))) > 0$ when agent-i precedes agent-j in the sequence $\sigma(\theta, s, TB(J*))$. The *NE* Condition 6 cannot hold if $\theta_i < \theta_j$ for any agent-i preceding agent-j. The *NE* condition restricts the sequencing discipline amongst agents with different unit waiting costs to be in decreasing order of their unit waiting costs. It is indifferent to how agents with the same unit waiting costs are served as long as they are served consecutively. Thus, the Just Sequencing rule (definition 9) is an envy-free sequencing rule, but it is not unique. By varying the linear order in the tie-breaking[12] rule of Just* sequencing rule (11), we obtain the entire

[11] Note that Ne implies ETE would hold for the stricter definition of equality of agents as well.

[12] The same variation in the tie-breaking rule for Just sequence only obtains the aggregate cost minimising sequences amongst the envyfree sequences.

class of envyfree sequencing rule. *No Envy* allows any possible tie-breaking rule. Thus, a mechanism can be envyfree *only if* the sequencing rule is such that the agents are served in non-increasing order of their unit waiting costs and some tie-breaking rule resolves any ties. If the sequencing rule is Just* sequencing rule, it depends on the transfer rule whether or not NE holds for the mechanism. We state without proof proposition 1.

Proposition 1. *A mechanism $\mu = (\sigma, \tau)$ is envyfree only if the sequencing rule $\sigma = \sigma^{J*} \in \boldsymbol{\sigma}^{J*}$.*

In queueing problems, *no-envy* implies queue efficiency (see Remark 2 in Yengin & Chun[11], see Svensson [22] for economies with indivisible goods, for queueing problems see Chun et al. [9]). However, in sequencing problems, non-increasing waiting cost sequencing differs from efficiency. Smith [23] shows that total waiting cost is minimized if the agents are served in non-increasing order with respect to their *urgency index: θ_i/s_i*. For $i, j \in N$, if $\theta_i/s_i = \theta_j/s_j$, agents-$i$ and j have equivalent urgency indices, and a tie-breaking rule is needed. If the processing times of all agents are not the same, then the *NE*-sequence is different from *efficient* sequence.

3.2. *No-Envy condition for adjacent agents*

Since the choice of agents in *NE* condition 6 is arbitrary, we may assume without loss of generality that agent-j is the immediate follower of agent-i in the sequence, i.e. $\sigma_i(\theta, s, TB(J*)) + 1 = \sigma_j(\theta, s, TB(J*))$ and therefore $S_j(\theta, s, TB(J*)) - S_i(\theta, s, TB(J*)) = s_j$. The *NE condition for adjacent agents i* and j such that $\sigma_i(\theta, s, TB(J*)) + 1 = \sigma_j(\theta, s, TB(J*))$ is:

$$\theta_i \cdot s_j \geq \tau_j(\theta, s, TB(J*)) - \tau_i(\theta, s, TB(J*)) \geq \theta_j \cdot s_j \tag{9}$$

Chun [2] have already pointed out that for queueing problems, it is sufficient to check the inequality between adjacent agents in the sequence. Proposition 2 generalises this observation to sequencing problems.

Proposition 2. *A mechanism $\mu = (\sigma, \tau)$ satisfies No Envy if and only if:*

$$\forall i \in N \text{ s.t. } \sigma_i(\theta, s, TB(J*)) + 1 = \sigma_j(\theta, s, TB(J*)), \quad \theta_i \cdot s_j \geq \tau_j(\theta, s, TB(J*)) - \tau_i(\theta, s, TB(J*)) \geq \theta_j \cdot s_j \tag{10}$$

Proof. We prove first that the adjacent no-envy condition 10 is sufficient for the mechanism to satisfy NE. Without loss of generality, we assume that agent-p precedes agent-q in the sequence. Let there be $m \geq 0$ agents between agent-p and agent-q in the adjacent-envy-free sequence: $(\ldots, i, i+1, \ldots, i+k, i+k+1, \ldots, i+m, j, \ldots)$. Proposition 1 dictates $\theta_i \geq \theta_k \geq \theta_j, \forall k \in N$ such that $\sigma_i(\theta, s, TB(J*))) \geq \sigma_k(\theta, s, TB(J*))) \geq \sigma_j(\theta, s, TB(J*)))$. By condition 6, we have:

$$\begin{aligned}
& \theta_i s_{i+1} \geq \tau_{i+1}(\theta, s, TB(J*)) - \tau_i(\theta, s, TB(J*))) \geq \theta_{i+1} s_{i+1} \\
& \vdots \\
\forall k \in \{1, \ldots, m\} \quad & \theta_{i+k} s_{i+k+1} \geq \tau_{i+k+1}(\theta, s, TB(J*))) - \tau_{i+k}(\theta, s, TB(J*))) \geq \theta_{i+k+1} s_{i+k+1} \\
& \vdots \\
& \theta_{i+m} s_j \geq \tau_j(\theta, s, TB(J*))) - \tau_{i+m}(\theta, s, TB(J*))) \geq \theta_j s_j
\end{aligned}$$

Summing up the inequalities, we get:

$$\sum_{k \in \{1,\ldots,m\}} \theta_{i+k-1} s_{i+k} \geq \tau_j(\theta, s, TB(J*)) - \tau_i(\theta, s, TB(J*)) \geq \sum_{k \in \{1,\ldots,m\}} \theta_{i+k} s_{i+k} \tag{11}$$

By Proposition 1, $\theta_i \geq \theta_{i+k}, \forall k\{1, \ldots, m\}$ and $\theta_{i+k} \geq \theta_j, \forall k \in \{1, \ldots, m\}$, and hence:

$$\theta_i \sum_{k\in\{1,\ldots,m\}} s_k \geq \sum_{k\in\{1,\ldots,m\}} \theta_k s_k \geq \tau_j(\theta, s, TB(J*)) - \tau_i(\theta, s, TB(J*)) \geq \sum_{k\in\{1,\ldots,m\}} \theta_k s_{k-1} \geq \theta_j \sum_{k\in\{1,\ldots,m\}} s_k \quad (12)$$

For agent-i and agent-j to not envy each other, only the following is needed:

$$\theta_i \sum_{k\in P_j(\sigma(\theta,s,TB(J*)))\setminus P_i(\sigma(\theta,s,TB(J*)))} s_k \geq \tau_j(\theta, s, TB(J*)) - \tau_i(\theta, s, TB(J*)) \geq \theta_j \sum_{k\in P_j(\sigma(\theta,s,TB(J*)))\setminus P_i(\sigma(\theta,s,TB(J*)))} s_k \quad (13)$$

Following proposition 1, the set of agents-i+k for all $k \in \{1, \ldots, m\}$ in condition 12 and the set $P_j(\sigma(\theta, s, TB(J*)))\setminus P_i(\sigma(\theta, s, TB(J*)))$, which is the set of all agents who precede agent-j in the sequence $\sigma(\theta, s, TB(J*))$ but do not precede agent-i (Agent-i belongs to this set) are the same. Hence, 12 $\implies$ 13. This completes the proof of the sufficiency of adjacent-envy-freeness.
The *necessity* part follows from the definition of *no envy* since even if one agent envies another (adjacent or not), *NE* does not hold by definition. □

3.3. *Envy-free transfer rules*

Following the proof of Proposition 2, for agent-i and agent-j to not envy each other, only 13 is needed, but it is a weaker condition than 12, which holds whenever 10 holds. Hence, given a Just* sequence σ^{J*}, the transfer rule(s) τ that allow the mechanism (σ^{J*}, τ) to satisfy no-envy are only those that satisfy condition 9.

4. Envy-free Strategyproofness

The Strategic notion we employ is DSIC Implementability (henceforth *Strategyproofness*). Due to the private information structure of sequencing problems, the desideratum of Strategyproofness is almost natural. We define the *ICJ* mechanism* in definition 12. The allocations recommended by the ICJ mechanism (see definition 10) developed in De & Mitra [4] and those recommended by the ICJ* mechanism are the same if there are no ties, or if all tied agents are tied in unit waiting costs as well as processing times and both mechanisms use the same linear order for tie-breaking. Otherwise, the recommendations are, in general, not the same.[13] The allocations recommended by the ICJ mechanism for any sequencing problem z_i^{ICJ} for every agent-$i \in N$ are a subset of the allocations recommended by the ICJ* mechanism z_i^{ICJ*}. Within the set $\boldsymbol{\sigma}^{J*}$, the just sequence is the most efficient. The trade-off is that the set of allocations obtained by varying the linear order used in tie-breaking does not form the set of all allocations that lead to each agent having the same welfare across all allocations. In this sense, even though the ICJ mechanism is ESV, it is not *Pareto indifferent*. ICJ* mechanism also recommends (in addition to recommending ICJ allocations) some other allocations which are not as efficient as the allocations recommended by the ICJ mechanism but still result in the same utility for all agents as they would receive under ICJ allocations. This implies that the sum of transfers required by the mechanism designer is the least in ICJ allocations, whereas ICJ* allocations allow for more expensive ways of implementing sequences that serve agents in non-increasing order of their unit waiting costs besides the ICJ allocation. We state our result with regard to implementing the Just* sequence(s). The ICJ Mechanism was introduced in De & Mitra [4], along with the result that Just

[13] They may be the same when TB(J) and TB(J*) coincidentally select the same sequence even when agents are not tied in both parameters.

sequencing is always Rawlsian. The just sequencing rule is implementable if and only if the mechanism (σ^J, τ) that implements it is an ICJ Mechanism (see De & Mitra [4]). The ICJ mechanisms are a subset of ICJ* mechanisms, and the two are identical when no two agents have the same waiting costs. They are also identical when all agents who have equal waiting costs also have equal processing times. In general, the two mechanisms differ when agents are tied in their waiting costs but not in processing times, except by coincidence. This leads us to state and prove lemma 1, which cannot be obtained from the results of De & Mitra [4].

Lemma 1. *A mechanism (σ^{J*}, τ) implements the sequencing rule $\sigma^{J*} \in \boldsymbol{\sigma}^{J*}$ in dominant strategies if and only if it is an Incentive Compatible Just* mechanism (ICJ* mechanism).*

Definition 17. ***Strategyproof (SP) Mechanisms**: A mechanism $\mu = (\sigma, \tau)$ is said to satisfy Strategyproofness or is Strategyproof (SP) if the transfer rule implements the sequencing rule in Dominant strategies.*

We defer the proof to Appendix A. The idea behind the proof is the following: Consider any sequencing problem. It is trivial to verify that $\boldsymbol{\sigma}^{J*} \subset \boldsymbol{\sigma}^{NI}$. Therefore, the set of mechanisms $(\boldsymbol{\sigma}^{J*}, \boldsymbol{\tau}) \subset (\boldsymbol{\sigma}^{NI}, \boldsymbol{\tau})$. Following De [27], $\boldsymbol{\sigma}^{NI}$ can be implemented only if the transfer rules are as per *Cut off based Mechanisms* in definition 7 (see De [27]). Using the definition of any $\sigma^{J*} \in \boldsymbol{\sigma}^{J*}$, the cut-offs are obtained to calculate the transfers. The cut-offs are the distinct waiting costs reported by all agents other than agent-i, whose transfer is being calculated. Based on this, the *cut-off based* transfer rule is calculated. We find that the rule obtained coincides with the *ICJ* transfer rule in De & Mitra [4] if the transfer rule is modified to include all possible followers of agent-i, including those with waiting cost equal to agent-i ($\theta_j = \theta_i$) but processing times more than agent-i's processing time ($s_j > s_i$) and may be followers of agent-i under some choice of tie-breaking rule in *Just** sequence σ^{J*} but are never followers of agent-i in a *just* sequence σ^J.

4.1. Strategyproof and envyfree mechanism

In this subsection, we study envy-free strategyproof mechanisms. We identify such mechanisms by imposing the *no-envy condition for adjacent agents* 10 on the transfer rule obtained from ICJ* mechanisms in equation 4. Without loss of generality, we label the agents based on the sequence position in a *Just** sequence, $\sigma^{J*}_k(\theta, s, TB(J*)) = k, \forall k \in N$. The *strategyproof no envy condition* is: $\theta_i s_{i+1} \geq h_{i+1}(\theta_{-(i+1)}) - \sum_{k \in F_{i+1}(\sigma^{J*}(\theta,s,TB(J*)))} s_k\theta_k - h_i(\theta_{-i}) - \sum_{k \in F_i(\sigma^{J*}(\theta,s,TB(J*)))} s_k\theta_k \geq \theta_i s_i \implies \theta_i s_{i+1} \geq h_{i+1}(\theta_{-(i+1)}) - h_i(\theta_{-i}) + \theta_{i+1}s_{i+1} \geq \theta_{i+1}s_{i+1}$.

$$(\theta_i - \theta_{i+1})s_{i+1} \geq h_{i+1}(\theta_{-(i+1)}) - h_i(\theta_{-i}) \geq 0. \tag{14}$$

We define another property called *anonymity*, which requires that the allocation is invariant with respect to the relabeling of agents, and based on that *anonymity* define a subclass of ICJ* mechanism- *Anonymous ICJ** mechanism. The *anonymity* property is analogous to the same property defined for queueing problems in Yengin & Chun [11].

Definition 18. *A mechanism $\mu = (\sigma, \tau)$ satisfies the anonymity property if for all $N \in \mathcal{N}, \theta \in \Theta^{|N|}, s \in \mathcal{S}$, every $TB(J*) \in TB$ and any bijection $\pi : N \to N$:*

$$U_i(\mu_i(\theta, s, TB(J*)); \theta_i, s_i) = U_{\pi(i)}(\mu_{\pi(i)}((\theta_{\pi(j)})_{j \in N}), s, TB(J*); \theta_{\pi(i)}, s_{\pi(i)}) \tag{15}$$

Definition 19. *A mechanism (σ^{J*}, τ^{J*}) is an Anonymous Incentive Compatible Just* mechanism (AICJ* mechanism) if the sequencing rule is the Just* sequencing rule $\sigma^{J*} \in \boldsymbol{\sigma}^{J*}$ and the transfer rule is such that for all $\theta \in \Theta^n$, $s \in \mathcal{S}$, any $TB(J*) \in TB$ and all $i \in N$,*

$$\tau_i^{J*}(\theta, s, TB(J*)) = h(\theta_{-i}, s) - \sum_{k \in F_i(\sigma^{J*}(\theta,s,TB(J*)))} s_k\theta_k \tag{16}$$

where the function $h(\theta_{-i}, s) : \Theta^{|N/\{i\}|} \times \mathbb{R}_{++}^{|N|} \rightarrow \mathbb{R}$ *is arbitrary. Also,* $\sum_{k\in A} s_k\theta_k = 0$ *whenever* $A = \phi$.

It is easily verified that the AICJ* mechanism satisfies *anonymity* property. In subsection 4.2, we present our result for the two-agent case and motivate the intuition behind the general case.

4.2. Two agent case

We present our main result for the two-agent case in theorem 1. The result completely characterises the strategyproof envy-free mechanisms for two-agent sequencing problems.

Theorem 1. *For any sequencing problem* Ω_N^s *where* $|N| = 2$*, let* $s_{min} = \min_{k\in N} s_k$*, a mechanism* (σ, τ) *is strategyproof and satisfies no-envy if and only if it is an AICJ* mechanism and the transfer rule is such that the function* $h(\theta_{-i}) : \Theta \rightarrow \mathbb{R}$ *satisfies* $s_{min} \geq \frac{\Delta h(\theta)}{\Delta\theta} \geq 0$*, for every possible* $\Delta\theta \in \mathbb{R}$.

Proof. Consider two agents 1 and 2, such that the waiting profile is $(\theta) = (\theta_1, \theta_2)$ where $\theta_1, \theta_2 \in \Theta$. The vector of processing times is (s_1, s_2). From Proposition 1, a mechanism is envyfree *only if* the sequencing rule is σ^{J*}. From lemma 1, a mechanism (σ^{J*}, τ) is strategyproof *only if* it is an ICJ* mechanism. A mechanism is strategyproof and satisfies no-envy *only if* it satisfies condition 6 and condition 4. Therefore, the transfer rule must be the ICJ* transfer rule subject to condition 14. Thus, $(\theta_i - \theta_{i+1})s_{i+1} \geq h_{i+1}(\theta_{-(i+1)}) - h_i(\theta_{-i}) \geq 0$, where $i = \arg\sigma_i(\theta, s, TB(J*)) = 1$ and $i + 1 = \arg\sigma_{i+1}(\theta, s, TB(J*)) = 2$. Now, we consider three cases.

Case 1(a): $\theta_1 = \theta_2$**. Tie-breaking rule serves agent-1 first**

In this case condition 14 demands: $(\theta_1 - \theta_2)s_2 \geq h_2(\theta_1) - h_1(\theta_2) \geq 0 \implies 0 \geq h_2(\theta_1) - h_1(\theta_1) \geq 0 \implies : h_2(\theta_1) = h_1(\theta_1) \ \ \forall\theta_1 \in \Theta$.

Case 1(b): $\theta_1 = \theta_2$**. Tie-breaking rule serves agent-2 first**

In this case condition 14 demands: $(\theta_1 - \theta_2)s_1 \geq h_2(\theta_1) - h_1(\theta_2) \geq 0 \implies 0 \geq h_2(\theta_1) - h_1(\theta_1) \geq 0 \implies : h_2(\theta_1) = h_1(\theta_1) \ \ \forall\theta_1 \in \Theta$. Therefore, for any mechanism to be strategyproof and satisfy no-envy in the two-agent case, it is *necessary* that the functions $h_1(\cdot)$ and $h_2(\cdot)$ are *anonymous* i.e. $h_1(x) = h_2(x)$ for all $x \in \Theta \subseteq \mathbb{R}_+$. We drop the agent-specific subscript in the following part of the proof.

Case 2: $\theta_1 > \theta_2$

In this case condition 14 demands: $(\theta_1 - \theta_2)s_2 \geq h(\theta_1) - h(\theta_2) \geq 0$. Since, $\theta_1 > \theta_2$, $h(\theta_1) - h(\theta_2) \geq 0$ demands that the $h(\cdot)$ function is *non-decreasing* in its argument i.e $\frac{\Delta h(\theta)}{\Delta\theta} \geq 0$. The condition $(\theta_1 - \theta_2)s_2 \geq h(\theta_1) - h(\theta_2)$ demands : $s_2 \geq \frac{\Delta h(\theta)}{\Delta\theta}$. Thus, it is *necessary* that: $s_2 \geq \frac{\Delta h(\theta)}{\Delta\theta} \geq 0$.

Case 3: $\theta_1 < \theta_2$

Repeating the calculations of Case 2, we obtain the *necessary* condition: $s_1 \geq \frac{\Delta h(\theta)}{\Delta\theta} \geq 0$.

For all the *necessary* conditions to hold, at all profiles $\theta \in \Theta^2$ and any vector of processing times $\{s_1, s_2\} \in \mathcal{S}$, it is *necessary* that: $s_{min} \geq \frac{\Delta h(\theta)}{\Delta\theta} \geq 0$, where $s_{min} = \min\{s_1, s_2\}$. This completes the *only if* part of the proof. For the *if* part, the ICJ* transfer rule satisfying $s_{min} \geq \frac{\Delta h(\theta)}{\Delta\theta} \geq 0$ is strategyproof from lemma 1. It is trivial to verify that it satisfies the *no-envy* condition. This completes the proof. □

4.3. General case

For any sequencing problem Ω^s with vector of processing times $s = (s_1, \ldots, s_{|N|})$, a mechanism $\mu = (\sigma^{J*}, \tau)$ is strategyproof and satisfies no envy only if the transfer rule $\tau(\theta, s, TB(J*))$is such that:

$$\tau_i(\theta, s, TB(J*)) = h(\theta_{-i}) - \sum_{k\in F_i(\sigma^{J*}(\theta,s,TB(J*)))} s_k\theta_k +, \quad \forall\theta \in \Theta^{|N|} \tag{17}$$

where the function $h(\theta_{-i}) : \Theta^{|N\setminus\{i\}|} \rightarrow \mathbb{R}$ is such that for any profiles $\theta' \geq \theta \in \Theta$ [14], $h(\theta_{-i}) + s_{min} \sum_{j\in N\setminus\{i\}}(\theta'_j - \theta_j) \geq h(\theta'_{-i}) \geq h(\theta_{-i})$, $\forall i \in N$, where $s_{min} = \min\{s_1, \ldots, s_n\}$. The result obtained here, when reduced to queueing problems, generates the result obtained in Chun & Yengin [26]. They obtain this result for envy-free, strategy-proof and monotonic mechanisms. We comment on the *cost-monotonicity* properties of mechanisms in the next section. Before we present our result, we need to adapt the definition of symmetric functions, as defined in Chun & Yengin [26].

Definition 20. *A function $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be symmetric if h(x)=h(y) whenever x and y are permutations of each other.*

Theorem 2. *For any sequencing problem Ω^s with the vector of processing times $s = (s_1, \ldots, s_{|N|})$, a strategyproof mechanism satisfies no envy if and only if 1,2 and 3 hold.*

1. *The mechanism is an AICJ* mechanism.*
2. *The function $h(\theta_{-i}) : \Theta^{|N\setminus\{i\}|} \rightarrow \mathbb{R}$ is such that for any profiles $\theta' \geq \theta \in \Theta$, $h(\theta_{-i}) + s_{min} \sum_{j\in N\setminus\{i\}}(\theta'_j - \theta_j) \geq h(\theta'_{-i}) \geq h(\theta_{-i})$, $\forall i \in N$, where $s_{min} = \min\{s_1, \ldots, s_n\}$.*
3. *$h(\theta_{-i}) : \Theta^{|N\setminus\{i\}|} \rightarrow \mathbb{R}$ is symmetric.*

Proof. The idea behind the proof is the same as the two-agent case. Since the mechanism satisfies no envy, it is necessary that the sequencing rule is any $\sigma^{J*} \in \boldsymbol{\sigma^{J*}}$ (Proposition 1). Since the mechanism is strategyproof, it is necessarily an ICJ* mechanism (Lemma 1). Due to Proposition 2, no envy among all pairs of adjacent agents in the sequence is necessary and sufficient. Consider a set of agents N with the vector of processing times $s \in \mathcal{S}$ and any profile $\theta = (\theta_1, \ldots, \theta_{|N|})$ such that $\theta_i = \theta_j$, $\forall i, j \in N$. Consider any agent-i. Using different tie-breaking rules will lead to different sequences $\sigma^{J*} \in \boldsymbol{\sigma^{J*}}$, and the vector θ_{-i} will get permuted from the change of sequences. Consider any two Just* sequencing rules that differ in tie-breaking and may allot different bundles to agent-i. Let θ_{-i} represent the profile of agents in $N \setminus \{i\}$ arranged as per their sequence positions in $J*(\theta, s, TB(J^{*1}))$, and θ'_{-i} represent the same when the sequence is $J*(\theta, s, TB(J^{*2}))$. Since the cut-off-based mechanism is ESV, the ICJ* mechanism is also ESV. This means that agent-i obtains the same utility across all ICJ* mechanisms that differ only in tie-breaking rules. Equating the utility of agent-i in both of the aforementioned sequences, we get $h_i(\theta_{-i}) = h_i(\theta'_{-i})$ whenever θ_{-i} and θ'_i are permutations of each other. Hence, it is necessary that the functions $h_i(x)$ are symmetric. This proves the necessity of 3[15].
Now consider any two agents-i and j. Then, for any of the sequences $\sigma^{J*}(\theta, s, TB(J*)) \in \boldsymbol{\sigma}^{J*}$, ETE demands (proposition 2):

$$
\begin{aligned}
U_i(\mu_i(\theta, s, TB(J*))) &= -\theta_i(s_i + \sum_{k\in P_i(\sigma^{J*}(\theta,s,TB(J*)))} s_k) - \sum_{k\in F_i(\sigma^{J*}(\theta,s,TB(J*)))} \theta_k s_k + h_i(\theta_{-i}) \\
&= -\theta_j(s_j + \sum_{k\in P_j(\sigma^{J*}(\theta,s,TB(J*)))} s_k) - \sum_{k\in F_j(\sigma^{J*}(\theta,s,TB(J*)))} \theta_k s_k + h_j(\theta_{-j}) = U_j(\mu_j(\theta, s, TB(J*))) \\
&\implies h_i(\theta_{-i}) = h_j(\theta_{-j}), \quad \forall i, j \in N
\end{aligned}
$$

Following the necessity of 3, all $h_i(\cdot)$ are symmetric. θ_{-j} can be written as some permutation of θ_{-i}, say θ'_{-i}.Then, $h_i(\theta_{-i}) = h_i(\theta'_{-i}) = h_i(\theta_{-j}) = h_j(\theta_{-j})$ for all $i, j \in N, \theta_{-i}, \theta'_{-i} \in \Theta^{|N\setminus\{i\}|}, \forall \theta_i = \theta_j \in \Theta$. The first

[14]By $\theta' \geq \theta$ we mean $\theta'_i \geq \theta_i, \forall i \in N$

[15]A function $h_i(x)$ is said to be symmetric if $h_i(x) = h_i(y)$ whenever x and y are permutations of each other. The functions $h_i(\cdot)$ must be symmetric. If this is not the case, then the mechanism is not ESV, and it can be shown that this will lead to a violation of implementability/strategyproofness.

equality follows from the fact that $h_i(\cdot)$ is symmetric, the second from the equivalence of θ'_{-i} and θ_{-j} through permutations, and the last equality follows from ETE. This proves the necessity of 1.

Consider a profile $\theta = (\theta_1, \theta_2, \ldots, \theta_{|N|})$ such that $\theta_1 > \theta_2 = \cdots = \theta_{|N|}$. Arbitrarily choose some agent-$j \in \{2, \cdots, |N|\}$. Let the $\mu = (\sigma^{J*}(\theta, s, TB(J*)), \tau)$ be an AICJ* mechanism with symmetric $h_i(\cdots), \forall i \in N$, and design TB(J*) to serve agent-j before any other agent-$k \in N \setminus \{1, j\}$. So, $\sigma_j^{J*}(\theta, s, TB(J*)) = 2$. From the no-envy condition 9 between agent-1 and agent-j, for the mechanism μ to satisfy no-envy and be strategy-proof it is necessary that $s_j(\theta_1 - \theta_j) \geq h(\theta_{-j}) - h(\theta_{-1}) \geq 0$, for all $j \in N \setminus \{1\}$. Since the choice of agent-j was arbitrary, necessary condition obtained is: $\tilde{s}_{min}(\theta_1 - \theta_j) \geq h(\theta_{-j}) - h(\theta_{-1}) \geq 0$, where $\tilde{s}_{min} = \min\{s_2, \cdots, s_{|N|}\} = \min_{k \in N \setminus \{1\}} s_k$. Now consider the profile where agent-1 and agent-2's unit waiting costs are swapped, $\hat{\theta} = (\theta_2, \theta_1, \ldots, \theta_l, \ldots, \theta_{|N|})$. Arbitrarily choose some agent-$l \in \{2, \cdots, |N|\}$. Let the $\mu = (\sigma^{J*}(\hat{\theta}, s, TB(J*l)), \tau)$ be an AICJ* mechanism with symmetric $h_i(\cdots), \forall i \in N$, and design TB(J*l) to serve agent-l before any other agent-$k \in N \setminus \{2, l\}$. This gives the necessary condition: $\hat{s}_{min}(\hat{\theta}_2 - \hat{\theta}_l) \geq h(\hat{\theta}_{-l}) - h(\hat{\theta}_{-2}) \geq 0$, where $\hat{s}_{min} = \min\{s_1, s_3, \cdots, s_{|N|}\} = \min_{k \in N \setminus \{2\}} s_k$. Noting that $\hat{\theta}_2 = \theta_1, \hat{\theta}_l = \theta_j, \theta_{-j} = \theta_{-l}$ and $\hat{\theta}_{-2} = \theta_{-1}$, this condition is: $\hat{s}_{min}(\theta_1 - \theta_j) \geq h(\theta_{-j}) - h(\theta_{-1})$. For mechanism μ satisfy no envy at both profiles $\theta, \hat{\theta} \in \Theta$, the necessary condition is:

$$s_{min}(\theta_1 - \theta_j) \geq h(\theta_{-j}) - h(\theta_{-1}) \geq 0 \text{ where } s_{min} = \min\{\hat{s}_{min}, \tilde{s}_{min}\} = \min\{s_1, s_2, \cdots, s_{|N|}\} \tag{18}$$

Condition 18 is a necessary condition for any pair of adjacent agents. More generally, we may write:

$$\forall \theta \in \Theta^{|N|},\ s \in \mathcal{S},\ \text{and } \forall TB(J*) \in TB \text{ and } \forall i, j \in N s.t. |\sigma_j^{J*}(\theta, s, TB(J*)) - \sigma_j^{J*}(\theta, s, TB(J*))| = 1,$$
$$s_{min}(\theta_i - \theta_j) \geq h(\theta_{-j}) - h(\theta_{-i}) \geq 0 \quad \text{where } s_{min} = \min\{s_1, s_2, \cdots, s_{|N|}\} \tag{19}$$

Finally consider profiles $\theta' \geq \theta \in \Theta^{|N|}$. Let $\theta^0 = \theta$, and for each $k \in \{1, 2, \ldots, |N|$, let θ^k be such that for each $j \in \{1, 2, \ldots, k\}$, $\theta_j^k = \theta'_j$ and for each $j \in \{k, k+1, \ldots, |N|\}$, $\theta_j^k = \theta_j$. Note that for each $k \in \{1, 2, \ldots, |N|$, $\theta_{-k}^k = \theta_{-k}^{k-1}$ and $\theta_k^k \geq \theta_k^{k-1}$ and $\theta^{|N|} = \theta'$. Then from condition 19, we have: $s_{min}(\theta_k^k - \theta_k^{k-1}) \geq h(\theta_{-i}^k) - h(\theta_{-i}^{k-1}) \geq 0$ for all $i \in N$. By transitivity of $\geq$, and recursion, $s_{min} \sum_{j \in N \setminus \{i\}} (\theta'_j - \theta_j) \geq h(\theta'_{-i}) - h(\theta_{-i}) \geq 0$. This proves the necessity of 2.

The sufficiency part is easy to verify and hence omitted. □

5. Cost Monotonicity

In this section, we consider axioms about how the mechanism responds to changes in the sequencing problem. Particularly, we examine some axioms of solidarity, requiring that all relevant agents' welfare be affected in the same direction when changes occur in variables over which they have no control. In this section, we examine the implication of changes in the waiting cost of one agent on other agents. In our sequencing framework, all agents are equally entitled to the service. If the aggregate waiting disutility increases, then society is worse off. Therefore, we could take the view that no agent should benefit from society getting worse off. *Negative Cost Monotonicity* (NCM, in short), introduced in Maniquet [3], requires that an increase in one agent's waiting cost should cause all other agents to be weakly worse off.

Definition 21. ***Negative Cost Monotonicity (NCM):*** *For each $N \in \mathcal{N}, j \in N$ for all profiles $\theta, \theta' \in \Theta^{|N|}$: $\theta'_j > \theta_j$ and $\theta'_k = \theta_k, \forall k \neq j \in N$ and each $i \in N \setminus \{j\}$:*

$$U_i(z_i(\theta, s); \theta_i) \geq U_i(z'_i(\theta', s); \theta'_i).$$

Another viewpoint is to hold each agent individually responsible for changes in his unit waiting cost. That is, if the waiting cost of an agent increases, this is bad news for society, but no other agent should be negatively affected. *Positive cost monotonicity*(PCM, in short), introduced in Chun [2], requires that an increase in one agent's unit waiting cost should cause other agents to be weakly better off.

Definition 22. ***Positive Cost Monotonicity (PCM):*** *For each $N \in \mathcal{N}$, $j \in N$ for all profiles $\theta, \theta' \in \Theta^{|N|}$: $\theta'_j > \theta_j$ and $\theta'_k = \theta_k, \forall k \neq j \in N$ and each $i \in N \setminus \{j\}$:*

$$U_i(z'_i(\theta', s)\theta'_i) \geq U_i(z_i(\theta, s); \theta_i).$$

Chun [2] study a queueing problem, where queue-efficiency is implied by no-envy, and obtain non-existence of mechanisms satisfying *queue-efficiency*[16], *budget-balance*[17], *no-envy* and either *negative* or *positive cost monotonicity*. Yengin & Chun [11] relax the budget balancedness condition and characterise the sub-classes of VCG Mechanisms satisfying either *negative* or *positive cost monotonicity* in queueing problems. In the case of queueing problems, the ICJ, VCG, ICJ^* mechanisms coincide but not in sequencing problems. The Solidarity properties of mechanisms for sequencing problems have not been investigated, particularly those of the ICJ mechanism. We define an independence axiom, as introduced in Chun [2], namely *Independence of Preceding Costs* (IPC, in short), which requires that an increase in the waiting cost of any of an agent's predecessors should not affect the agent.

Definition 23. ***Independence of Preceding Costs (IPC):*** *For each $N \in \mathcal{N}$, $j \in N$ for all profiles $\theta, \theta' \in \Theta^{|N|}$: $\theta'_j \neq \theta_j$, $\theta'_k = \theta_k, \forall k \neq j \in N$ and all $z(\theta, s) = (\sigma(\theta, s), \tau(\theta, s)$ and each $i \in N$ where $\sigma_j(\theta, s) < \sigma_i(\theta, s)$:*

$$U_i(z'_i(\theta', s)\theta'_i) = U_i(z_i(\theta, s); \theta_i).$$

We present our results for the ICJ* mechanism[18].

Theorem 3. *For any sequencing problem Ω^s, the following statements are equivalent:*

1. *The mechanism is strategyproof and satisfies no-envy and negative cost monotonicity.*
2. *The mechanism is strategyproof and satisfies no-envy and Independence of preceding costs.*
3. *The mechanism is $(\sigma^{J*}, \tau^{AICJ*})$ where, for some constant C, the transfer rule satisfies 20.*

$$-\sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))} (s_k - s_{min})\theta_k + C \geq \tau_i^{AICJ*}(\theta, s, TB(J*)) \geq -\sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))} s_k\theta_k + C \tag{20}$$

Proof. Theorem 2 identifies the class of mechanisms satisfying strategyproofness and no-envy. Consider any sequencing problem Ω^s_N with profile θ. The utility of any agent-$i \in N$ in an SP mechanism satisfying NE is given by:

$$U_i(\sigma^{J*}(\theta, s, TB(J*)), \tau^{AICJ*}(\theta, s, TB(J*)) = -\theta_i \cdot S_i(\sigma^J * (\theta, s, TB(J*))) + h(\theta_{-i}, s) - \sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))} s_k\theta_k \tag{21}$$

where $h(\theta_{-i})$ is symmetric and $\forall \theta' \geq \theta \in \Theta$, and $\forall i \in N$ $h(\theta_{-i}) + (\min_{k \in N} s_k) \sum_{j \in N \setminus \{i\}} (\theta'_j - \theta_j) \geq h(\theta'_{-i}) \geq h(\theta_{-i})$. Let agent-j be any predecessor of agent-i in the sequence $\sigma^J * (\theta, s, TB(J*))$. Consider another profile

[16] Coincides with agents being served in non-increasing order of their unit waiting costs for queueing problems.

[17] A mechanism satisfies *Budget-balancedness* if the sum of monetary transfers is zero.

[18] $ICJ \subseteq ICJ*$, so our results hold for ICJ mechanisms.

θ' such that $\theta'_j > \theta_j$ and $\theta'_k = \theta_k$ for all $k \neq j \in N$. Then, $S_i(\sigma^J * (\theta, s, TB(J*))) = S_i(\sigma^J * (\theta', s, TB(J*)))$. Suppose the mechanism $(\sigma^{J*}, \tau^{AICJ*})$ also satisfies NCM. Then,

$$U_i(\sigma^{J*}(\theta, s, TB(J*)), \tau^{AICJ*}(\theta, s, TB(J*)) \geq U_i(\sigma^{J*}(\theta', s, TB(J*)), \tau^{AICJ*}(\theta', s, TB(J*))$$
$$\implies \tau^{AICJ*}(\theta, s, TB(J*) - \tau^{AICJ*}(\theta', s, TB(J*) \geq 0$$
$$\implies h(\theta_{-i}) - h(\theta'_{-i}) \geq 0.$$

NCM demands that $h(\theta_{-i}) \geq h(\theta'_{-i})$, while we have $h(\theta'_{-i}) \geq h(\theta_{-i})$. Therefore, $h(\theta_{-i})$ is independent of the reports of predecessors of agent-i, i.e. $(\theta_{-i})_k, \forall k \in \{1, \ldots, i-1\}$. Hence, SP, NE and NCM together imply that $U_i(\sigma^{J*}(\theta, s, TB(J*)), \tau^{AICJ*}(\theta, s, TB(J*)) = U_i(\sigma^{J*}(\theta', s, TB(J*)), \tau^{AICJ*}(\theta', s, TB(J*))$, whenever the waiting cost of any predecessor of agent-i increases. This is true for every agent-$i \in N$. Hence, SP, NE and NCM imply SP, NE and IPC.
Let $\tilde{\theta} = (\theta_1, \ldots, \theta_i, 0, \ldots, 0)$ be a profile where all agents following agent-i have unit waiting costs zero. The transfer of agent-i $\tau_i^{AICJ*}(\tilde{\theta}, s, TB(J*)) = h(\tilde{\theta}_{-i}) = constant$, say C. Then for any profile $\theta \geq \tilde{\theta}$ such that the set of followers of agent-i $F_i(\sigma(\tilde{\theta}, s, TB(J*))) = F_i(\sigma(\tilde{\theta}, s, TB(J*)))$, we have $s_{min} \sum_{k \in F_i(\sigma(\tilde{\theta}, s, TB(J*)))}(\theta_k - \tilde{\theta}_k) = s_{min} \sum_{k \in F_i(\sigma(\tilde{\theta}, s, TB(J*)))} \theta_k + C \geq h(\theta_{-i}) \geq C$. Thus, a mechanism satisfying SP, NE and IPC has a transfer rule satisfying: $-\sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))} s_k\theta_k + s_{min} \sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))} \theta_k + C \geq \tau_i^{AICJ*}(\theta, s, TB(J*)) \geq -\sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))} s_k\theta_k + C$, where C is a constant.
Now consider a mechanism $(\sigma^{J*}, \tau^{AICJ*})$ where for some constant C, the transfer rule satisfies 20. Since the mechanism is a subclass of AICJ* mechanisms and satisfies all conditions in Theorem 2, it is SP and satisfies NE. Any increase in the waiting cost of an agent's predecessor neither changes the completion time nor the transfer of an agent. If the waiting cost of any of the followers of an agent increases, it decreases the transfer of the agent, since $s_k - s_{min} \geq 0, \forall k \in N$ and may increase the agent's job completion time. Hence, the mechanism satisfies NCM. This completes the proof. □

We obtain a negative result with regards to the search for mechanisms satisfying SP, NE and *Positive-cost Monotonicity*

Theorem 4. *For any sequencing problem Ω^s, no mechanism satisfies Strategyproofness, No-envy, and positive cost monotonicity.*

Proof. From Theorem 2, an envy-free and strategyproof mechanism must be an AICJ* mechanism with symmetric $h(\theta_{-i})$ functions that satisfy for any profiles $\theta' \geq \theta \in \Theta$, $h(\theta_{-i}) + s_{min} \sum_{j \in N \setminus \{i\}}(\theta'_j - \theta_j) \geq h(\theta'_{-i}) \geq h(\theta_{-i}), \forall i \in N$, where $s_{min} = \min\{s_1, \ldots, s_n\}$. From condition 20, $\tau_i^{AICJ*}(\theta', s, TB(J*)) - \tau_i^{AICJ*}(\theta, s, TB(J*)) \leq -\sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))}(s_k - s_{min})\theta_k$, and $\tau_i^{AICJ*}(\theta', s, TB(J*)) - \tau_i^{AICJ*}(\theta, s, TB(J*)) \geq 0, \forall \theta', \theta \in \Theta$. The two conditions hold together for all agents $i \in N$ at all profiles $\forall \theta', \theta \in \Theta$ conditions only if $s_{min} - s_k = 0, k \in N$. This implies that the sequencing problem is a queueing problem. The required subclass is already identified in Chun & Yengin [26]. Hence, for the general class of sequencing problems, we obtain a negative result. □

6. Conclusion

Sequencing problems have garnered significant attention in recent years. However, a critical fairness principle, no envy, has yet to be comprehensively analyzed in this context. Our paper answers this question. Further, along with fairness, we study the strategic aspect of the problem. We identify the class of mechanisms that are strategyproof and satisfy no envy. We then study how the mechanism responds to changes in the problem. Particularly, the solidarity compliance of mechanisms has not been investigated for sequencing problems. We investigate two solidarity axioms regarding how agents are affected when the unit waiting costs of other agents change. Our first contribution identifies the sequencing rules allowing envy-free allocations with suitable transfer rules. We show that envyfreeness amongst all adjacent pairs of agents in the sequence is sufficient to ensure *no-envy*, while it is necessary by definition of *no-envy*. De & Mitra [4] have already pointed out that the Just sequence is always Rawlsian. We show that the Just Sequencing rule admits envy-free allocations. But we take a more liberal stance on the tie-breaking rule used by the mechanism and identify the complete set of sequencing rules that admit envy-free allocations. The Just* sequencing rules that serve agents in non-increasing order of their unit waiting costs are the only envy-free sequences. De & Mitra [4] identified the ICJ mechanism, which implements the Just sequencing rule. We add to the literature and identify a larger class than the ICJ mechanism, the ICJ* mechanism, which implements all Just* sequences in dominated strategies. We next identify the necessary and sufficient condition for a mechanism to be strategyproof and satisfy no envy in the case of two agents. Then, we identify the class of mechanisms satisfying strategyproofness and no-envy for the general sequencing problem. Maniquet [3] introduced the notion of Negative Cost Monotonicity in the queuing problems. We extend the same notion to sequencing problems. We characterise the class of strategyproof mechanisms satisfying no envy and negative cost monotonicity and prove its equivalence with the class of strategyproof mechanisms satisfying no envy and independence of preceding cost. The independence axiom was introduced in Chun [2]. All of our results hold not for one sequencing rule but the entire set of sequencing rules that admit envy-free allocations. By changing the tie-breaking rule, our results hold for the entire set of mechanisms. In this sense, the results hold for the Pareto indifferent correspondence pair of a set of sequencing rules and a corresponding set of transfer rules. Regarding queueing problems, Chun [2] discuss the importance of imposing Pareto indifference to characterise more than one mechanism. We make similar observations for sequencing problems. Yengin & Chun [11] relax the budget balancedness condition and characterise the sub-classes of VCG Mechanisms satisfying either *negative* or *positive cost monotonicity* in queueing problems. In the case of queueing problems, the ICJ, VCG, ICJ^* mechanisms coincide but not in sequencing problems. Our characterisation result adds to this literature. Further, we show that mechanisms satisfying Strategyproofness, no envy and positive cost monotonicity may exist only for a subclass of sequencing problems, the queueing problem. We establish the non-existence of such mechanisms for the general sequencing problem. Such characterisation for queueing problems has been done in Chun & Yengin [26].

Appendix A. Proof of Lemma 1

Proof. Following De [27] (pp. 49), consider any $\sigma^{J*}(\theta, s, TB(J*)) \in \boldsymbol{\sigma}^{J*}(\theta, s, TB(J*)) \subset \boldsymbol{\sigma}^{NI}$ and any agent $i \in N$. Fix a profile θ_{-i}. Suppose that the number of different completion time(s) that agent faces as θ_i varies over Θ is T. This does not consider the different completion times due to different tie-breaking rules. For each $i \in N$, we first select any function $h_i(\theta_{-i}) : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$ and then, given $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, we consider the waiting cost cut off vector $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$ where $0 := \theta_i^{(0)} < \theta_i^{(1)}(\theta_{-i}) < \ldots < \theta_i^{(T-1)}(\theta_{-i}) < \theta_i^{(T)} =: \infty$ such that for any $t \in \{1, 2, \ldots, T-1\}$, $S_i(\sigma^{J*}((\theta_i^t, \theta_{-i}), s, TB(J*))) := \bar{S}(t, \theta_{-i})$ for all $\theta_i^t \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i}))$. Define $D_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - \bar{S}(t, \theta_{-i})$ and $\bar{D}_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - S_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*)))$ for any $t \in \{1, 2, \ldots, T-1\}$. . Observe that the difference in the definitions of $D_t(\theta_{-i})$ and $\bar{D}_t(\theta_{-i})$ lies in the second term. For $D_t(\theta_{-i})$, $\bar{S}(t, \theta_{-i})$ is the completion time of agent-i when his waiting cost is any number θ_i^t that lies in the open interval $(\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i}))$. For $\bar{D}_t(\theta_{-i})$, $S_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*)))$ is the completion time of agent-i when his waiting cost is exactly $\theta_i^{(t)}(\theta_{-i})$ which is a cut off point. Depending on the tie-breaking rule, $\bar{S}(t, \theta_{-i})$ and $S_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*)))$ may or may not be different, and hence for completeness of the analysis, the distinction between $D_t(\theta_{-i})$ and $\bar{D}_t(\theta_{-i})$ is necessary. The cut-off points $\theta_i^{(t)}(\theta_{-i})$ are the distinct unit waiting costs reported by all the agents other than agent-i. Thus, $\theta_i^{(t)}(\theta_{-i}) \in \theta_{-i}$ such that $\theta_i^{(t-1)}(\theta_{-i}) \neq \theta_i^{(t)}(\theta_{-i})$ for any $t \in \{1, \ldots, T\}$ and hence $T \leq |N| - 1$. For the sequence $\sigma^{J*}(\theta, s, TB(J*))$, the set of agents-$k \in N$ for whom $\theta_k < \theta_i$ is always a subset of the follower set of agent-i: $F_i(\sigma^{J*}((\theta_i, \theta_{-i}), s, TB(J*)))$. When $\hat{\theta}_i = \theta_i^t \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i}))$, agent-$i$'s report is not tied with any other agents report since their reports are the cut-off points, therefore the set of agents-$k \in N$ for whom $\theta_k < \theta_i^t$ is the follower set of agent-i: $F_i(\sigma^{J*}((\theta^t{}_i, \theta_{-i}), s, TB(J*)))$. Therefore, for $\theta_i \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i})), t = \{1, 2, \ldots, T-1\}$:

$$D_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - \bar{S}(t, \theta_{-i}) = \sum_{k \in N : \theta_k = \theta_i^{(t)}} s_k$$

$$I_i(\theta_i, \theta_{-i}) = \sum_{r=t}^{T-1} \theta_i^{(r)}(\theta_{-i}) D_r(\theta_{-i}) = \sum_{r=t}^{T-1} \theta_i^{(r)}(\theta_{-i}) \sum_{k \in N : \theta_k = \theta_i^{(r)}} s_k = \sum_{r=t}^{T-1} \sum_{k \in N : \theta_k = \theta_i^{(r)}} \theta_i^{(r)} s_k = \sum_{k \in F_i(\sigma^{J*}((\theta_i, \theta_{-i}, s, TB(J*))))} \theta_k s_k$$

This holds true for all $i \in N, \theta_i \in \Theta, \theta_{-i} \in \Theta^{|N\setminus\{i\}|}$ and for any tie breaking rule $TB(J*) \in TB$. Also,

$$\bar{D}_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - S_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*))) = S_i(\sigma^{J*}((\theta_i^{(t+1)}, \theta_{-i}), s, TB(J*))) - S_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*)))$$

Note that the first term involves a report in the open interval and is thus invariant under the choice of any tie-breaking rule. Since the sets $\{i\} \cup P_i(\sigma^{J*}((\theta_i^{t+1}, \theta_{-i}), s, TB(J*))) = N \setminus F_i(\sigma^{J*}((\theta_i^{t+1}, \theta_{-i}), s, TB(J*)))$ and $\{i\} \cup P_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*))) = N \setminus F_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*)))$.

$$\bar{D}_t(\theta_{-i}) = \sum_{k \in \{i\} \cup P_i(\sigma^{J*}((\theta_i^{t+1}), \theta_{-i}), s, TB(J*))} s_k - \sum_{k \in \{i\} \cup P_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*))} s_k$$

$$= \sum_{k \in N \setminus F_i(\sigma^{J*}((\theta_i^{t+1}, \theta_{-i}), s, TB(J*)))} s_k - \sum_{k \in N \setminus F_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*)))} s_k = \sum_{k \in F_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*)))} s_k - \sum_{k \in F_i(\sigma^{J*}((\theta_i^{t+1}, \theta_{-i}), TB(J*)))} s_k$$

$$= \sum_{k \in F_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*))) \cap \theta_k = \theta_i^{(t)}} s_k$$

Hence, when $\theta_i^t \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i}))$ (In the following calculation, we abuse notation and denote $\theta_i^{(t)}(\theta_{-i})$ as $\theta_i^{(t)}$)

$$I_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i}) = I_i(\theta_i^t, \theta_{-i}) - \theta_i^{(t)} \bar{D}_t(\theta_{-i}) + \theta_i^{(t)} D_t(\theta_{-i})$$

$$= I_i(\theta_i^t, \theta_{-i}) - \theta_i^{(t)}[D_t(\theta_{-i}) - \bar{D}_t(\theta_{-i})] = I_i(\theta_i^t, \theta_{-i}) - \theta_i^{(t)}[\sum_{k \in N: \theta_k = \theta_i^{(t)}} s_k - \sum_{k \in F_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*))) \cap \theta_k = \theta_i^{(t)}} s_k]$$

$$= I_i(\theta_i^t, \theta_{-i}) - \sum_{k \in P_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*))) \cap \theta_k = \theta_i^{(t)}} \theta_i^{(t)} s_k = \sum_{k \in F_i(\sigma^{J*}((\theta_i, \theta_{-i}, s, TB(J*))))} \theta_k s_k - \sum_{k \in P_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*))) \cap \theta_k = \theta_i^{(t)}} \theta_k s_k$$

$$= \sum_{k \in F_i(\sigma^{J*}((\theta_i^{(t)}, \theta_{-i}), s, TB(J*)))} \theta_k s_k$$

Hence, $\forall \theta_i \in \Theta$, we have:

$$I_i(\theta_i, \theta_{-i}) = \begin{cases} 0 & \text{if } \theta_i \in (\theta_i^{(T)}, \theta_i^{T-1}(\theta_{-i})), \\ \sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))} \theta_k s_k & \text{if } \theta_i \in (\theta_i^{T-1}(\theta_{-i}), \theta_i^0) \end{cases} \tag{A.1}$$

Following the convention, $\sum_{k \in A}(\cdot) = 0$, whenever $A = \phi$, We can write:

$$\tau_i^{J*}(\theta, s, TB(J*)) = h_i(\theta_{-i}, s) - \sum_{k \in F_i(\sigma^{J*}(\theta, s, TB(J*)))} s_k \theta_k$$

□ □

This proves that a mechanism implements any sequence $\sigma^{J*} \in \boldsymbol{\sigma}^{J*}$ *only if* it is an *ICJ* mechanism*. The *if* part only involves verification and is left to the reader. Note that the transfer is exactly the same as the *ICJ* transfer rule. We did not expand the transfer rule. We only expanded the scope of applicability of the ICJ transfer rule and renamed it as *ICJ** transfer rule. For the *Just* sequences, both transfer rules are the same. For the set of *Just** rules, the ICJ transfer rule will give the same transfers, but De & Mitra [4] did not show that all *Just** rules are implementable by the *ICJ* transfer rule. Our contribution in proving lemma 1, therefore, although rigorous, is not of much novelty. However, for the sake of completeness of our arguments in this paper, it was necessary to prove lemma 1.

Appendix B. ICJ* example

Consider a three-agent problem. The agents are $N = \{a, b, c\}$ and the reported profile is $\theta = (\theta_a, \theta_b, \theta_c)$ where $\theta_a = \theta_b = \theta_c = 5$ and the vector of processing times is $s = (s_a = 3, s_b = 6, s_c = 6)$. In total, there are $3! = 6$ possible sequences. Since all agents are tied in waiting costs, all six sequences are *Just** sequences but only the two sequences serving agent-a first (*abc*) and (*acb*) are *Just* sequences. The tables below summarise the allocations in all six sequences.
The tie-breaking rule TB_{bc} (respectively TB_{cb}) is such that it resolves the tie between agents-*b* and *c*, in favour of *b* (respectively *c*). Because $s_a < s_b = s_c$, agent-*a* is always served before the other two agents under these two sequences recommended by the *ICJ Mechanism*. Note that the mechanism is *Essentially Single Valued* i.e. no matter which sequence is recommended, all agents have the same utility under both allocations. The *ICJ* Mechanism* recommends the above two allocations, as well as four more allocations. The *ICJ* Mechanism* resolves the tie without resorting to the criterion in *Just* sequencing. Hence, *Just* sequencing only needs to resolve ties amongst agents who are tied in waiting costs as well as processing

Parameter	Agent-a	Agent-b	Agent-c
Waiting cost	$-5(3)$	$-5(3+6)$	$-5(3+6+6)$
Transfer	$-5(6)-5(6)+h(5,5)$	$-5(6)+h(5,5)$	$h(5,5)$
Allocation (S_i, τ_i)	$(3, -60+h(5,5))$	$(9, -30+h(5,5))$	$(15, h(5,5))$
Utility$(U_i(S_i, \tau_i))$	$-75+h(5,5)$	$-75+h(5,5)$	$-75+h(5,5)$

Table B.1: $\sigma_a^{J*}(\theta, s, TB_{bc}) = 1, \sigma_b^{J*}(\theta, s, TB_{bc}) = 2, \sigma_c^{J*}(\theta, s, TB_{bc}) = 3$

Parameter	Agent-a	Agent-b	Agent-c
Waiting cost	$-5(3)$	$-5(3+6+6)$	$-5(3+6)$
Transfer	$-5(6)-5(6)+h(5,5)$	$h(5,5)$	$-5(6)+h(5,5)$
Allocation (S_i, τ_i)	$(3, -60+h(5,5))$	$(15, h(5,5))$	$(9, -30+h(5,5))$
Utility$(U_i(S_i, \tau_i))$	$-75+h(5,5)$	$-75+h(5,5)$	$-75+h(5,5)$

Table B.2: $\sigma_a^{J*}(\theta, s, TB_{cb}) = 1, \sigma_b^{J*}(\theta, s, TB_{cb}) = 3, \sigma_c^{J*}(\theta, s, TB_{cb}) = 2$

times and thus selects a subset of all possible tie-breaking rules, which may result in 6 possible sequences. The tie-breaking rules can be based on any criterion, but the set of all possible outcomes of application of any tie-breaking rule is only 6. Consider the tie-breaking rule TB_{bca} serving agents in the order b followed by c followed by a. Likewise, consider the rules TB_{cba}, TB_{cab} and TB_{bac}.

Parameter	Agent-a	Agent-b	Agent-c
Waiting cost	$-5(6+6+3)$	$-5(6)$	$-5(6+6)$
Transfer	$h(5,5)$	$-5(6)-5(3)+h(5,5)$	$-5(3)+h(5,5)$
Allocation (S_i, τ_i)	$(15, h(5,5))$	$(6, -5(6)-5(3)+h(5,5))$	$(12, -5(3)+h(5,5))$
Utility$(U_i(S_i, \tau_i))$	$-75+h(5,5)$	$-75+h(5,5)$	$-75+h(5,5)$

Table B.3: $\sigma_a^{J*}(\theta, s, TB_{bca}) = 3, \sigma_b^{J*}(\theta, s, TB_{bca}) = 1, \sigma_c^{J*}(\theta, s, TB_{bca}) = 2$

Note that the aggregate disutility in waiting costs $=\sum_{k\in N} \theta_k S_k(\theta, s, TB(J*))$ measured as a sum of waiting costs of all agents is -135 for the sequences recommended by ICJ Mechanism in Table B.1 and Table B.2. The same is -165 for sequences recommended by ICJ* Mechanism in Table B.3 and Table B.4, and it is -150 for sequences recommended by ICJ* Mechanism in Table B.5 and Table B.6. The advantage of ICJ over the ICJ* mechanism is that it only recommends the aggregate cost-minimising sequences amongst those recommended by the ICJ* mechanism. Still, the ICJ* mechanism is Pareto Indifferent, whilst the ICJ mechanism is not. The justification of the bias of the ICJ mechanism towards cost minimisation amongst recommendable allocations is difficult. Agents must pay more in transfers under recommendations by the ICJ mechanism than under the other allocations in the ICJ* mechanism. In the quasi-linear setup we study, if welfare is the same, agents have no preference for any combination of waiting times and transfers that achieve that welfare. The allocations not recommended by ICJ but recommended by ICJ* mechanism lead to all agents achieving the same utility with lesser transfers. Where transfer bounds are relevant, say when agents have a limited endowment of money, the other four recommendations may bear an advantage over those recommended by ICJ. Whether to bias a mechanism in favour of lesser disutility in waiting or in favour of lesser money availability with agents is another question that we do not delve into. The fact that ICJ $\subseteq ICJ*$ mechanisms should be reason enough to work with ICJ* mechanisms when the justification for efficiency is not readily available. Both mechanisms move considerably away from the notion of efficiency. This partial return to efficiency at the cost of Pareto Indifference needs more justification.

Parameter	Agent-a	Agent-b	Agent-c
Waiting cost	$-5(6+6+3)$	$-5(6+6)$	$-5(6)$
Transfer	$h(5,5)$	$-5(3)+h(5,5)$	$-5(6)-5(3)+h(5,5)$
Allocation (S_i, τ_i)	$(15, h(5,5))$	$(12, -5(3)+h(5,5))$	$(6, -5(6)-5(3)+h(5,5))$
Utility$(U_i(S_i, \tau_i))$	$-75+h(5,5)$	$-75+h(5,5)$	$-75+h(5,5)$

Table B.4: $\sigma_a^{J*}(\theta, s, TB_{cba}) = 3, \sigma_b^{J*}(\theta, s, TB_{cba}) = 2, \sigma_c^{J*}(\theta, s, TB_{cba}) = 1$

Parameter	Agent-a	Agent-b	Agent-c
Waiting cost	$-5(6+3)$	$-5(6)$	$-5(6+3+6)$
Transfer	$-5(6)+h(5,5)$	$-5(3)-5(6)+h(5,5)$	$h(5,5)$
Allocation (S_i, τ_i)	$(9, -5(6)+h(5,5))$	$(6, -5(6)-5(3)+h(5,5))$	$(15, h(5,5))$
Utility$(U_i(S_i, \tau_i))$	$-75+h(5,5)$	$-75+h(5,5)$	$-75+h(5,5)$

Table B.5: $\sigma_a^{J*}(\theta, s, TB_{bac}) = 2, \sigma_b^{J*}(\theta, s, TB_{bac}) = 1, \sigma_c^{J*}(\theta, s, TB_{bac}) = 3$

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Parameter	Agent-a	Agent-b	Agent-c
Waiting cost	$-5(6+3)$	$-5(6+3+6)$	$-5(6)$
Transfer	$-5(6)+h(5,5)$	$h(5,5)$	$-5(3)-5(6)+h(5,5)$
Allocation (S_i,τ_i)	$(9,-5(6)+h(5,5))$	$(15,h(5,5))$	$(6,-5(6)-5(3)+h(5,5))$
Utility$(U_i(S_i,\tau_i))$	$-75+h(5,5)$	$-75+h(5,5)$	$-75+h(5,5)$

Table B.6: $\sigma_a^{J*}(\theta,s,TB_{cab})=2, \sigma_b^{J*}(\theta,s,TB_{cab})=3, \sigma_c^{J*}(\theta,s,TB_{cab})=1$

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